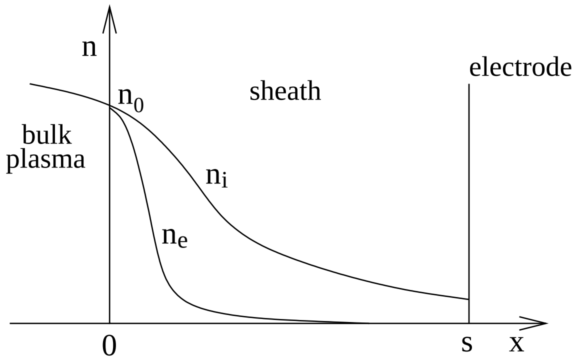
DC sheath

1 Collisionless sheath



$$\Phi(0) = 0 \quad (1)$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} \approx 0 \quad (2)$$

$$n_e = n_0\,\mathrm{e}^{\frac{q\Phi}{kT_e}} \quad (3)$$

$$n_i v_i = n_0 v_B \quad (4)$$

$$\frac{1}{2}m_i v_i^2 + q\Phi = \frac{1}{2}m_i v_B^2 \quad (5)$$

We start with the Poisson equation $\triangle\Phi = -\frac{\rho}{\varepsilon_0}$:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = -\frac{q}{\varepsilon_0}(n_i - n_e) = \frac{n_0 q}{\varepsilon_0}\left(\mathrm{e}^{q\Phi/kT_e} - \frac{1}{\sqrt{1-\frac{2q\Phi}{m_i\,v_B^2}}}\right) \quad (6)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right)^2 = 2\,\frac{\mathrm{d}\Phi}{\mathrm{d}x}\,\frac{n_0 q}{\varepsilon_0}\left(\mathrm{e}^{q\Phi/kT_e} - \frac{1}{\sqrt{1-\frac{2q\Phi}{m_i\,v_B^2}}}\right)$$

$$E^2 - E_0^2 = \frac{2n_0 q}{\varepsilon_0}\int_0^{\Phi}\left(\mathrm{e}^{q\Phi'/kT_e} - \frac{1}{\sqrt{1-\frac{2q\Phi'}{m_i\,v_B^2}}}\right)\mathrm{d}\Phi' \quad (7)$$

In order to shorten the notation, we will tranform the equation to dimensionless quantities

$$\varphi = \frac{q\Phi}{kT_e}$$

$$\xi = \frac{x}{\lambda_D}$$

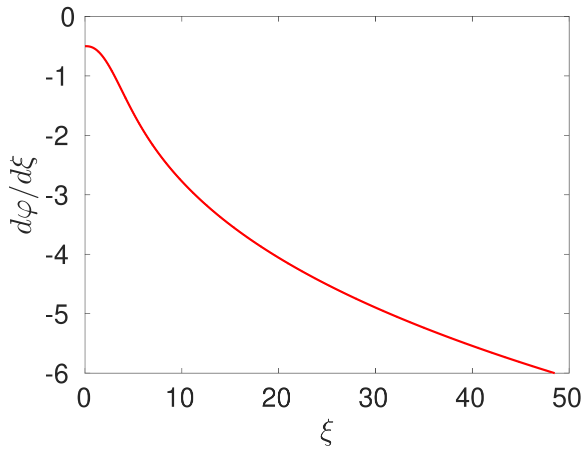
$$\lambda_D = \frac{1}{q}\sqrt{\frac{\varepsilon_0 kT_e}{n_0}}$$

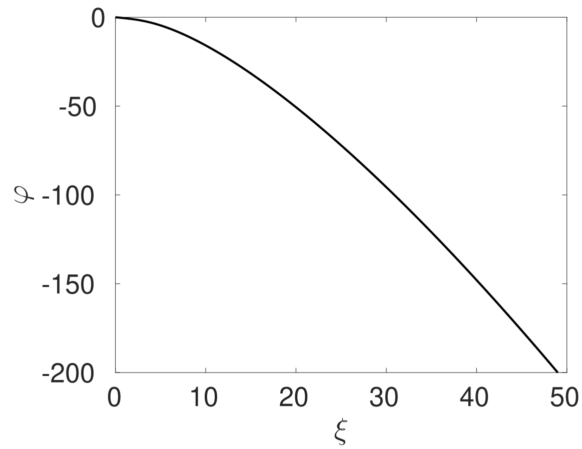
The eq. (7) can be transformed with the use of (10) to

