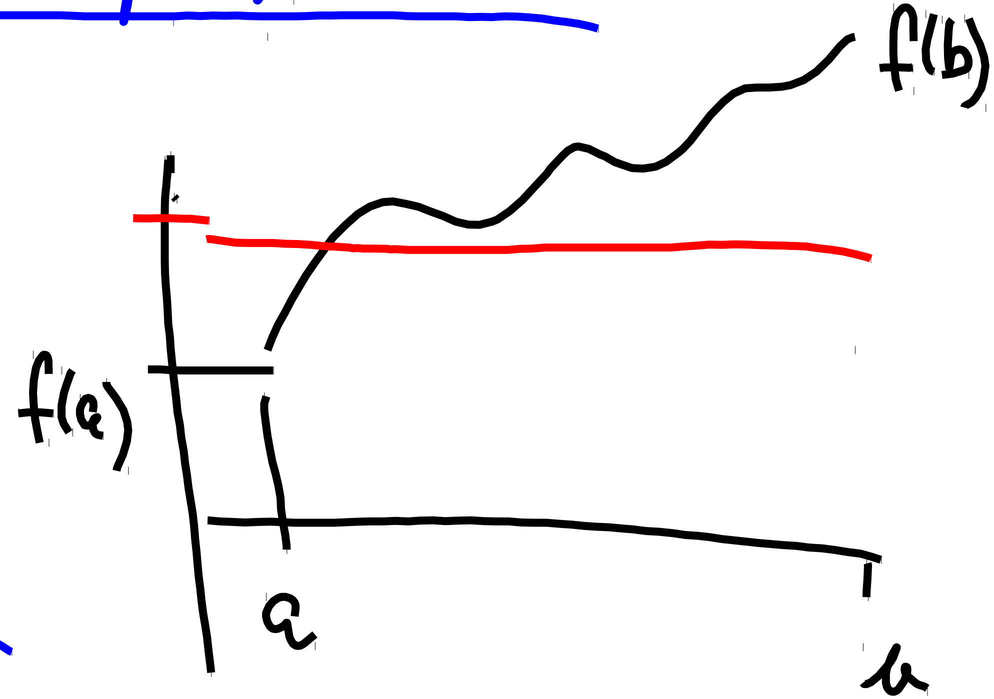
Dodatek ke spojitosti

Věta o nabývání mezihodnot



Nechť $f : [a,b] \to \mathbb{R}$ je spojitá a $f(a) < f(b)$.
Pak pro každé $c \in [f(a), f(b)]$ existuje $x_0 \in [a,b]$ $f(x_0) = c$

Věta neplatí v rac. číslech

$$f(x) = x^2 - 2 \qquad [0,3]$$

$$f(0) = -2 < f(3) = 7$$

$0 \in [-2,7]$ $\qquad f(x_0) = x_0^2 - 2 = 0$

$x_0 \in [0,3]$ $\qquad x_0 = \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Příklad Řešte rovnici $\quad 2^x = 11 - x$

