① $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots\dots$

$$S_k = \sum_{n=1}^{k} n = \underbrace{1 + \dots\dots + k}_{\text{aritmetická řada}} = \frac{(k+1)k}{2}$$

$$\lim_{k \to \infty} S_k = \lim_{k \to \infty} \frac{k(k+1)}{2} = \frac{\infty \cdot \infty}{2} = \underline{\underline{\infty}}$$

řada diverguje (D)

② $\sum_{n=0}^{\infty} A = A + A + \dots$ , $A \in \mathbb{R}$

$$S_k = \sum_{n=0}^{k} A = \underbrace{A + \dots + A}_{k+1} = (k+1)A$$

$$\lim_{k \to \infty} S_k = \lim_{k \to \infty} (k+1) A = \infty \cdot A = \begin{cases} \infty & , \underline{A > 0} \quad D \\ -\infty & , \underline{A < 0} \quad D \\ \underset{\text{N.V.}}{0 \cdot \infty} & , A = 0 \end{cases}$$

$\underline{A = 0}$ ~> $S_k = \sum_{n=0}^{k} 0 = 0$ $\lim_{k \to \infty} S_k = 0$ K

$A \in \mathbb{R} \setminus \{0\}$ řada diverguje , $A = 0$ řada konverguje řada konverguje

③ $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{11}{18}$ $\quad S_k = \frac{1}{4} + \frac{1}{2\cdot5} + \frac{1}{3\cdot6} + \dots + \frac{1}{k(k+3)}$

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \qquad A = ? = B \quad / \, n(n+3)$$

reálný kořen

$$1 = A(n+3) + B\cdot n \rightsquigarrow 1) \text{ soustava}$$

$n = 0 \rightsquigarrow 1 = 3A$

$A = \frac{1}{3}$

$n^1: \; 0 = A + B$

$n^0: \; 1 = 3A$

$n = -3 \rightsquigarrow 1 = -3B$

$B = -\frac{1}{3}$

$$S_k = \sum_{n=1}^{k} \frac{\frac{1}{3}}{n} - \frac{\frac{1}{3}}{n+3} = \boxed{\frac{1}{3}} - \cancel{\frac{1}{12}} + \boxed{\frac{1}{6}} - \cancel{\frac{1}{15}} + \boxed{\frac{1}{9}} - \cancel{\frac{1}{18}} + \cancel{\frac{1}{12}} - \cancel{\frac{1}{21}} + \cancel{\frac{1}{15}} - \cancel{\frac{1}{24}} + \dots$$

$$+ \cancel{\frac{1}{3(k-3)}} - \cancel{\frac{1}{3k}} + \cancel{\frac{1}{3(k-2)}} - \boxed{\frac{1}{3(k+1)}} + \cancel{\frac{1}{3(k-1)}} - \boxed{\frac{1}{3(k+2)}} + \cancel{\frac{1}{3k}} - \boxed{\frac{1}{3(k+3)}}$$

$$S_k = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \overset{\to 0}{\frac{1}{3(k+1)}} - \overset{\to 0}{\frac{1}{3(k+2)}} - \overset{\to 0}{\frac{1}{3(k+3)}}$$

$$\xrightarrow{k \to \infty} \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{6+3+2}{18} = \boxed{\frac{11}{18}}$$

řada je konvergentní

④ $$\underbrace{\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)}_{=-\ln 2} = \ln\left(1-\frac{1}{4}\right)+\ln\left(1-\frac{1}{9}\right)+$$

$$+\ln\left(1-\frac{1}{16}\right)+\dots$$

$$\ln\left(1-\frac{1}{n^2}\right) = \ln\frac{n^2-1}{n^2} = \ln\underbrace{(n^2-1)}_{(n+1)\cdot(n-1)} - \ln n^2 =$$

$$= \ln(n+1)+\ln(n-1)-2\ln n$$

$$S_k = \sum_{n=2}^{k} \ln(n+1)+\ln(n-1)-2\ln n = \cancel{\ln 3}+\boxed{\ln 1}-\overset{1}{2}\boxed{\cancel{\ln 2}}+$$

$$+\cancel{\ln 4}+\cancel{\ln 2}-2\cancel{\ln 3}+$$

$$+\cancel{\ln 5}+\cancel{\ln 3}-2\cancel{\ln 4}$$

$$+\cancel{\ln 6}+\cancel{\ln 4}-2\cancel{\ln 5}+\dots$$

$$+\cancel{\ln k}+\cancel{\ln(k-2)}-2\cancel{\ln(k-1)}+$$

$$+\boxed{\ln(k+1)}+\cancel{\ln(k-1)}-\overset{1}{\cancel{2}}\boxed{\ln k}$$

$$S_k = \overset{=0}{\ln 1} - \ln 2 + \underbrace{\ln(k+1)}_{\infty} - \underbrace{\ln k}_{-\infty} \xrightarrow{k\to\infty}$$

