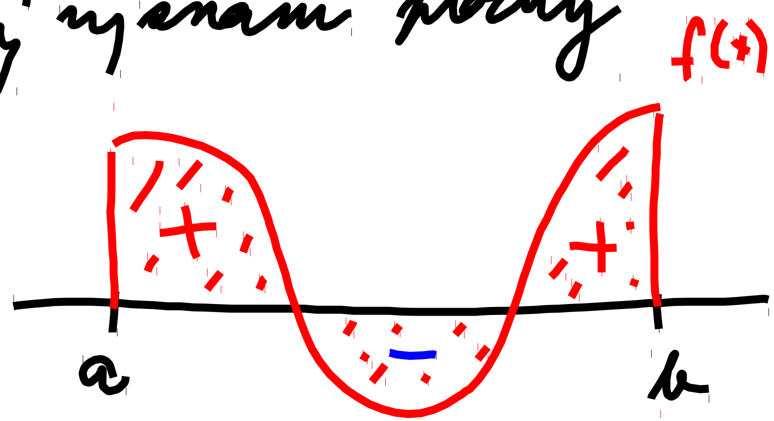
# Geometrické aplikace určitého integrálu

f je spojitá funkce na $[a,b]$ a $F(x)$ je její primitivní funkce

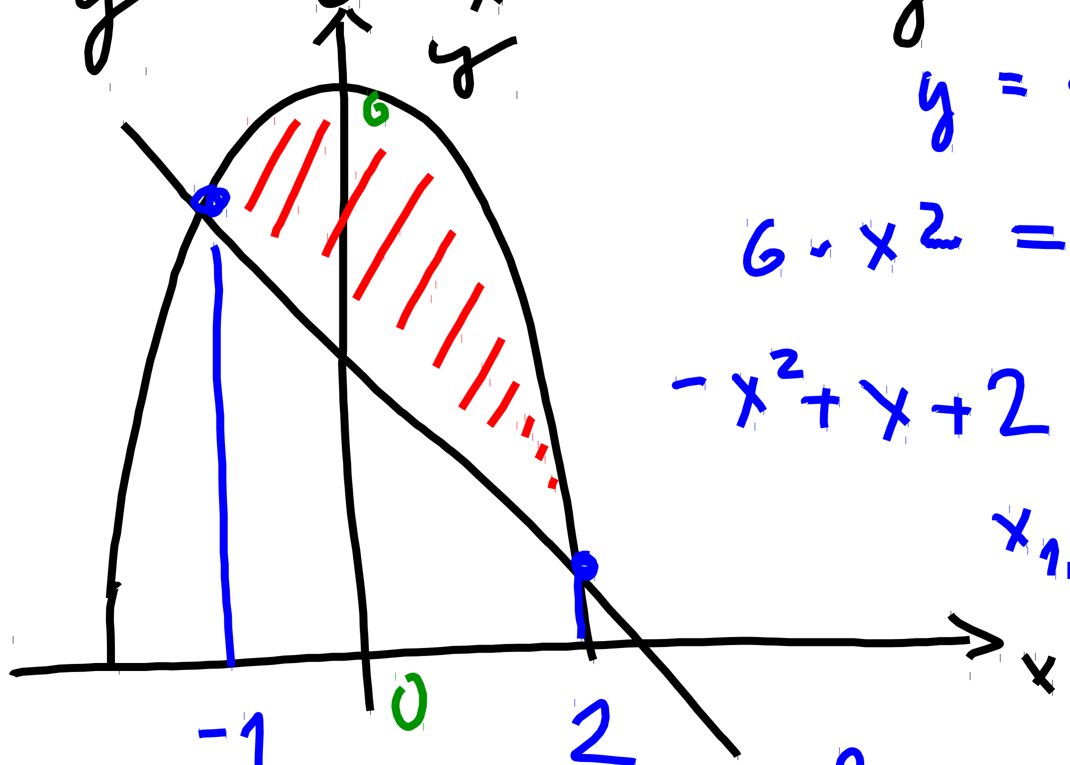
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Geometrický význam plochy