$2^x + x = 13$

$f(x) = 2^x + x$ je rostoucí $\qquad 2^x + x = 11$

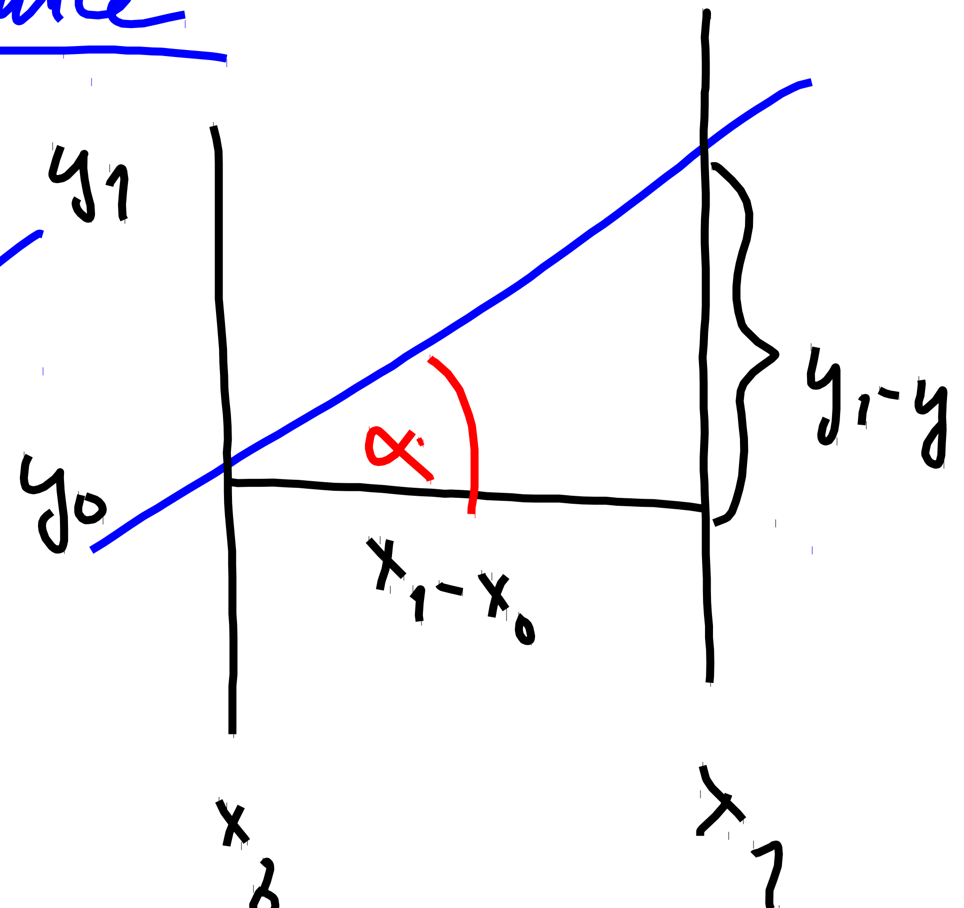
Víme, že existuje právě jedno řešení

na $\mathbb{R}$, spojitá. Protože je f rostoucí, může hodnota 11 nabývat v jediném čísle. $f(0) = 1 \quad f(4) = 2^4 + 4 = 20$

$11 \in (1,20)$ víme, že existuje x_0 $f(x_0) = 11$, $\qquad x_0 = 3$.

Derivace funkce

Směrnice přímky procházející body $[x_0, y_0]$, $[x_1, y_1]$



$$k = \frac{y_1 - y_0}{x_1 - x_0} = \operatorname{tg} \alpha$$

Definice derivace Nechť interval $(x_0-\Delta, x_0+\Delta)$, $\Delta>0$, leží v def. oboru funkce f. Derivace f v bodě x_0 je limita

