① $\displaystyle\iint_M \frac{1}{x^2y^2}\,dx\,dy = \underline{1}$ $\quad M: [1,\infty)^2$

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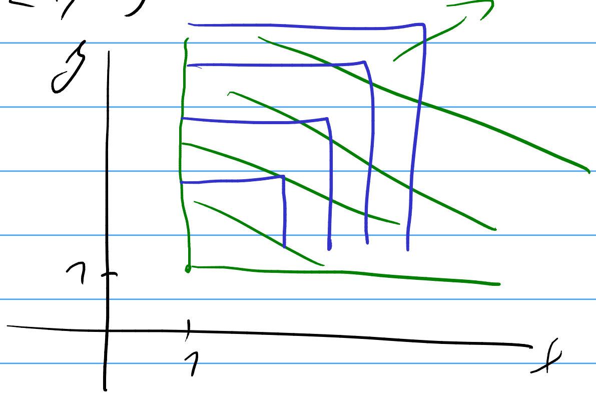


$$I_n = \iint_{M_n} \frac{1}{x^2y^2}\,dx\,dy = \int_1^{1+n}\int_1^{1+n} \frac{1}{x^2y^2}\,dx\,dy =$$

$M_n: [1,1+n]^2,\ n\in N$

$$= \underbrace{\int_1^{1+n}\frac{1}{x^2}\,dx}_{A}\ \underbrace{\int_1^{1+n}\frac{1}{y^2}\,dy}_{A} = \left[\int_1^{1+n}\frac{1}{x^2}\,dx\right]^2 = \left(\left[-\frac{1}{x}\right]_1^{1+n}\right)^2 =$$

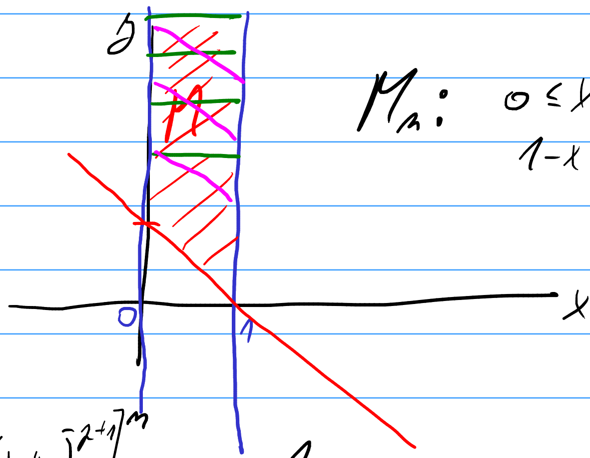
$$= \left(-\frac{1}{1+n}+1\right)^2 = I_n$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty}\left(1-\cancelto{0}{\frac{1}{1+n}}\right)^2 = 1^2 = \underline{\underline{1}} \quad \text{konverguje}$$

② $\displaystyle\iint_M \frac{dx\,dy}{(x+y)^p}$

$y \geq 1-x$

$M: x+y \geq 1,\ 0 \leq x \leq 1,\ p > 0$



$M_n:\ 0 \leq x \leq 1$
$\quad 1-x \leq y \leq n$, $n \in \mathbb{N}$

$$I_n = \iint_{M_n} \frac{dx\,dy}{(x+y)^p} =$$

$$= \int_0^1 \int_{1-x}^{n} \underbrace{\frac{1}{(x+y)^p}}_{\frac{1}{x+y}} \overset{(x+y)^{-p}}{} \,dy\,dx \overset{p \neq 1}{=} \int_0^1 \left[\frac{(x+y)^{-p+1}}{-p+1}\right]_{1-x}^{n} dx = \int_0^1 \frac{1}{(1-p)(x+n)^{p-1}} - \frac{1}{1-p}\,dx =$$

$$\overset{p \neq 2}{=} \left[\frac{1}{(1-p)(2-p)(x+n)^{p-2}}\right]_0^1 + \frac{1}{p-1} = \frac{1}{(1-p)(2-p)(1+n)^{p-2}} - \frac{1}{(1-p)(2-p)n^{p-2}} + \frac{1}{p-1}$$

$p > 2$: $\displaystyle\lim_{n\to\infty} I_n = \frac{1}{p-1}$

$p < 2$: $\displaystyle\lim_{n\to\infty} \frac{(n+1)^{2-p} - n^{2-p}}{(1-p)(2-p)} = \lim_{n\to\infty} \frac{n^{2-p}\left[\left(1+\frac{1}{n}\right)^{2-p} - 1\right]}{(1-p)(2-p)}$ $\quad \infty \cdot 0$ N.V