Příklad

$y = 6 - x^2$ $\qquad$ $x + y = 4$




$y = 4 - x$

$6 - x^2 = 4 - x$

$-x^2 + x + 2 = 0$

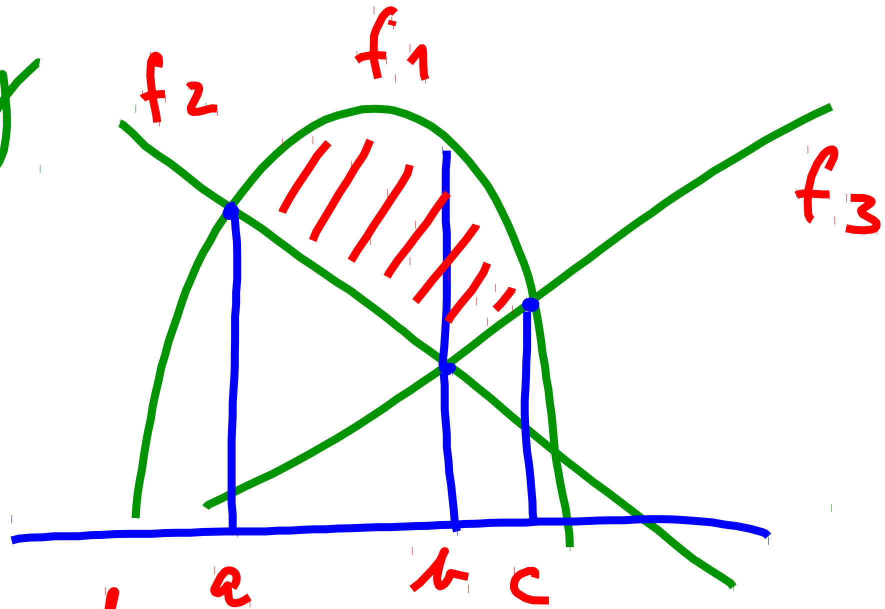
$x_{1,2} = \frac{-1 \pm 3}{-2} = \begin{matrix} -1 \\ 2 \end{matrix}$

$$S = \int_{-1}^{2} (6 - x^2)\,dx - \int_{-1}^{2} (4 - x)\,dx = \int_{-1}^{2} (2 + x - x^2)\,dx$$

$$= \left[ 2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2} = \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 8 - 3 - \frac{1}{2} = \frac{9}{2}$$

Složitější úlohy



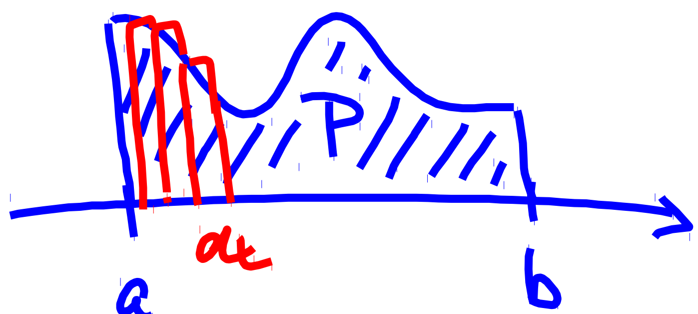

$$S = \int_a^c f_1(x)\,dx - \int_a^b f_2(x)\,dx - \int_b^c f_3(x)\,dx$$

## Objem rotačního tělesa

Máme spojitou a nezápornou funkci $f$ na $[a,b]$.