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

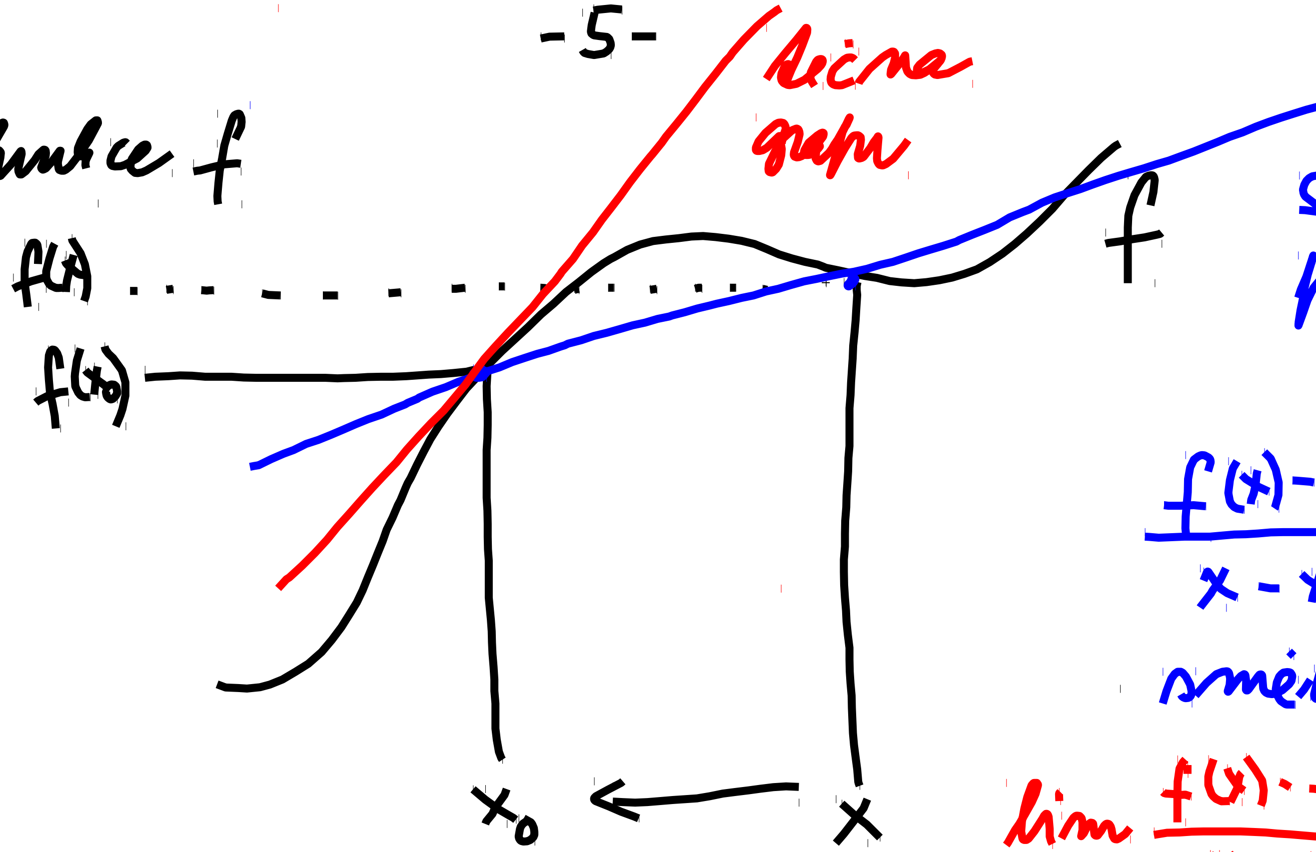
Derivace zprava

$$\lim_{x\to x_0+}\frac{f(x)-f(x_0)}{x-x_0} = f'_+(x_0)$$

Derivace zleva

$$\lim_{x\to x_0-}\frac{f(x)-f(x_0)}{x-x_0} = f'_-(x_0)$$

Graf funkce f



sečna grafu funkce f

$$\frac{f(x)-f(x_0)}{x-x_0} \text{ je směrnice sečny}$$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \text{ je směrnice}$$

tečny v bodě $[x_0, f(x_0)]$ ke grafu funkce f

Vlastnosti

1) Funkce má v x_0 nejvýše jednu derivaci

2) Má-li funkce f na intervalu (a,b) v každém bodě derivaci, je spojitá.

3) $$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

Derivace elementárních funkcí

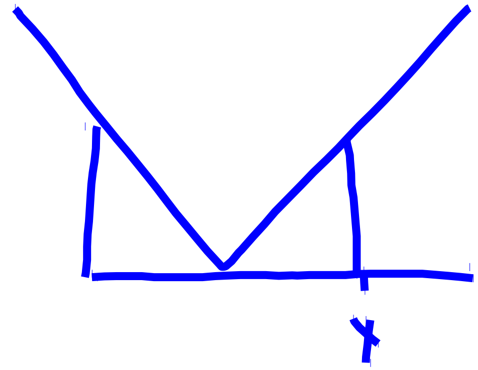
(1) Konstantní funkce $f(x)=k$ $\qquad (\text{konst})'=0$

$$f'(x_0)=\lim_{h\to 0}\frac{k-k}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

(2) $f(x)=ax+b$ $\qquad (ax+b)'=a$

$$f'(x_0)=\lim_{h\to 0}\frac{a(x_0+h)+b-ax_0-b}{h} = \lim \frac{ah}{h} = \lim a = a$$

(3) $f(x) = |x|$



$x > 0$ $\quad f'(x) = \lim\limits_{h \to 0} \dfrac{|x+h| - |x|}{h}$

$$= \lim_{h \to 0} \frac{x+h-x}{h} = 1$$

$x < 0$ $\quad f'(x) = \lim\limits_{h \to 0} \dfrac{|x+h| - |x|}{h} = \lim \dfrac{-(x+h)-(-x)}{h} = -1$

$x = 0$

$$f'_+(0) = \lim_{h \to 0_+} \frac{|0+h| - |0|}{h} = \lim \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0_-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0_-} \frac{-h}{h} = -1$$

$f'(0)$ neexistuje.