N.V

$$\lim_{k\to\infty} \ln\frac{k+1}{k} = \lim_{k\to\infty} \ln\left(1+\frac{1}{k}\right) =$$

$$= \ln\left(\lim_{k\to\infty} 1+\overset{\nearrow 0}{\frac{1}{k}}\right) = \ln 1 = 0$$

$$S_k \to -\ln 2$$

⑤ $$\boxed{\sum_{n=1}^{\infty} \frac{1}{2^n}} = \underbrace{\overset{\frac{1}{2}}{\frac{1}{2}}}_{a_0} + \underbrace{\overset{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{4}}}_{a_0\cdot q} + \underbrace{\overset{\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2}{\frac{1}{8}}}_{a_0\cdot q^2} + \overset{\frac{1}{2}\cdot\left(\frac{1}{2}\right)^3}{\ldots\ldots} = \frac{a_0}{1-q} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \boxed{1}$$

$$a_0 \xrightarrow{\frac{1}{2}} a_0\cdot q \xrightarrow{\frac{1}{2}} a_0\cdot q^2$$

$$\sum_{n=0}^{\infty} a_0 q^n = a_0 + a_0\cdot q + \underbrace{+ a_0\cdot q^2}$$

$$\underbrace{\left(\frac{1}{2}\right)^{n+1}}_{a_{n+1}} = \frac{1}{2}\underbrace{\left(\frac{1}{2}\right)^n}_{a_n}$$

$$a_{n+1} = \frac{1}{2} a_n$$

$$q = \frac{a_0 q}{a_0} = \frac{\frac{1}{4}}{\frac{1}{2}} =$$

$$q = \frac{a_0\cdot q^2}{a_0\cdot q} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$

$$|q| < 1$$

$\frac{1}{2} < 1$ ✓ řada je konvergentní

⑥ $$\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \underbrace{\frac{5}{6}}_{a_0} + \underbrace{\frac{13}{36}}_{a_0 q} + \underbrace{\frac{35}{216}}_{a_0 \cdot q^2} + \dots$$

není geom. řada

$$q = \frac{\frac{13}{36}}{\frac{5}{6}} = \frac{13}{5\cdot 6} = \frac{13}{30}$$

$$q = \frac{\frac{35}{216}}{\frac{13}{36}} = \frac{35}{13\cdot 6} = \frac{35}{78}$$

řada konverguje

$$\sum_{n=1}^{\infty} \frac{3^n}{6^n} + \frac{2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{9} + \frac{1}{8} + \frac{1}{27} + \dots$$

$\frac{1}{2}$ $\frac{1}{3}$

$$\overbrace{\sum^{\infty} a_n}^{=A} \ K \ \& \ \overbrace{\sum b_n}^{=B} \ K$$

$$\Rightarrow \underbrace{\sum^{\infty} a_n + b_n}_{A+B} \ K$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$q = \frac{1}{2} < 1$ řada konverguje

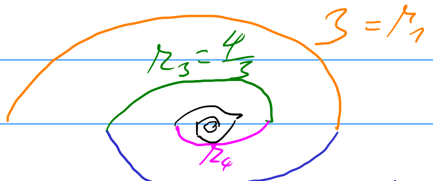
$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3}\right)^2 + \dots$$

$|q| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$ řada konv.

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n = 1 + \frac{1}{2} = \underline{\underline{\frac{3}{2}}}$$

⑦ nekonečná spirála je tvořena z půlkružnic první má poloměr 3cm. Každá další půlkružnice má poloměr vždy o $\frac{1}{3}$ předchozího poloměru menší. Jaká je délka spirály?



$r_2 = 3 - \frac{1}{3} \cdot 3 = 2$

$$l = o_1 + o_2 + o_3 + \dots = \sum_{n=1}^{\infty} o_n$$

$$2 - \frac{2}{3} = \frac{4}{3}$$

$$r_4 = \frac{4}{3} - \frac{4}{3} \cdot \frac{1}{3} = \frac{4}{3}\left(1 - \frac{1}{3}\right) = \frac{8}{9}$$

$$o_k = 2\pi r \rightarrow o_{pk} = \pi r$$

$$l = \pi r_1 + \pi r_2 + \dots = \sum_{i=1}^{\infty} \pi r_i = 3\pi + 2\pi + \frac{4}{3}\pi + \frac{8}{9}\pi + \dots$$

$$q = \frac{2\pi}{3\pi} = \frac{2}{3}$$

$$q = \frac{\frac{4}{3}\pi}{2\pi} = \frac{4}{6} = \frac{2}{3}$$

$$r_n = r_{n-1} - \frac{1}{3} r_{n-1} = r_{n-1} \cdot \frac{2}{3}$$

$|q| = \left|\frac{2}{3}\right| < 1$

řada konverguje

$$l = \frac{\pi 3}{1 - \frac{2}{3}} = \frac{3\pi}{\frac{1}{3}} = \boxed{9\pi}$$

řada diverguje $\cancel{K}$ ?

roste rychleji

⑧ $$\sum_{n=2}^{\infty} \underbrace{\frac{n}{\ln n}}_{a_n} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\ln n} = \left|\frac{\infty}{\infty}\right| = \infty \neq 0$$

$$\lim_{x\to\infty} \frac{x}{\ln x} \overset{l'H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$$

nutná podm. konv.

