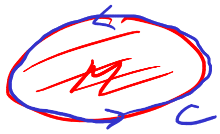
C je orien. hladká hranice M

(1) spočtěte pomocí Greenovy věty obsah kruhu s poloměrem r

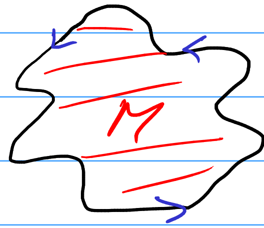


$x = r\cos t$
$y = r\sin t$
$t \in [0, 2\pi]$
souhlasná

$$S = \iint_M 1\,dx\,dy = \int_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy =$$

P, Q aby $Q_x - P_y = 1$

$Q = \frac{x}{2},\ P = -\frac{y}{2}$

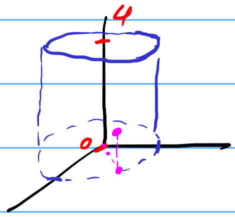


$$\oint_C P\,dx + Q\,dy = \iint_M (Q_x - P_y)\,dx\,dy$$

$$= +\int_0^{2\pi} -\frac{r\sin t}{2}\cdot(-r\sin t) + \frac{r\cos t}{2}\cdot(r\cos t)\,dt =$$

$$= \frac{1}{2}\int_0^{2\pi} \underbrace{r^2\sin^2 t + r^2\cos^2 t}_{=r^2}\,dt = \frac{r^2}{2}\underbrace{\int_0^{2\pi} 1\,dt}_{2\pi} = \pi r^2$$

(2) parametrizirajte $S: \underline{x^2+y^2=\boxed{4}}$, $\underline{0 \le z \le 4}$