② $M_n:\ 0 \leq x \leq 1$

$\quad 1-x \leq y \leq n-x$, $n \geq 2$, $\in \mathbb{N}$

$$\overset{p\neq 1}{\tilde{I}_n} = \int_0^1 \frac{1}{(1-p)\,n^{p-1}} - \frac{1}{1-p}\,dx = \frac{1}{1-p}\left[\cancel{\frac{1}{n^{p-1}}}^{\,p>1:\ \to 0} - 1\right]$$

$\boxed{p > 1}$ $\displaystyle\lim_{n\to\infty} \tilde{I}_n = \boxed{\frac{1}{p-1}\ K}$ $\qquad$ $\boxed{p < 1}$ $\displaystyle\lim_{n\to\infty} \tilde{I}_n = \lim_{n\to\infty} \frac{1}{1-p}\left[n^{1-p} - 1\right] = \infty$ $\boxed{D}$

$p = 1$: $\displaystyle\tilde{I}_n = \int_0^1 \left[\ln(x+y)\right]_{1-x}^{n-x} dx = \int_0^1 \ln n - \ln 1\,dx = \ln n \xrightarrow{n\to\infty} \boxed{\infty\ D}$

(3) $\boxed{\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\cos(x^2+y^2)\,dx\,dy = \frac{\pi}{2}}$ K

$M_n:\ x^2+y^2=\rho^2 \qquad \varphi\in[0,2\pi]$

$x^2+y^2\le n^2 \qquad \rho\in[0,n]$

$$I_n = \iint\limits_{M_n} dx\,dy = \frac{1}{2}\int_0^{2\pi}\int_0^{n} 2\rho\cdot e^{-\rho^2}\cdot\cos\rho^2\,d\rho\,d\varphi = \left|\begin{matrix} t=\rho^2 \\ dt = 2\rho\,d\rho\end{matrix}\right|$$

$$= \frac{2\pi}{2}\cdot\underbrace{\int_0^{n^2} e^{-t}\cos t\,dt}_{A} = \pi\cdot\left[\frac{e^{-n^2}\sin n^2 - e^{-n^2}\cos n^2 + 1}{2}\right]$$

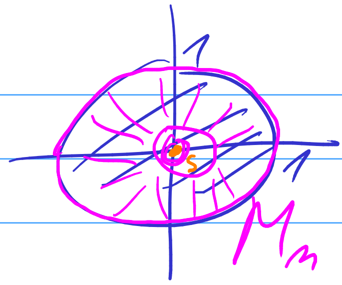
$$A = \int_0^{n^2} e^{-t}\cos t\,dt = \left|\begin{matrix} e^{-t} & -e^{-t} \\ \cos t & \sin t\end{matrix}\right| = \Big[\sin t\, e^{-t}\Big]_0^{n^2} + \int_0^{n^2} e^{-t}\sin t\,dt = \left|\begin{matrix} e^{-t} & -e^{-t} \\ \sin t & -\cos t\end{matrix}\right|$$

$$= \underbrace{e^{-n^2}\sin n^2 - \Big[\cos t\, e^{-t}\Big]_0^{n^2}}_{=2A} - \underbrace{\int_0^{n^2} e^{-t}\cos t\,dt}_{A}$$

$$I_n = \pi\cdot\left[\frac{\cancel{e^{-n^2}\cdot \text{OHR}\,\sin n^2} - \cancel{e^{-n^2}\cdot\text{OHR}\,\cos n^2} + 1}{2}\right] \xrightarrow{n\to\infty} \pi\left[\frac{0+0+1}{2}\right] = \frac{\pi}{2}$$

(4) $\iint\limits_M \frac{1}{\underbrace{x^2+y^2}_{\text{není def. } 0}}\,dx\,dy$ $\qquad M:\ x^2+y^2 \le 1$

$M_n:\ \frac{1}{n^2} \le x^2+y^2 \le 1$, $\quad n \in \mathbb{N},\ n \ge 2$



$$I_n = \iint\limits_{M_n} \frac{1}{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_{\frac{1}{n}}^{1} \frac{\cancel{\rho}}{\rho^{\cancel{2}}}\,d\rho\,d\varphi = 2\pi \int_{\frac{1}{n}}^{1} \frac{1}{\rho}\,d\rho =$$

$$= 2\pi\left[\ln\rho\right]_{\frac{1}{n}}^{1} = 2\pi\left[\cancelto{0}{\ln 1} - \ln\frac{1}{n}\right]$$