Uvažujme oblast nad intervalem $[a,b]$ a pod grafem funkce $f$

podgraf $$P = \{(x,y)\in\mathbb{R}^2;\ x\in[a,b],\ 0\le y\le f(x)\}$$

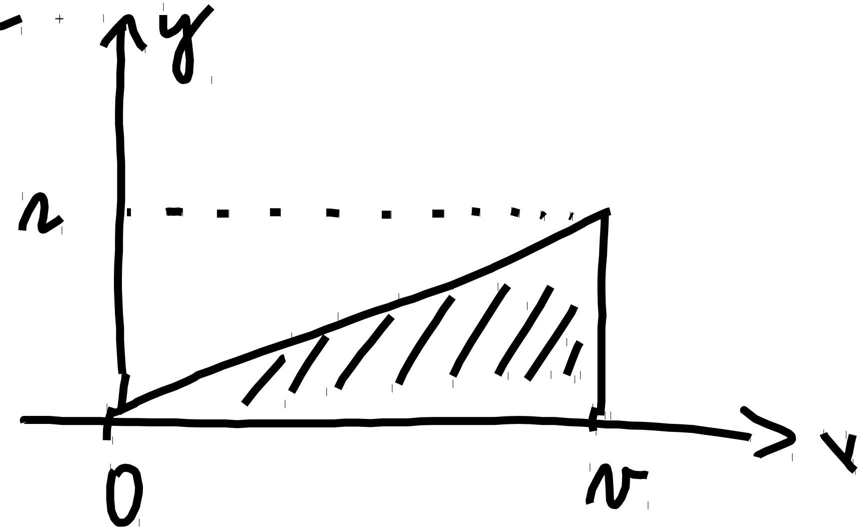



Rotací kolem osy $x$ podgrafem $P$ dostaneme těleso

$$T = \{(x,y,z)\in\mathbb{R}^3;\ x\in[a,b],\ y^2+z^2\le f^2(x)\}$$

Jeho objem je $$V = \pi\int_a^b f^2(x)\,dx$$

**Příklad** Objem kuželu o výšce $v$ a poloměru podstavy $r$

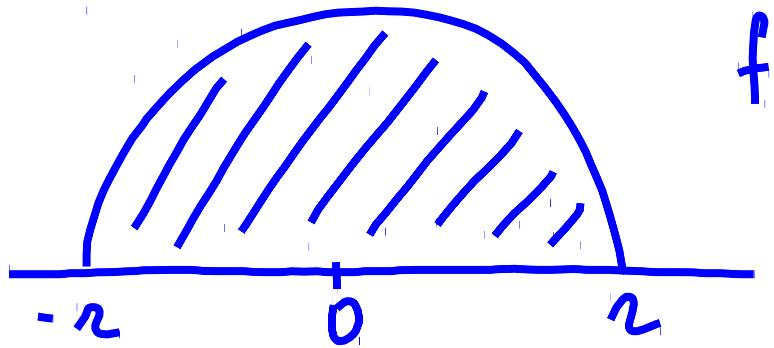



$$f(x) = \frac{r}{v} x$$

$$V = \pi \int_0^v \frac{r^2}{v^2} x^2 \, dx = \frac{\pi r^2}{v^2} \left[ \frac{x^3}{3} \right]_0^v = \frac{\pi r^2}{v^2} \frac{v^3}{3}$$

$$= \frac{1}{3} \pi r^2 v$$

**Příklad**: objem koule o poloměru $r$




$$f(x) = \sqrt{r^2 - x^2}$$

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\, dx = \pi \left[ r^2 x - \frac{x^3}{3} \right]_{-r}^{r} =$$

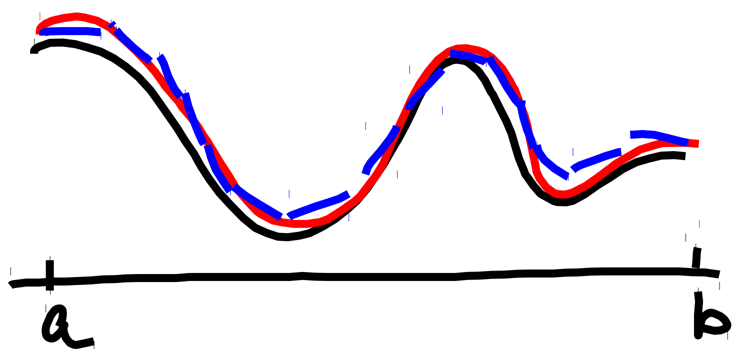
$$= \pi \left[ \left( r^3 - \frac{r^3}{3} \right) - \left( -r^3 + \frac{r^3}{3} \right) \right] = \pi \frac{4}{3} r^3 = \frac{4}{3} \pi r^3$$