$x = 2\cos u$

$y = 2\sin u$

$z = v$

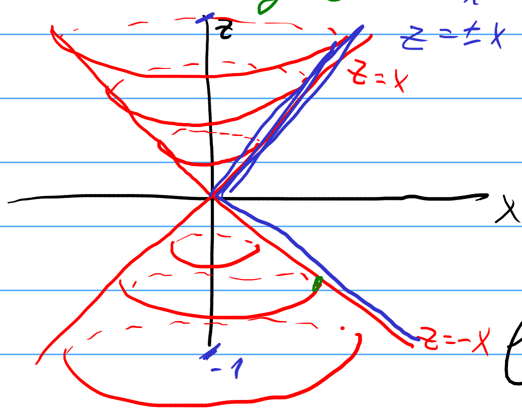
$M: 0 \le v \le 4$

$u \in [0, 2\pi]$

③ $S: z^2 = x^2 + y^2 \quad , \; -1 \le z \le 1$

$y = 0: \; z^2 = x^2$

$z = \pm x$



$x = u \cos v$

$y = u \cos v$

$z^2 = u^2 \rightsquigarrow z = \pm u$

$u = |z|$

$u = \pm z$

$\boxed{u \ge 0}$

① $S_1: z = u$

$z \ge 0$

$\cancel{-1 \le} \; z \le 1$

$u \le 1$

$\boxed{u \in [0, 1]}$

$\boxed{v \in [0, 2\pi]}$

$S_2: \; z = -u$

$z \le 0$

$-1 \le z \; \cancel{\le 1}$

$-1 \le -u$

$u \le 1$

$\underline{u \in [0, 1]}$

② $x = |w| \cos v$

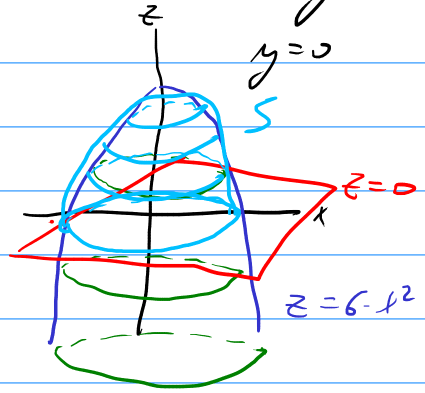
$y = |w| \sin v$

$z = w$

$w \in [-1, 1]$

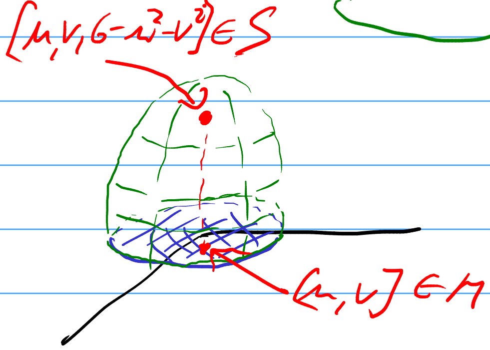
$v \in [0, 2\pi]$

④ $\iint\limits_{1} 1\,dS$, $S: z = \overbrace{6-x^2-y^2}^{g(x,y)}$ nad rovinou $\boxed{z=0}$ $\longrightarrow$ $z \geq 0$



$$x = u \qquad [u,v] \in M$$
$$y = v$$
$$z = 6-u^2-v^2$$

$$6-x^2-y^2 \geq 0$$
$$x^2+y^2 \leq 6$$
$$M: \underline{u^2+v^2 \leq 6}$$



$$n = (-g_x, -g_y, 1) = (2u, 2v, 1) \qquad \|n\| = \sqrt{4u^2+4v^2+1}$$

$$\iint\limits_S 1\,dS = \iint\limits_{u^2+v^2 \leq 6} 1 \cdot \sqrt{4u^2+4v^2+1}\, du\, dv =$$

$$= \int_0^{\sqrt{6}}\int_0^{2\pi} \rho\sqrt{\underbrace{4\rho^2+1}_{t}}\, d\varphi\, d\rho = \left| \begin{array}{l} t = 4\rho^2+1 \\ dt = 8\rho\, d\rho \end{array} \right|$$

(5) $$\iint_S \frac{1}{\sqrt{R^2-z^2}}\, dS \qquad S: \underbrace{x^2+y^2=R^2}_{\text{válec poloměr } R},\quad 0\le z\le H$$

$$R>H>0$$

$$S: \begin{array}{l} x = R\cos u \\ y = R\sin u \\ z = v \end{array} \qquad M: \begin{array}{l} u\in[0,2\pi] \\ v\in[0,H] \end{array}$$

$$n = (x_u, y_u, z_u)\times(x_v, y_v, z_v) = (-R\sin u, R\cos u, 0)\times(0,0,1) = (R\cos u, R\sin u, 0)$$

$$\begin{array}{cccccc} R\cos u & 0 & -R\sin u & R\cos u \\ 0 & 1 & 0 & 0 \end{array}$$

$$\underbrace{\quad}_{R\cos u} \quad \underbrace{\quad}_{R\sin u} \quad \underbrace{\quad}_{0}$$

$$I = \iint_M \frac{1}{\sqrt{R^2-v^2}}\cdot\sqrt{\underbrace{R^2\cos^2 u + R^2\sin^2 u}_{R^2} + 0^2}\, du\, dv =$$

$$= \int_0^H\int_0^{2\pi} \frac{R}{\sqrt{R^2-v^2}}\, du\, dv = 2\pi\int_0^H \frac{\cancel{R}\,1}{\sqrt{\cancel{R^2}-v^2}}\, dv =$$

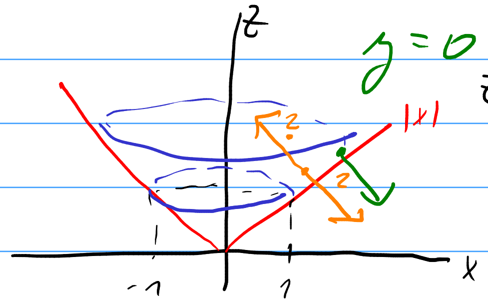
$$= 2\pi\int_0^H \frac{1}{\sqrt{1-\left(\frac{v}{R}\right)^2}}\, dv = \left| \begin{array}{l} s = \frac{v}{R} \\ ds = \frac{1}{R}dv \end{array}\right| = 2\pi R\int_0^{\frac{H}{R}} \frac{1}{\sqrt{1-s^2}}\, ds =$$

$$= 2\pi R\left[\arcsin s\right]_0^{\frac{H}{R}} = \underline{\underline{2\pi R \arcsin\left(\frac{H}{R}\right)}}$$