(4) $f(x) = x^2$

$$f'(x_0) = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0} \frac{x_0^2+2hx_0+h^2-x_0^2}{h}$$

$$= \lim \frac{2hx_0}{h} + \lim \frac{h^2}{h} = \lim_{h\to 0} 2x_0 + \lim_{h\to 0} h$$

$$= 2x_0 + 0 = 2x_0$$

$$(x^2)' = 2x$$

(5) $f(x) = \frac{1}{x}$

$$f'(x_0) = \lim_{h\to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} = \lim_{h\to 0} \frac{x_0 - (x_0+h)}{(x_0+h)x_0 h} =$$

$$= \lim_{h\to 0} \frac{-1}{(x_0+h)x_0} = -\frac{1}{x_0^2}$$

$$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

(6) Nicht $n \in \mathbb{Z}$ $\quad f(x) = x^n$

$$f'(x) = n\,x^{n-1}$$

$n = 2 \qquad (x^2)' = 2x$

$n = -1 \qquad x^{-1} = \frac{1}{x} \qquad (x^{-1})' = \left(\frac{1}{x}\right)' = \frac{-1}{x^2} = (-1)\,x^{-2}$

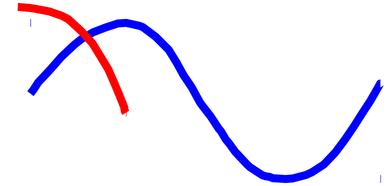
(7) $f(x) = \sin x$ $\qquad (\sin x)' = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2 \sin \frac{x+h-x}{2} \cdot \cos \frac{x+h+x}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) = 1 \cdot \cos x = \cos x$$

(8) Podobně $(\cos x)' = -\sin x$



(9) Derivace funkce $f(x) = e^x$

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$

$$= \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{\overset{!}{=}1} \cdot \lim_{h\to 0} e^x = 1 \cdot e^x = e^x \qquad (e^x)' = e^x$$

(10) $f(x) = \ln x$

$$f'(x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \cdot \frac{1}{x} = \lim_{y \to 0} \frac{\ln(1+y)}{y} \cdot \lim_{h \to 0} \frac{1}{x}$$