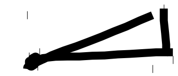
## Délka křivky

Nechť $f$ je spojitá funkce na intervalu $[a,b]$

Délka grafu

$$G = \{(x, f(x)) \in \mathbb{R}^2 ; \ x \in [a,b]\}$$



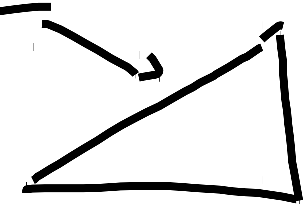
$f(x)$

směrnice je $f'(x)$

$dx$

$x$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

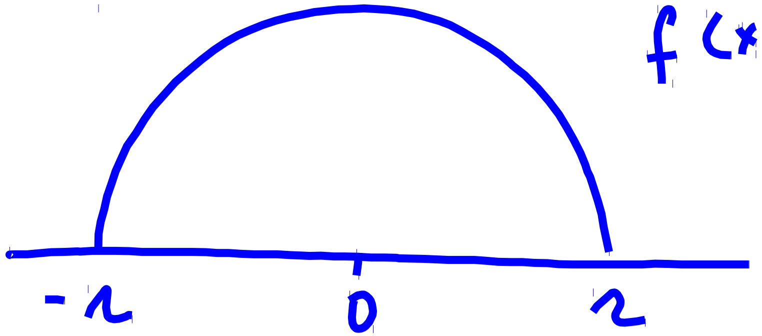


$f'(x)\,dx$

$dx$

$$\sqrt{(dx)^2 + (f'(x)\,dx)^2} = \sqrt{1 + (f'(x))^2}\, dx$$

Příklad Spočítáme délku půlkružnice o poloměru $r$




$$f(x)=\sqrt{r^2-x^2}$$

$$f'(x)=\frac{1}{2}\frac{1}{\sqrt{r^2-x^2}}(-2x)$$

$$\left(f'(x)\right)^2=\frac{x^2}{r^2-x^2}$$

$$L=\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=\int_{-r}^{r}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx=$$

$$=r\int_{-r}^{r}\frac{1}{\sqrt{r^2-x^2}}\,dx=r\int_{-1}^{1}\frac{r\,dz}{\sqrt{r^2-r^2z^2}}=r\int_{-1}^{1}\frac{dz}{\sqrt{1-z^2}}$$

$x\in[-r,r]$ $\quad x=rz$

$z\in[-1,1]$ $\quad dx=r\,dz$

$$=r\left[\arcsin z\right]_{-1}^{1}=r\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\pi r$$

## Povrch pláště rotačního tělesa

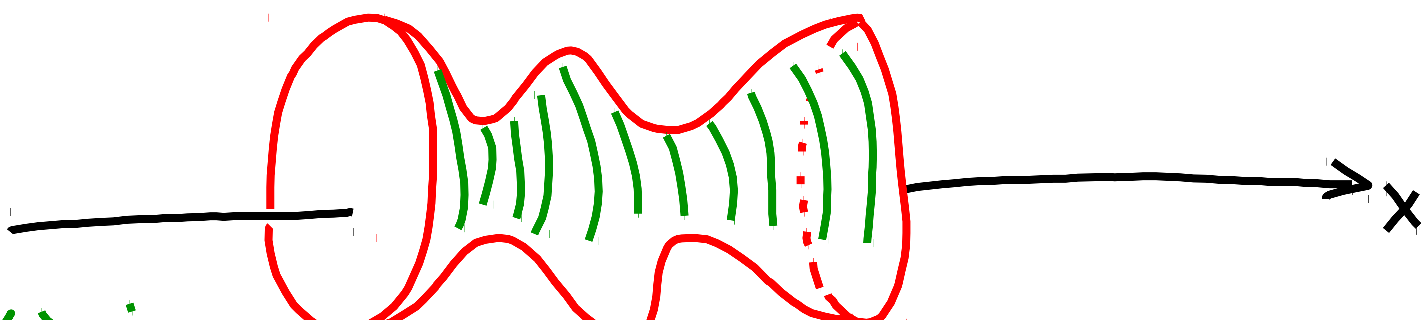
Mějme $f$ spojitou a nezápornou na $[a,b]$.