⑥ $\iint\limits_S \underset{P}{y}\,dy\,dz + \underset{Q}{z}\,dx\,dz + \underset{R}{x^2}\,dx\,dy \qquad S:\ z = \overbrace{\sqrt{x^2+y^2}}^{g(x,y)}$

$z \le 2$

a normálový vektor míří vně



$y = 0$

$z = \sqrt{x^2} = |x|$

$x = u \qquad \rightsquigarrow \underline{n} = \left(-\frac{\cancel{2}u}{\cancel{2}\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1\right)$

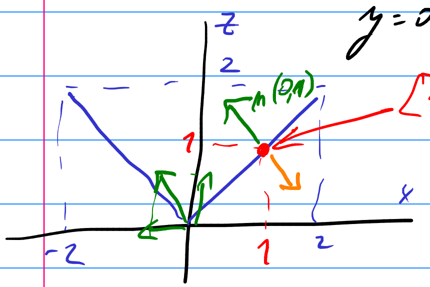
$y = v$

$z = \sqrt{u^2+v^2} \qquad M:\ z = \sqrt{u^2+v^2} \le 2$

$u^2+v^2 \le 4$

$F \cdot n$

$$= -\iint\limits_{u^2+v^2\le 4} v\cdot\left(-\frac{u}{\sqrt{u^2+v^2}}\right) + \cancel{\sqrt{u^2+v^2}}\cdot\left(-\frac{v}{\cancel{\sqrt{u^2+v^2}}}\right) + u^2\cdot 1\,du\,dv =$$



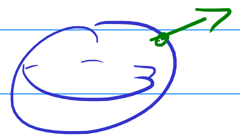
$y = 0 \ (u,v) = (1,0)$

$[1,0,1] \rightsquigarrow n(1,0) = \left(-\frac{1}{\sqrt{1^2+0^2}}, -\frac{0}{\sqrt{1^2+0^2}}, 1\right) = (-1,0,1)$

nesouhlasná param.

$$= -\int_0^{2\pi}\int_0^2 \left(-\frac{\rho^2 \overbrace{\cos\varphi\sin\varphi}^{\frac{\sin 2\varphi}{2}}}{\rho} - \rho\sin\varphi + \rho^2 \overbrace{\cos^2\varphi}^{\frac{1+\cos 2\varphi}{2}}\right)\rho\,d\rho\,d\varphi = \dots$$

⑦ $\iint_S x\,dy\,dz + 0\cdot dx\,dz + y\,dx\,dy$ $\quad S: x^2+y^2+z^2=4,\ z\geq 0,\ y\leq 0$



normálový vekt. míří ven

nesouhlasné

$x = 2\cos u \sin v$

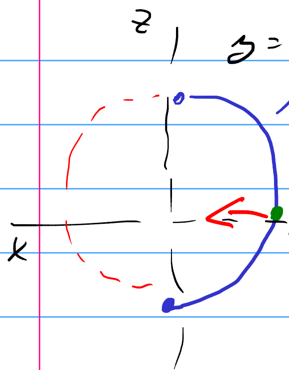
$y = 2\sin u \sin v \rightsquigarrow \sin u \leq 0 \rightsquigarrow u\in[\pi, 2\pi]$

$z = 2\cos v \geq 0 \quad v\in[0,\frac{\pi}{2}]$

$$\begin{array}{cccc} y & z & x & y \\ 2\cos u\sin v & 0 & -2\sin u\sin v & 2\cos u\sin v \\ 2\sin u\cos v & -2\sin v & 2\cos u\cos v & 2\sin u\cos v \end{array}$$