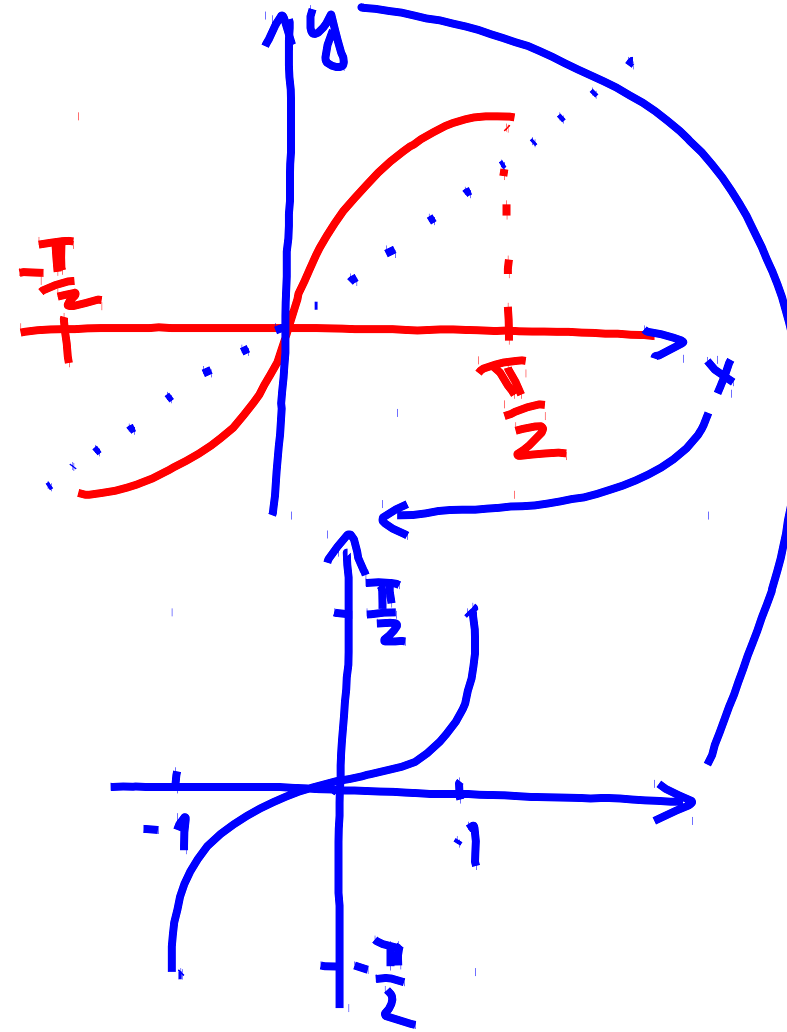
$$= 1 \cdot \frac{1}{x} = \frac{1}{x}$$

$$(\ln x)' = \frac{1}{x}$$

Odbočka

sin : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$

arcsin : $[-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

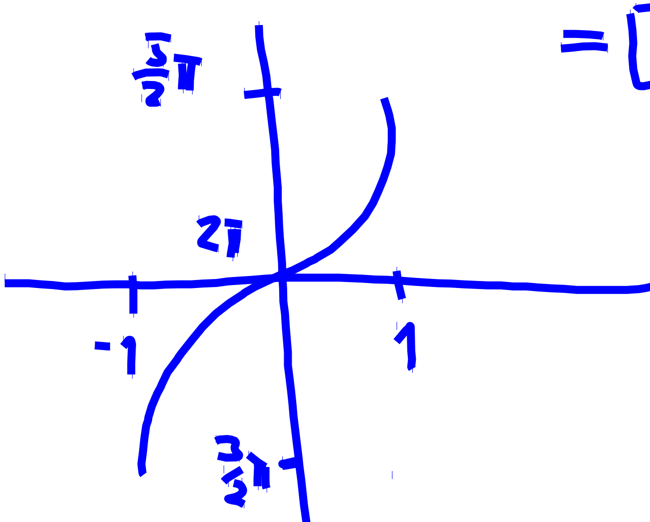


Def. obo. obor hodnot, graf

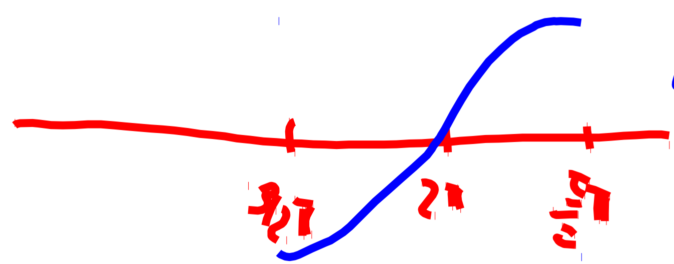
$f(x) = \arcsin x + 2\pi$

$D(f) = [-1, 1]$ $\quad H(f) = \left[-\frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 2\pi\right]$

$= \left[\frac{3}{2}\pi, \frac{5}{2}\pi\right]$



Co je inverzní funkce k f ?



sin : $\left[\frac{3}{2}\pi, \frac{5}{2}\pi\right] \longrightarrow [-1, 1]$

$f(x) = \arcsin x + 2\pi$

$g(x) = \sin x \quad : \left[\frac{3}{2}\pi, \frac{5}{2}\pi\right] \to [-1, 1]$

$x \in [-1, 1]$

$g(f(x)) = \sin\left(\arcsin x + 2\pi\right) = \sin(\arcsin x) = x$

$y \in \left[\frac{3}{2}\pi, \frac{5}{2}\pi\right]$

$y - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$f(g(y)) = \arcsin(\sin y) + 2\pi$