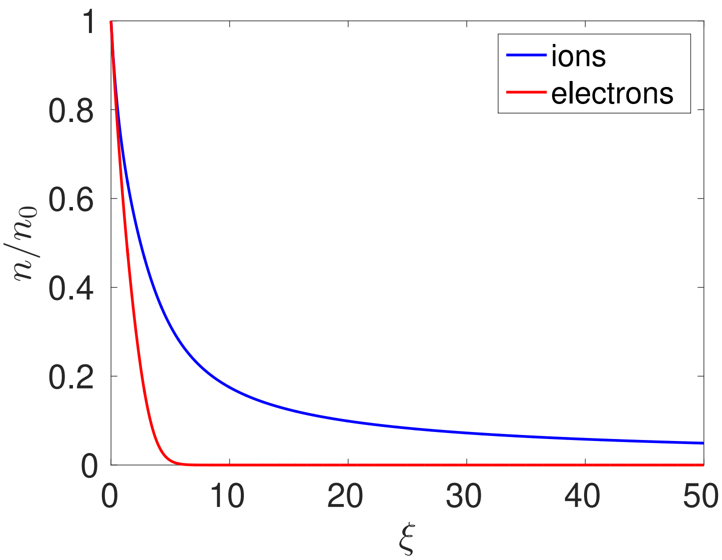
$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 - \left(\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\right)^2 = 2\int_0^{\varphi}\left(\mathrm{e}^{\varphi'} - \frac{1}{\sqrt{1-2\varphi'}}\right)\mathrm{d}\varphi' = 2\left(\mathrm{e}^{\varphi} - 1 + \sqrt{1-2\varphi} - 1\right) \quad (8)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = -\sqrt{\left(\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right|_{\xi=0}\right)^2 + 2\left(\mathrm{e}^{\varphi} + \sqrt{1-2\varphi} - 2\right)} \quad (9)$$

An example of the solution calculated by means of eq. (9) is shown bellow:







2 Bohm velocity

Because the left-hand part of the eq. (7) is not negative and because $\mathrm{d}\Phi' < 0$, the following inequality must be valid:

$$\mathrm{e}^{q\Phi/kT_e} - \frac{1}{\sqrt{1 - \frac{2q\Phi}{m_i\, v_B^2}}} \;\le\; 0,$$

which leads to

$$v_B^2 \;\ge\; \frac{1}{m_i}\,\frac{2q\Phi}{1 - \mathrm{e}^{-2q\Phi/kTe}}.$$

This inequality must be valid also for small velues of the potential Φ, thus

$$v_B^2 \;=\; \frac{kT_e}{m_i}. \tag{10}$$

The ion velocity $\sqrt{kT_e/m_i}$ is called the Bohm velocity. It is the drift velocity of ions at the plasma-sheath border.

3 Floating potential

If there is no net electric current flowing through the sheath, the sheath voltage Φ_{fl} can be calculated from the equality of the electron and ion flow:

$$\begin{aligned}
\frac{1}{4} n_0 \sqrt{\frac{8kT_e}{\pi m_e}}\, \mathrm{e}^{\frac{q\Phi_{fl}}{kT_e}} &= n_0 \sqrt{\frac{kT_e}{m_i}} \\
q\Phi_{fl} &= -\frac{kT_e}{2} \ln \frac{m_i}{2\pi m_e} \qquad (11)
\end{aligned}$$

4 Child-Langmuir law for collisionless sheath

The eq. (7) can be modified for assumptions $n_e \approx 0$, $\frac{1}{2} m_i\, v_B^2 \ll q\Phi$ a $E_0 \ll E$ and with the help of $j = qn_0 v_B$ to

$$\begin{aligned}
E^2 &\approx -\frac{2n_0 q}{\varepsilon_0} \int_0^\Phi \frac{\mathrm{d}\Phi'}{\sqrt{-\frac{2q\Phi'}{m_i\, v_B^2}}} = -\frac{\sqrt{2qm_i}}{\varepsilon_0} n_0\, v_B \int_0^\Phi \frac{\mathrm{d}\Phi'}{\sqrt{-\Phi'}} = \frac{j}{\varepsilon_0} \sqrt{\frac{2m_i}{q}}\, 2\sqrt{-\Phi} \\
\frac{\mathrm{d}\Phi}{\mathrm{d}x} &= -\sqrt{\frac{j}{\varepsilon_0} 2 \sqrt{\frac{2m_i}{q}}}\, (-\Phi)^{\frac{1}{4}} \\
(-\Phi)^{\frac{3}{4}} &= \frac{3}{2} \sqrt{\frac{j}{\varepsilon_0} \sqrt{\frac{m_i}{2q}}}\, x \\
j &= \frac{4\varepsilon_0}{9} \sqrt{\frac{2q}{m_i}}\, \frac{U_{sh}^{\frac{3}{2}}}{s^2} \qquad (12)
\end{aligned}$$

5 Collisional sheath

$$j = q\, n_i\, v_i = q\, n_i\, \mu_i\, E \qquad (13)$$

We will assume a constant ion mobility μ_i:

$$\begin{aligned}
\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} &= -\frac{qn_i}{\varepsilon_0} = \frac{j}{\varepsilon_0\, \mu_i\, \frac{\mathrm{d}\Phi}{\mathrm{d}x}} \\
2\, \frac{\mathrm{d}\Phi}{\mathrm{d}x}\, \frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} &= \frac{2j}{\varepsilon_0\, \mu_i} \\
\frac{\mathrm{d}\Phi}{\mathrm{d}x} &= -\sqrt{\frac{2j}{\varepsilon_0\, \mu_i}\, x} \\
-\Phi &= \frac{2}{3} \sqrt{\frac{2j}{\varepsilon_0\, \mu_i}}\, x^{\frac{3}{2}} \\
j &= \frac{9\varepsilon_0\, \mu_i}{8}\, \frac{U_{sh}^2}{s^3} \qquad (14)
\end{aligned}$$

6 Sheath with homogeneous ion concentration (matrix sheath)

$$E = \frac{qn_0\, x}{\varepsilon_0} \tag{15}$$

$$U_{sh} = \frac{qn_0\, s^2}{2\varepsilon_0} \tag{16}$$