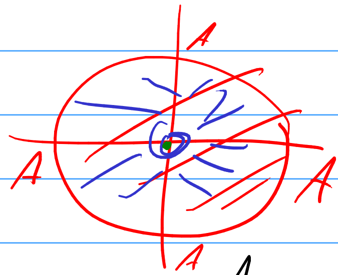
$$= 2\pi\left[+\ln n\right]$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} 2\pi \ln n = \underline{\infty}$$

Diverguje

(5) $\iint_M (x^2+y^2)\ln(x^2+y^2)\,dx\,dy$ $\qquad$ $\boxed{\pi A^4 \ln A - \frac{\pi A^4}{4}}$ $\qquad$ $M:\ x^2+y^2 \le A^2,\ A>0$

není def. v 0 $\qquad$ $\boxed{K}$



$$I_n = \iint_{M_n} dx\,dy = \int_0^{2\pi}\int_{\frac{1}{n}}^{A} \rho\cdot\rho^2\cdot\ln\rho^2\,d\rho\,d\varphi = 4\pi\int_{\frac{1}{n}}^{A}\rho^3\ln\rho\,d\rho$$

$$\begin{vmatrix} \ln\rho & \frac{1}{\rho} \\ \rho^3 & \frac{\rho^4}{4} \end{vmatrix}$$

$$= 4\pi\left[\left[\frac{\rho^4}{\boxed{4}}\ln\rho\right]_{\frac{1}{n}}^{A} - \int_{\frac{1}{n}}^{A}\frac{\rho^3}{\boxed{4}}\,d\rho\right] =$$

$$= \pi\cdot\left(A^4\ln A - \frac{1}{n^4}\ln\frac{1}{n} - \left[\frac{\rho^4}{4}\right]_{\frac{1}{n}}^{A}\right) = \underbrace{\pi\left(A^4\ln A - \frac{\ln n}{n^4} - \frac{A^4}{4} + \frac{1}{4n^4}\right)}_{I_n}$$

n^{-1}

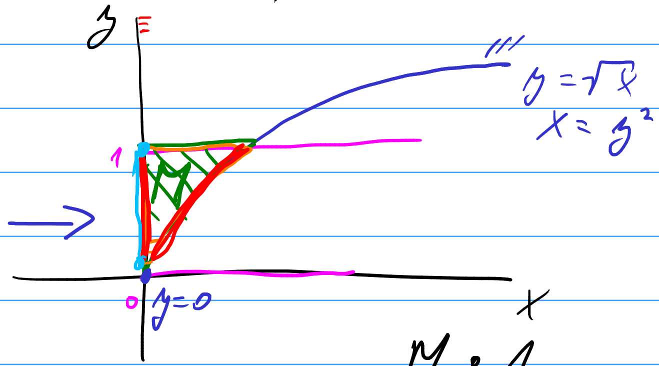
$$\lim_{n\to\infty} I_n = \lim_{n\to\infty}\pi\left(A^4\ln A - \cancelto{0}{\frac{\ln n}{n^4}} - \frac{A^4}{4} + \cancelto{0}{\frac{1}{4n^4}}\right) = \pi A^4\ln A - \frac{\pi A^4}{4}$$

$$\int_0^{2\pi}\int_0^{A}\rho^3\ln\rho^3 = \left[\rho^4\ln\rho\right]_0^A - \left[\cancelto{0}{\rho^4}\right]_0^A$$

$$\lim_{\rho\to 0^+}\rho^4\ln\rho = 0$$

⑥ $$\boxed{\iint_M e^{\frac{x}{y}}\,dx\,dy = \frac{1}{2}}$$ $M:\ 0 \le y \le 1,\ x \ge 0,\ y \ge \sqrt{x}$

$y = 0$



$$I_n = \iint_{M_n} e^{\frac{x}{y}}\,dx\,dy = \int_{\frac{1}{n}}^{1}\int_{0}^{y^2} e^{\frac{x}{y}}\,dx\,dy =$$

$M_n:\ \frac{1}{n} \le y \le 1$, $x \ge 0,\ y \ge \sqrt{x}$

$\left| \begin{aligned} z &= \frac{x}{y} \\ dz &= \frac{1}{y}dx \end{aligned} \right|$

$$\int e^{Ax}\,dx = \frac{e^{Ax}}{A}$$

$$= \int_{\frac{1}{n}}^{1} \left[y \cdot e^{\frac{x}{y}} \right]_0^{y^2} dy = \int_{\frac{1}{n}}^{1} y e^{y} - y\,dy =$$