$= \underbrace{\arcsin(\sin(y - 2\pi))}_{y - 2\pi} + 2\pi$

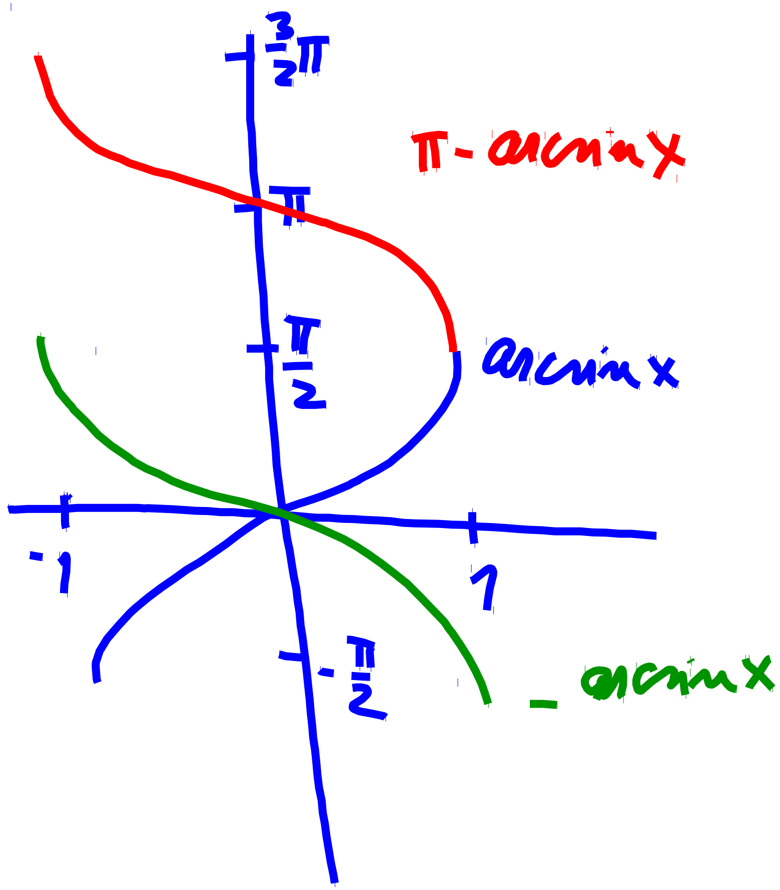
$y - 2\pi \quad + 2\pi \quad = y$

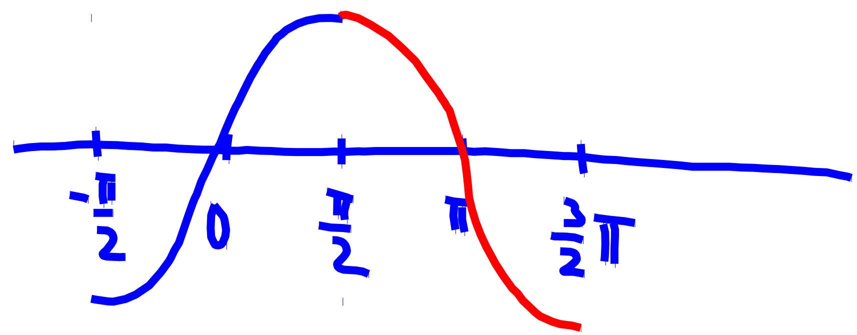
$f(x) = \pi - \arcsin x$

$D(f) = [-1, 1]$

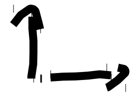
$H(f) = \left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$

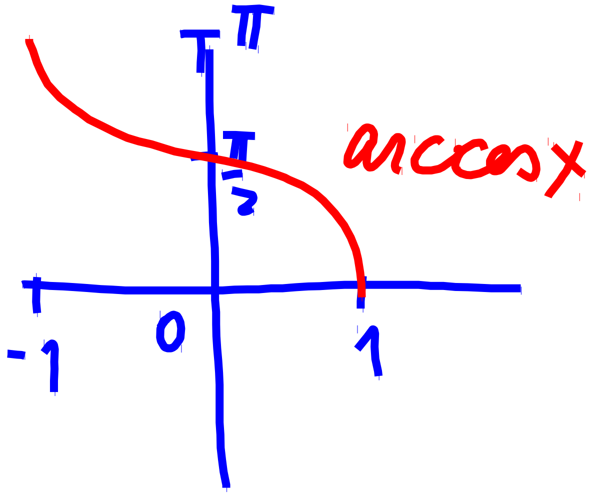
inverzní funkce k sin : $\left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$