Uvažujme těleso vzniklé rotací podgrafu funkce $f$

$$T = \{ (x,y,z) \in \mathbb{R}^3 ;\ x \in [a,b],\ y^2+z^2 \leq f^2(x) \}$$

a jeho plášť

$$R = \{ (x,y,z) \in \mathbb{R}^3 ;\ x \in [a,b],\ y^2+z^2 = f^2(x) \}$$




Povrch pláště je

$$S = 2\pi \int_a^b f(x) \sqrt{1+(f'(x))^2}\, dx$$

$\sqrt{dx^2 + f'(x)^2 dx^2}$

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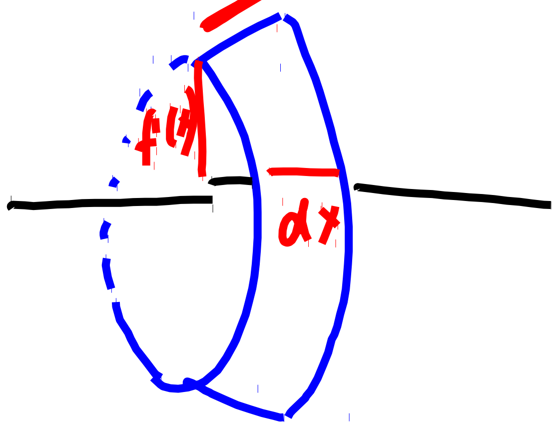
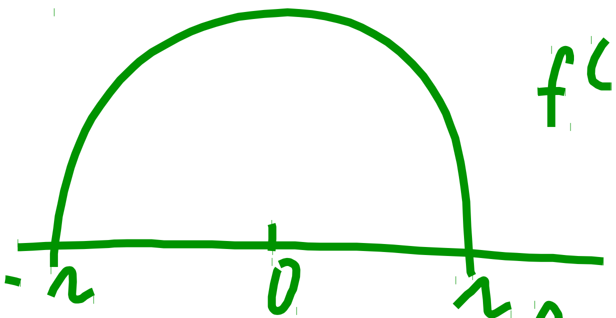



## Povrch koule




$f(x) = \sqrt{r^2 - x^2}$

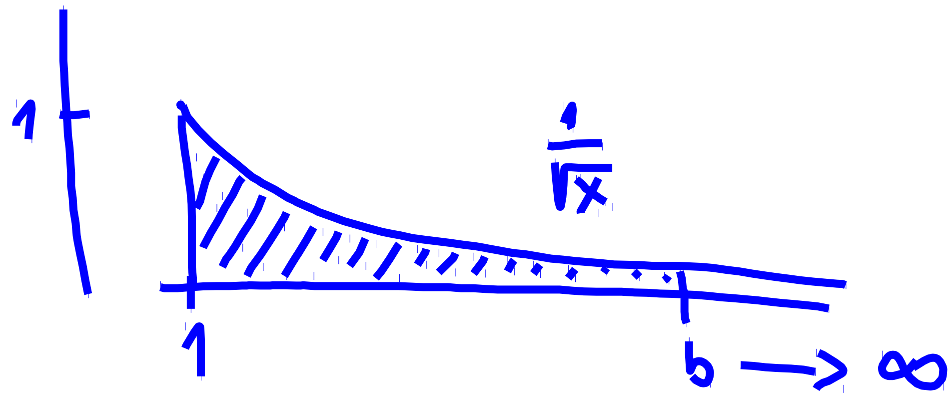
$$S = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \, \frac{\sqrt{r^2}}{\sqrt{r^2 - x^2}} \, dx = 2\pi r \int_{-r}^{r} 1 \, dx$$

předchozí úloha

$$= 2\pi r \left[ \cancel{x} \right]_{-r}^{r} = 2\pi r (r + r) = \underline{\underline{4\pi r^2}}$$

# Nevlastní integrály

$$f(x) = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$




Primitivní funkce

je $$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x} = F(x)$$

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = F(x) - F(1) = 2\sqrt{b} - 2$$