$\left| \begin{matrix} y & 1 \\ e^y & e^y \end{matrix} \right|$

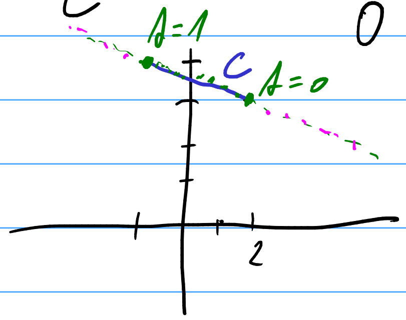
$$= \left[y e^{y} \right]_{\frac{1}{n}}^{1} - \left[e^{y} \right]_{\frac{1}{n}}^{1} - \left[\frac{y^2}{2} \right]_{\frac{1}{n}}^{1} = \underbrace{e - \frac{e^{\frac{1}{n}}}{n} - e + e^{\frac{1}{n}} - \frac{1}{2} + \frac{1}{2n^2}}_{I_n}$$

$$\lim_{n\to\infty} \cancel{e} - \frac{e^{\frac{1}{n}}}{n} - \cancel{e} + e^{\frac{1}{n}} - \frac{1}{2} + \cancel{\frac{1}{2n^2}} = e^0 - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$\frac{1}{n} \to 0$, $\frac{e^{\frac{1}{n}}}{n} \to 0$, $\frac{1}{2n^2} \to 0$

⑦ určete délku úsečky spojující body [2,3] a [-1,4]

$$l = \int_C 1\, ds = \int_0^1 1 \cdot \sqrt{10}\, dt = \sqrt{10}$$



$t \in [0,1]$

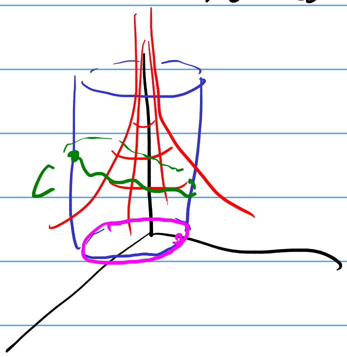
$$C: \begin{aligned} x &= 2 + t\cdot(-3) = 2-3t = \alpha(t) \\ y &= 3 + t\cdot 1 = 3+t = \beta(t) \end{aligned}$$

$x' = -3$

$y' = 1$

$$\sqrt{(x')^2+(y')^2} = \sqrt{(-3)^2+1^2} = \sqrt{10}$$

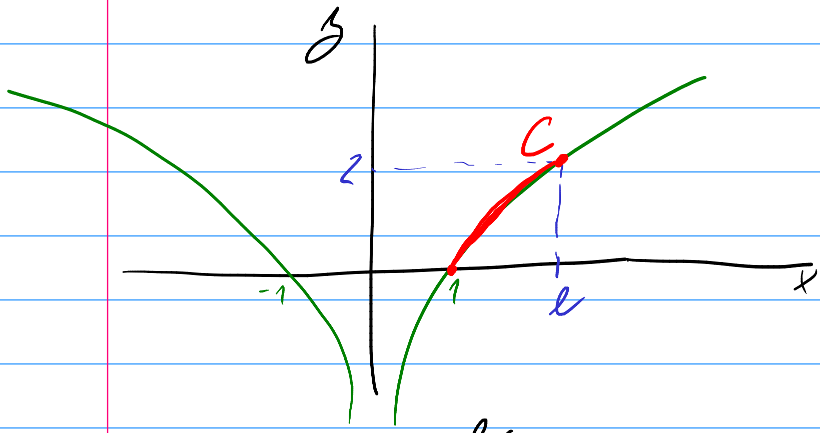
(8) parametrizujte křivku C, která vznikne průnikem válce $x^2+y^2=4$ a plochy $z=\frac{1}{x^2+y^2}$



$$C: \boxed{\begin{aligned} x &= 2\cdot\cos t \\ y &= 2\cdot\sin t \qquad t\in[0,2\pi] \\ z &= \frac{1}{4} \end{aligned}}$$

$$z=\frac{1}{x^2+y^2}=\frac{1}{4\cos^2 t+4\sin^2 t}=\frac{1}{4}$$

⑨ spočtěte integrál $\int_C \frac{x^2}{5}\, ds$, kde křivka C je dána $y = \ln x^2$
a spojuje body $[e,2]$ & $[1,0]$