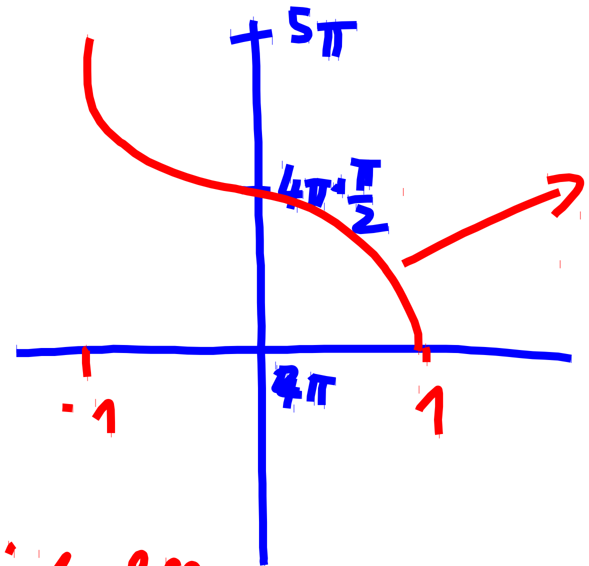




$f(x) = \arccos x + 4\pi$ 　 arccos je inverze ke $\cos : [0,\pi] \to [-1,1]$





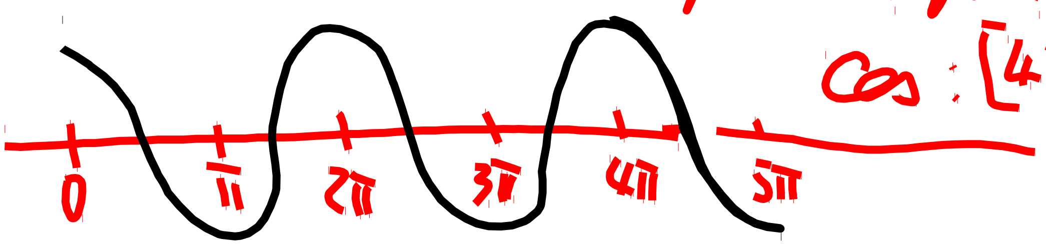


$D(f) = [-1,1]$

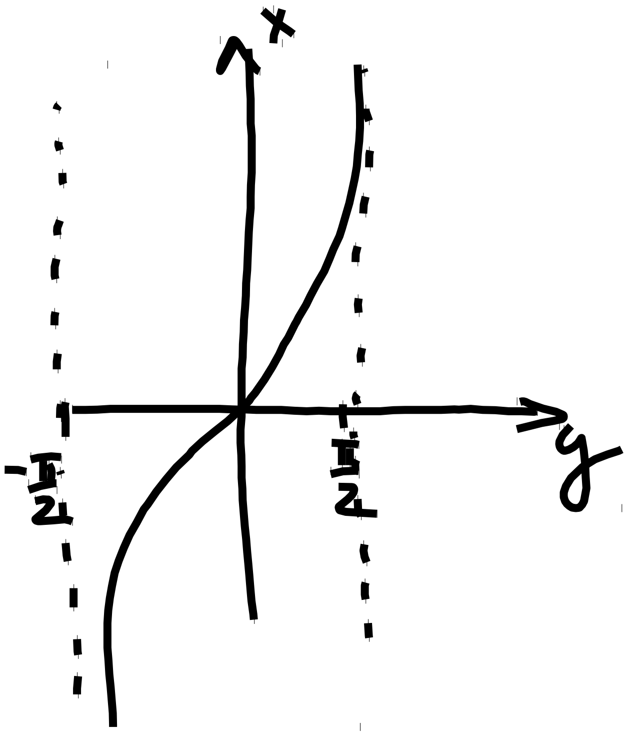
$H(f) = [4\pi, 5\pi]$

posunutí funkce

$\cos : [4\pi, 5\pi] \to [-1,1]$ 　 $x = \cos y$



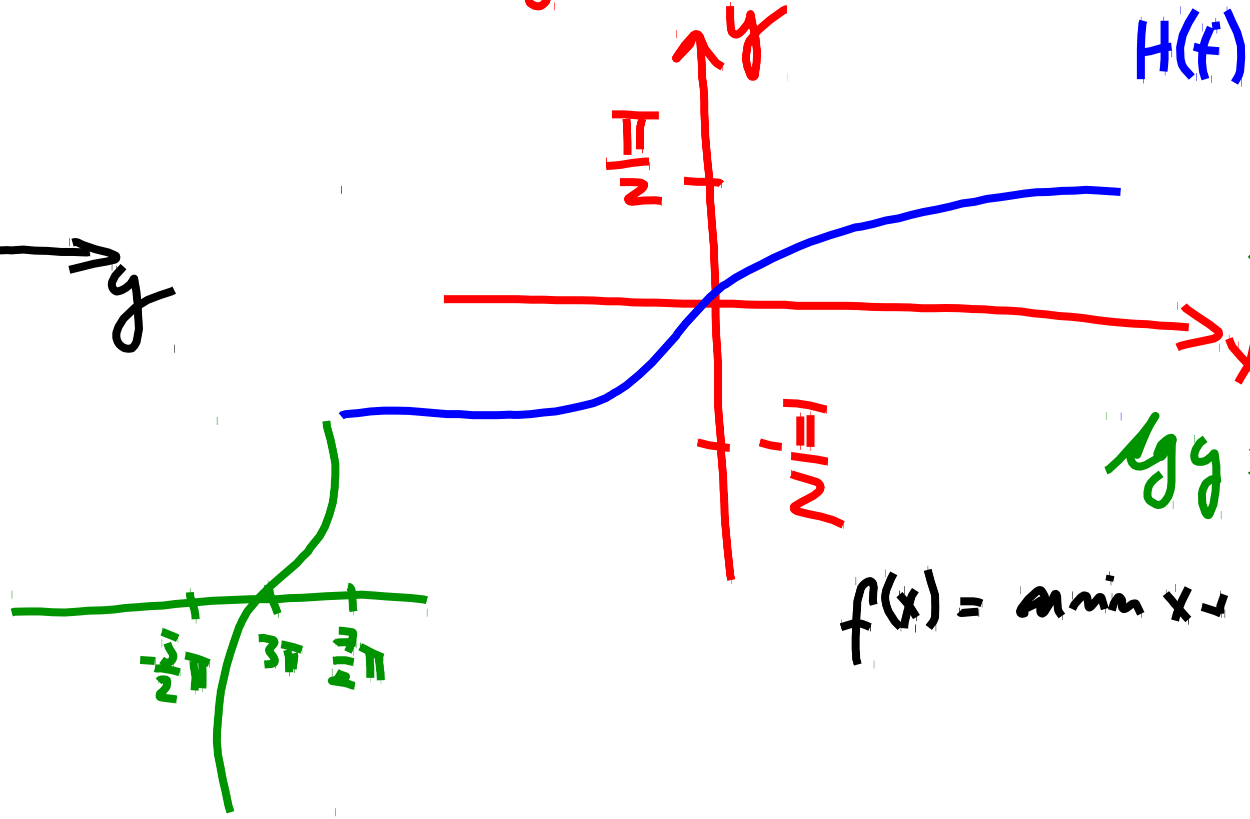
arctg x inverse di tg y $: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$



arctg x

$f(x) = \text{arctg}\, x + 3\pi$

$H(f) = \left(-\frac{\pi}{2} + 3\pi, \frac{\pi}{2} + 3\pi\right)$



inverse

tg y $: \left(\frac{5}{2}\pi, \frac{7}{2}\pi\right)$

$f(x) = \arcsin x + \arccos x = \frac{\pi}{2}$

Zpět k derivacím

Pravidla pro počítání derivací

① $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