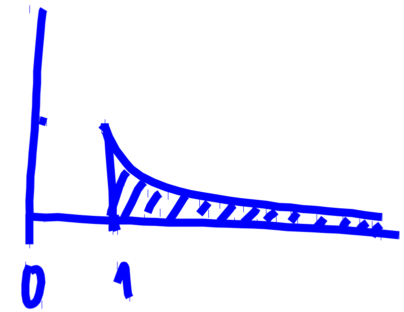
$$\lim_{b \to \infty} 2\sqrt{b} - 2 = \infty$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

diverguje

$f(x) = \frac{1}{x^{3/2}}$ primitivní funkce je $\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} = -2\frac{1}{\sqrt{x}}$

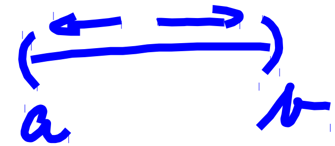
$$\int_1^b f(x)\,dx = \int_1^b \frac{dx}{x^{3/2}} = \left[-2\frac{1}{\sqrt{x}}\right]_1^b = \frac{-2}{\sqrt{b}} + 2$$




$\lim_{b\to\infty} -\frac{2}{\sqrt{b}} = 0$, proto definujeme

$$\int_1^\infty \frac{dx}{x^{3/2}}\,dx = \lim_{b\to\infty} \frac{-2}{\sqrt{b}} + 2 = 2$$

Integrál konverguje.




Nechť $f$ je spojitá na otevřeném intervalu $(a,b)$.

Nechť $F$ je primitivní funkce k $f$ na $(a,b)$.

Jestliže existuje

$$\lim_{x\to b_-} F(x) \in \mathbb{R} \quad \text{a} \quad \lim_{x\to a_+} F(x) \in \mathbb{R},$$

pak definujeme

$$\int_a^b f(x)\,dx = \lim_{x\to b_-} F(x) - \lim_{x\to a_+} F(x)$$

Říkáme, že integrál konverguje.

$\int_1^\infty \frac{1}{x^\alpha}\,dx$ a chceme zjistit pro která $\alpha > 0$ integrál konverguje a pro která diverguje

$\alpha = 1$ prim. funkce k $\frac{1}{x}$ je $\ln|x|$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{x\to\infty} \ln|x| - \ln 1 = \infty - 0 = \infty$$

integrál diverguje

$\alpha > 1$ prim. funkce je $F(x) = \frac{x^{1-\alpha}}{1-\alpha} = \frac{1}{(1-\alpha)\,x^{\alpha-1}}$

$\frac{1}{x^\alpha} = x^{-\alpha}$

$$\lim_{x\to\infty} F(x) = 0$$

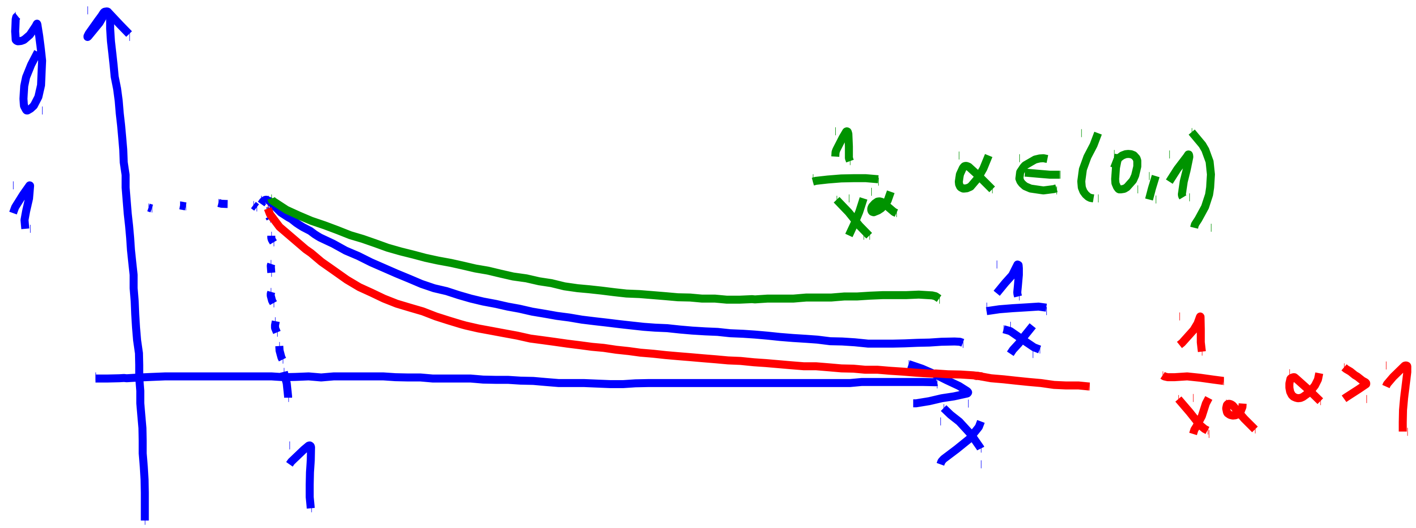
a proto $\int_1^\infty \frac{1}{x^\alpha}\,dx = 0 - \frac{1}{1-\alpha} = \frac{1}{\alpha-1}$ konverguje