⑨ $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n!}}_{\geq 0}$ K/D? $\lim_{n\to\infty} \frac{1}{n!} = \frac{1}{\infty} = 0$

$$\sum^{\infty} \frac{1}{n^p} \begin{cases} K, & p>1 \\ D, & p \leq 1 \end{cases}$$

podílové krit / odmocninové krit.

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n!}}_{} = \underset{=}{\overset{\leq 1}{1}} + \underset{=}{\overset{\leq \frac{1}{2}}{\frac{1}{2}}} + \overset{\leq \frac{1}{4}}{\frac{1}{6}} + \overset{\leq \frac{1}{8}}{\frac{1}{24}} + \overset{\leq \frac{1}{16}}{\frac{1}{120}} + \dots$$

$$\underset{D}{b_n} \leq \boxed{\underset{\frac{1}{n!}}{a_n} \leq c_n} \ \forall n \geq n_0 \quad \underset{K}{}$$

~~$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ D~~

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}, \ |q| = \frac{1}{2} < 1 \quad K$$

podle srovnávacího krit $\sum^{\infty} \frac{1}{n!}$ konverguje

~~$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$~~

rac. lom. fce

⑩ $\sum_{n=1}^{\infty} \boxed{\frac{1}{2n-1}}$ K/D?

$\geq 0$

$= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots$

$\lim_{n \to \infty} \frac{1}{2n-1} = 0$ → nic nevíme

odmocninové, podílové krit. → selže

integrální krit ✓

limitní srovnávací krit

$a_n = \frac{1}{2n-1}$

$b_n = \frac{1}{n^r}$

$$L = \lim_{n \to \infty} \frac{\frac{1}{2n-1}}{\frac{1}{n^r}} = \lim_{n \to \infty} \frac{n^r}{2n-1} = \begin{cases} \boxed{\frac{1}{2} \quad r = 1} \\ \infty \quad r > 1 \\ 0 \quad r < 1 \end{cases}$$

chceme $r$, aby $L \in (0, \infty)$

$$\sum^{\infty} a_n \ K(D) \Leftarrow \sum^{\infty} b_n \ K(D)$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ harm. řada diverguje

$L = \frac{1}{2} \in (0, \infty)$ ✓

$\Rightarrow$ limitní srovnávací kritérium říká $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverguje

⑪ $$\sum_{n=3}^{\infty} \frac{\sqrt[5]{\overset{\sim n^2}{n^2+1}} + 2\sqrt[7]{\overset{\sim n^3}{n^3-n}}}{\underbrace{\sqrt{\sqrt{n}+\sqrt[3]{n^5}}}_{\geq 0 \; \sim n^{\frac{5}{3}}} + \sqrt[4]{n^7}}$$ K/D ?

⇝ "skoro rac. lom. fce"

- podílové + odmocninové krit selžou
- integrální taky selže
- nutná podmínka konvergence

$\sqrt{A+B} \neq \sqrt{A} + \sqrt{B}$

$$\lim_{n\to\infty} \frac{\dfrac{\sqrt[5]{n^2+1} + 2\sqrt[7]{n^3-n}}{\sqrt{\sqrt{n}+\sqrt[3]{n^5}}+\sqrt[4]{n^7}}}{\dfrac{1}{n^2}}$$

$$\left|\frac{\dfrac{n^{\frac{2}{5}}+n^{0}+2\left(n^{\boxed{\frac{3}{7}}}-n^{\frac{1}{7}}\right)}{n^{\frac{1}{4}}+n^{\frac{5}{6}}+n^{\boxed{\frac{7}{4}}}}}{\dfrac{n^{\boxed{0}}}{n^{\boxed{2}}}}\right| =$$

$$\boxed{\frac{3}{7}} - \frac{2}{5} = \frac{15-14}{35} = \frac{1}{35} > 0$$

$$= \lim_{n\to\infty} \frac{\dfrac{n^{\frac{3}{7}}\left(\sqrt[5]{\cancel{\dfrac{n^2+1}{n^{\frac{15}{7}}}}}\to 0 + 2\sqrt[7]{1-\cancel{\dfrac{1}{n^2}}}\to 0\right)}{n^{\frac{7}{4}}\left(\cancel{\dfrac{n^{\frac{1}{2}}}{n^{\frac{7}{2}}}}\to 0 + \cancel{\dfrac{n^{\frac{5}{3}}}{n^{\frac{21}{6}}}}\to 0 + 1\right)}}{\dfrac{1}{n^p}} = \lim_{n\to\infty} \frac{\dfrac{2n^{\frac{3}{7}}}{n^{\frac{7}{4}}}}{\dfrac{1}{n^p}} =$$

$$= \lim_{n\to\infty} \frac{\dfrac{2}{n^{\frac{7}{4}-\frac{3}{7}}}}{\dfrac{1}{n^p}} = \boxed{2} \in (0,\infty)$$

pro $p = \frac{7}{4} - \frac{3}{7} = \frac{49-12}{28} = \frac{37}{28} > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{37}{28}}}$ řada konverguje

podle limitního srovnávací kritéria původní řada taky konverguje