② $\left(f(x) \cdot g(x)\right)' = f'(x)\, g(x) + f(x)\, g'(x)$

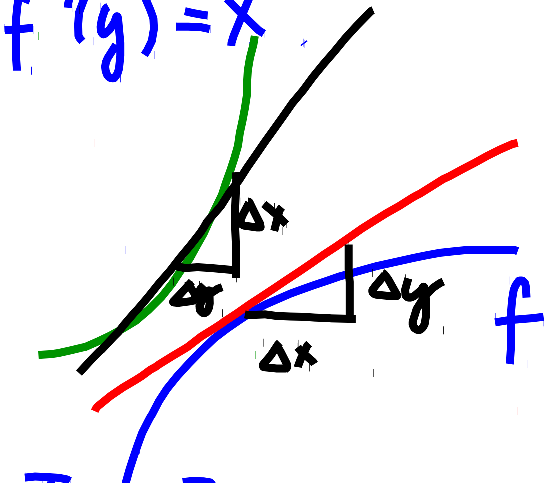
③ nechť $g(x) \neq 0$. Potom

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x)\, g'(x)}{g^2(x)}$$

④ Derivace složené funkce

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

⑤ Derivace inverzní funkce: Nechť funkce f má inverzní funkci f^{-1} a nechť $f'(x) \neq 0$, $f^{-1}(y) = x$



$$\left(f^{-1}(y)\right)' = \frac{1}{f'\left(f^{-1}(y)\right)}$$

arcsin x je inverzní k sin y $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$

$$(\arcsin x)' = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} =$$

$\cos^2 x + \sin^2 x = 1$

$|\cos x| = \sqrt{1 - \sin^2 x}$

$\cos = \sqrt{1-\sin^2}$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$$