$\alpha \in (0,1)$

$f(x) = \dfrac{1}{x^\alpha}$ prim. funkcija je $F(x) = \dfrac{x^{1-\alpha}}{1-\alpha}$ $\quad 1-\alpha > 0$

$$\lim_{x \to \infty} F(x) = \infty$$

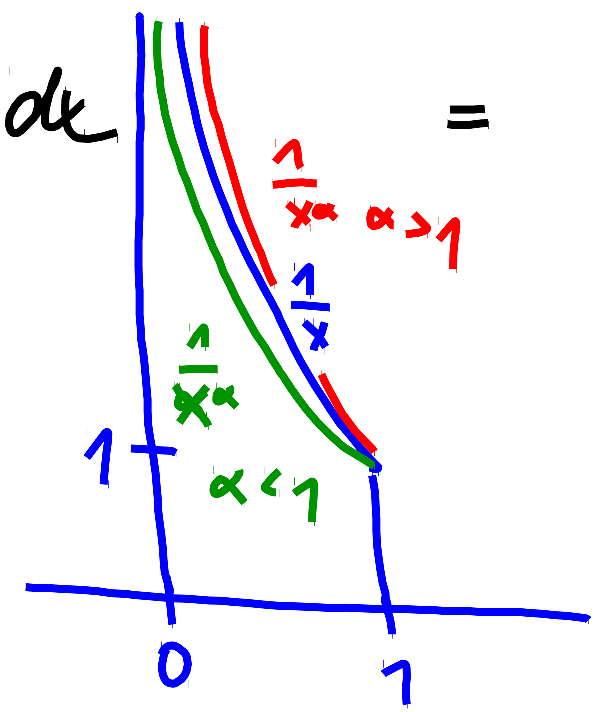
$$\int_1^\infty \frac{dx}{x^\alpha} = \infty - \frac{1}{1-\alpha} = \infty$$ integral divergira

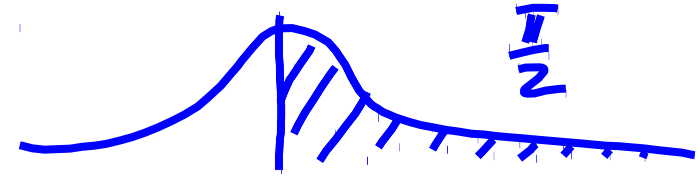


Vezmeme interval $(0,1]$

a na něm funkce $\frac{1}{x^\alpha}$

| | $\alpha \in (0,1)$ | $\alpha = 1$ | $\alpha \in (1,\infty)$ |
|---|---|---|---|
| prim. funkce $F(x)$ | $\frac{x^{1-\alpha}}{1-\alpha}$ | $\ln x$ | $\frac{1}{(1-\alpha)\, x^{\alpha-1}}$ |
| $\lim\limits_{x\to 0+} F(x) =$ | $0$ | $-\infty$ | $-\infty$ |
| $\int_0^1 \frac{1}{x^\alpha}\,dx =$ | $\frac{1}{1-\alpha} - 0 = \frac{1}{1-\alpha}$ | $0-(-\infty) = \infty$ | $\frac{1}{1-\alpha} - (-\infty) = \infty$ |
| | konverguje | | diverguje |