$C: \quad x = t \qquad t \in [1,e]$

$y = \ln x^2 = \ln t^2$

$x' = 1 \quad y' = \frac{2}{t}$

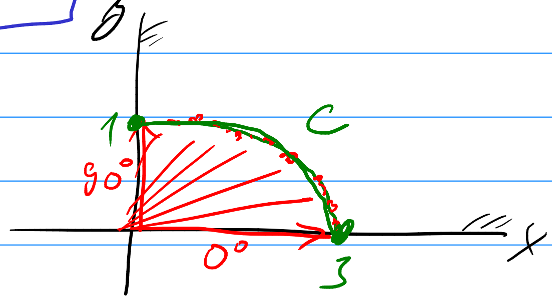
$$\int_C \frac{x^2}{5}\, ds = \int_1^e \frac{t^2}{5} \cdot \sqrt{1 + \frac{4}{t^2}}\, dt = \frac{1}{5}\int_1^e t^{\cancel{2}} \frac{\sqrt{t^2+4}}{\cancel{t}}\, dt =$$

$$= \frac{1}{5}\int_1^e t\sqrt{t^2+4}\, dt = \left| \begin{array}{l} z = t^2+4 \\ dz = 2t\, dt \end{array} \right| =$$

$$= \frac{1}{10}\int_5^{e^2+4} \underbrace{\sqrt{z}}_{z^{\frac{1}{2}}} \cdot dz = \frac{1}{10}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_5^{e^2+4} = \frac{2}{30}\sqrt[3]{(e^2+4)^2} - \frac{2}{30}\sqrt[3]{25}$$

(10) $\int_C xy\,ds =$ $\quad C: \boxed{\frac{x^2}{9}+y^2=1}, x\geq 0, y\geq 0$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$$= \int_0^{\frac{\pi}{2}} 3\cos t \sin t \sqrt{\underbrace{9\sin^2 t+\cos^2 t}_{9-8\cos^2 t}}\,dt$$

$$= 3\int_0^{\frac{\pi}{2}} \sin t \cos t \sqrt{9-8\cos^2 t}\,dt =$$

$x = 3\cos t$

$y = \sin t$

$t \in \left[0, \frac{\pi}{2}\right]$

$\frac{x^2}{9}+y^2=1$

$\frac{\cancel{9}\cos^2 t}{\cancel{9}}+\sin^2 t=1$

$$= \left| \begin{array}{l} z=\cos t \\ dz=-\sin t \end{array} \right|$$

$x' = -3\sin t$

$y' = \cos t$

$$= 3\int_0^1 z\sqrt{9-8z^2}\,dz$$

$$\left| \begin{array}{l} A=9-8z^2 \\ dA=-16z\,dz \end{array} \right| = +\frac{3}{16}\int_1^9 \sqrt{A}\,dA =$$

$$= \frac{\cancel{3}}{\cancel{16}_{8}}\left[\frac{A^{\frac{3}{2}}}{\frac{\cancel{3}}{2}}\right]_1^9 = \frac{1}{8}\left[\underbrace{27-1}_{26}\right] = \frac{13}{4}$$

(11) $\int_C 3\sqrt{x^2+y^2}\,dt\,dz$ $C: \boxed{\begin{aligned} x &= t\cos t \\ y &= t\sin t \\ z &= t \end{aligned}}$ pro $t \in [0, \pi]$

$x' = \cos t - t\sin t$
$y' = \sin t + t\cos t$
$z' = 1$

$= 3\int_0^{\pi} \sqrt{\underbrace{t^2\cos^2 t + t^2\sin^2 t}_{t^2}} \cdot \sqrt{2+t^2}\,dt =$

$(x')^2 + (y')^2 + (z')^2 = \cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t + 1 = 2 + t^2$

$\cos^2 t + \sin^2 t = 1$, $\sin^2 t + \cos^2 t = 1$

$= 3\int_0^{\pi} t\cdot\sqrt{2+t^2}\,dt = \left| \begin{aligned} z &= 2+t^2 \\ dz &= 2t\,dt \end{aligned} \right| =$

$= \frac{3}{2}\int_2^{2+\pi^2} \sqrt{z}\,dz = \frac{3}{2}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^{2+\pi^2} = \underline{\underline{\left[2+\pi^2\right]^{\frac{3}{2}} - 2^{\frac{3}{2}}}}$