$n = \left(-4\cos u\sin^2 v,\ -4\sin u\sin^2 v,\ -4\sin v\cos v\sin^2 u - 4\sin v\cos v\cos^2 u\right)$; $\sin^2 u + \cos^2 u = 1$

$= -4\sin v\cos v)$



$u=0,\ v\in[0,\pi]$

$[u=0, v=\frac{\pi}{2}]$ $\quad n = (-4\cdot1\cdot1^2, -4\cdot0\cdot1^2, -4\cdot1\cdot0) =$

$= (-4,0,0)$

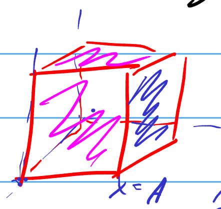
$$= +\int_0^{\frac{\pi}{2}}\int_\pi^{2\pi} +8\cos^2 u\sin^3 v + 8\sin u\sin^2 v\cdot\cos v\,du\,dv = \dots$$

$\cos^2 u = \frac{1+\cos 2u}{2}$

(8) $\iint\limits_S \overset{P}{x^2}\,dy\,dz + \overset{Q}{y^2}\,dx\,dz \quad +R=0$ $\quad$ S: hranice $[0,A]^3$, $A>0$

normálový vektor míří ven

G-O



6 param.

$y\in[0,A]$

$z\in[0,A]$

$$= +\iiint\limits_{[0,A]^3} \overset{2x}{P_x} + \overset{2y}{Q_y} + \overset{0}{R_z}\,dx\,dy\,dz =$$

$$= \int_0^A\int_0^A\int_0^A 2x\,dx\,dy\,dz + \int_0^A\int_0^A\int_0^A 2y\,dx\,dy\,dz =$$

$$= 4\cdot\int_0^A\int_0^A\int_0^A x\,dx\,dy\,dz = 4A^2\left[\frac{x^2}{2}\right]_0^A = \underline{\underline{2A^4}}$$

(9) $\iint_S \overset{Q}{y}\,dx\,dz$ $\quad S: x^2+y^2+z^2=1$ a normálový vekt. míří dovnitř

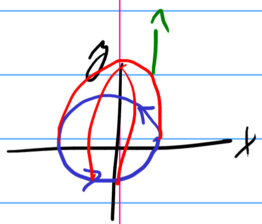
$$= -\iiint_{x^2+y^2+z^2\le 1} 1\,dx\,dy\,dz = -V(\text{koule s poloměrem } 1) = -\frac{4}{3}\pi$$

$\frac{4}{3}\pi r^2$

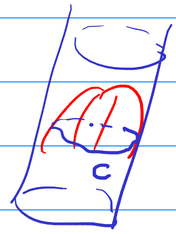
(10) $\int\limits_C (x+y)\,dx + zy^2\,dy + z\,dz =$ $\qquad n = \left(+\frac{2x}{4}, \frac{2y}{4}, 1\right)$

C je hranica plochy $S: z = \frac{2-x^2-y^2}{4}$ vykrojená $x^2+y^2 \le 4$

orientovaná súhlasne s S



C: kruž. $z = \frac{2-4}{4} = -2$



$$= +\iint\limits_S (0-y^2)\,dy\,dz + (0-0)\,dx\,dz + (0-1)\,dx\,dy =$$

$$= \iint\limits_S -y^2\,dy\,dz - dx\,dy = \pm \iint\limits_{x^2+y^2\le 4} -y^2\cdot\frac{x}{2} - 1\cdot 1\,dx\,dy =$$

$$= \pm \int_0^{2\pi}\int_0^2 -\rho^4\frac{\sin^2\varphi\cos\varphi}{2} - \rho\,d\rho\,d\varphi = \pm\left[\frac{\rho^5}{10}\right]_0^2 \underbrace{\left[\frac{\sin^3\varphi}{3}\right]_0^{2\pi}}_{=0}$$

$$\mp 2\pi\left[\frac{\rho^2}{2}\right]_0^2 = \mp 4\pi$$