1) $\sin x$, $x_0 = 0$, $\sin x^2$

$$T_n = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$f = \sin x \rightsquigarrow f(0) = \sin 0 = 0 \quad k=0$

$f' = \cos x \rightsquigarrow 1 \quad k=1 \leftarrow n=0$

$f'' = -\sin x \quad 0 \quad k=2$

$f''' = -\cos x \quad -1 \quad k=3 \leftarrow n=1$

$f^{(4)} = \sin x \quad 0 \quad k=4 \qquad k=2n+1$

$1 \quad k=5 \leftarrow n=2$

0

-1

0

1

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

② $$\cos^2 x = \cos x \cdot \cos x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{\ldots}\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{\ldots}\right)$$

$a_0 = 1$

$a_1 = 0 = a_3 = a_5 = \ldots = a_{2k+1}$

$a_2 = -\frac{1}{2} - \frac{1}{2} = -1$

$a_4 = \frac{1}{24} + \frac{1}{24} + \left(-\frac{1}{2}\right)^2 = \frac{1}{12} + \frac{1}{4} = \frac{1+3}{12} = \frac{1}{3}$

$a_6 = -\frac{1}{6!} \cdot 2 - \cancel{\frac{1}{2}} \cdot \frac{1}{24} \cdot \cancel{2} = \ldots$

$$\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \left(\frac{2}{6!} + \frac{1}{24}\right)x^6 \pm \ldots$$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)2^n}$ $\qquad \ln(1+x)=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n \quad /\cdot(-1)$

$x\in(-1,1)$

$\leftarrow x=\frac{1}{2}$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)}x^n$ $\qquad -\ln(1+x)=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^n \quad /\cdot x$

$\underline{R=1}$ $\quad \frac{1}{2}\in(-1,1)$

$$-x\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{n+1} \quad /\int dx$$

$\left| \begin{array}{ll} u=\ln(1+x) & u'=\frac{1}{1+x} \\ v'=x & v=\frac{x^2}{2} \end{array}\right|$ $\displaystyle (-)\int x\ln(1+x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)}x^{n+2}$ $\quad /:x^2$

$$=-\frac{x^2}{2}\ln(1+x)+\frac{1}{2}\int\frac{x^2}{1+x}\,dx=-\frac{x^2}{2}\ln(1+x)+\frac{1}{2}\int x-1+\frac{1}{x+1}\,dx=$$

$x^2:x+1=x-1+\frac{1}{x+1}$

$-x$

$+1$

$$=-\frac{x^2}{2}\ln(1+x)+\frac{x^2}{4}-\frac{x}{2}+\frac{1}{2}\ln(x+1)+c$$

$x=0 \leadsto 0=0+0-0+0+c$

$c=0$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)}x^n=-\frac{\ln(1+x)}{2}+\frac{1}{4}-\frac{1}{2x}+\frac{\ln(x+1)}{2x^2}$$

$\swarrow x=\frac{1}{2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+2)2^n}=-\frac{\ln\frac{3}{2}}{2}+\frac{1}{4}-1+2\ln\frac{3}{2}=$$

$$=\underline{\underline{\frac{3}{2}\ln\frac{3}{2}-\frac{3}{4}}}$$

(4) $$\lim_{x\to 0} \frac{e^x \sin x - x(1+x)}{x^3} = \left|\frac{0}{0}\right| = (*)$$

$$e^x \sin x = x + x^2 + \frac{x^3}{2} - \frac{x^3}{3!} + \cancel{\frac{x^4}{6}} - \cancel{\frac{x^4}{6}} + \frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$(*) = \lim_{x\to 0} \frac{\cancel{x} + \cancel{x^2} + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \frac{5+10+1}{120}x^5 + \dots - \cancel{x} - \cancel{x^2}}{x^3} =$$

$$= \lim_{x\to 0} \underbrace{\frac{3-1}{6} + \frac{16}{120}x^2 + \dots}_{} = \frac{2}{6} = \frac{1}{3}$$

(5) $\int\limits_{\frac{1}{10}}^{2} \frac{\cos x}{x^2}\,dx$ odhadněte pomocí 1.–3 nenulových členů M. řady

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

$$\int\limits_{\frac{1}{10}}^{2} \frac{1 - \frac{x^2}{2} + \frac{x^4}{24}}{x^2}\,dx = \int\limits_{\frac{1}{10}}^{2} \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{24}\,dx =$$

$$= \left[-\frac{1}{x} - \frac{x}{2} + \frac{x^3}{72} \right]_{\frac{1}{10}}^{2} = -\frac{1}{2} - 1 + \frac{1}{9} - \left(-10 - \frac{1}{20} + \frac{1}{7200} \right) =$$

$$= -\frac{3}{2} + \frac{1}{9} + 10 + \frac{1}{20} - \frac{1}{7200} = \dots$$