# Příklady

(A) $\int_0^\infty \frac{1}{x^2+1}\,dx = \left[\operatorname{arctg} x\right]_0^\infty = \lim_{x\to\infty} \operatorname{arctg} x - \operatorname{arctg} 0$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

(B) $\int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = \lim_{x\to\infty}(-e^{-x}) - (-e^{0}) = 0+1 = 1$

(C) $\int_0^\infty \sin x\,dx = \left[-\cos x\right]_0^\infty = \underbrace{\lim_{x\to\infty}(-\cos x)}_{\text{neexistuje}} - \cos 0$

diverguje

(D) $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \left[-2\sqrt{1-x}\right]_0^1 = -2\cdot 0 - (-2) = 2$

$t = 1-x$

(E) $$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\arcsin x\right]_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}$$

__Příklad__ $$\int_0^1 x \ln x \, dx$$

prim. funkce existuje na $[0,1]$

i integrál konverguje

$x \ln x$ definována na $(0,1]$

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \underset{\text{l'Hosp.}}{=} \lim \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} (-x) = 0$$

My to chceme ale spočítat!

$$\int_0^1 x \ln x \, dx = \lim_{a\to 0^+} \int_a^1 x \ln x \, dx \overset{\text{per parles}}{=} \lim_{a\to 0^+} \left[\frac{x^2}{2}\ln x\right]_a^1 -$$

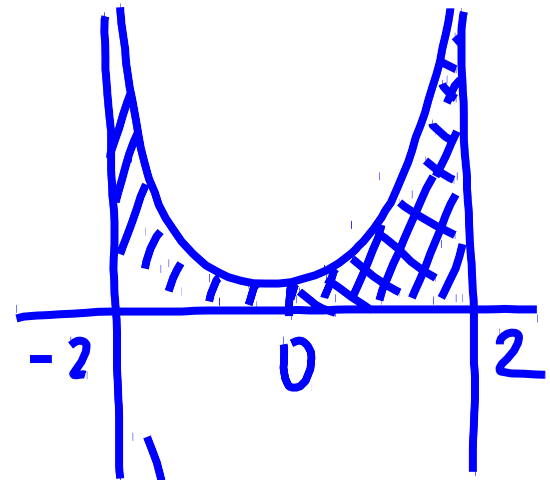
$$- \lim_{a\to 0^+} \int_a^1 \frac{x^2}{2}\,\frac{1}{x}\, dx = \lim_{a\to 0^+} \left[\frac{x^2}{2}\ln x\right]_a^1$$

$$- \lim_{a\to 0^+} \left[\frac{x^2}{4}\right]_a^1 = \lim_{a\to 0^+} \left(0 - \frac{a^2}{2}\ln a\right)$$

$$- \lim_{a\to 0^+} \left(\frac{1}{4} - \frac{a^2}{4}\right) = 0 - \frac{1}{4} = -\frac{1}{4}$$

Příklad

$$\int_{-2}^{2} \frac{1}{4-x^2}\,dx$$




$$\int_0^2 \frac{dx}{4-x^2} = \int_0^2 \left(\frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x}\right)dx =$$

$$= \frac{1}{4} \lim_{a\to 2_-} \Big[-\ln|2-x|\Big]_0^a + \frac{1}{4}\Big[\ln(2+x)\Big]_0^2$$

$$\frac{1}{4}\left[\lim_{a\to 2_-}\underbrace{\left(-\ln|2-a|\right)}_{\infty} - \ln 2\right] + \frac{1}{4}\Big[\ln 4 - \ln 2\Big] = \infty$$

diverguje

$\int_{-2}^{0}$ obdobně