①

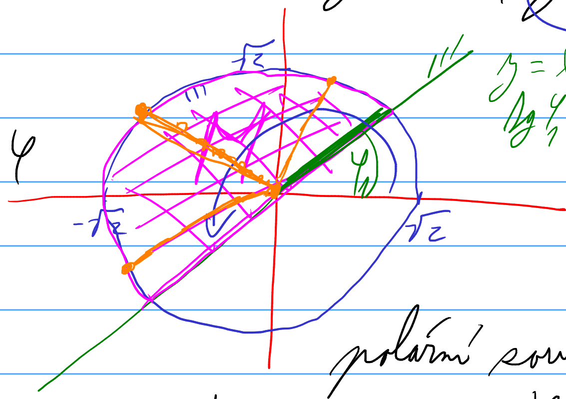
Transformujte $\iint_M f(x,y)\,dx\,dy =$ $\quad M: \underbrace{x^2+y^2}_{\rho^2} \le 2,\ y \ge x$

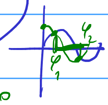
$\rho^2 \le 2$

$\rho\sin\varphi \ge \rho\cos\varphi$

$\sin\varphi \ge \cos\varphi$

$$= \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi}\int_0^{\sqrt{2}} f(\rho\cos\varphi, \rho\sin\varphi)\cdot\rho\,d\rho\,d\varphi$$





$y = x$

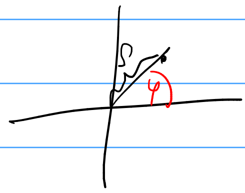
$\operatorname{tg}\varphi_1 = 1$

$45°$

$\varphi_2 = \varphi_1 + \pi$

polární souř.

OBECNÝ



$J = \rho \qquad x = \rho\cos\varphi \qquad \varphi\in[0,2\pi]$

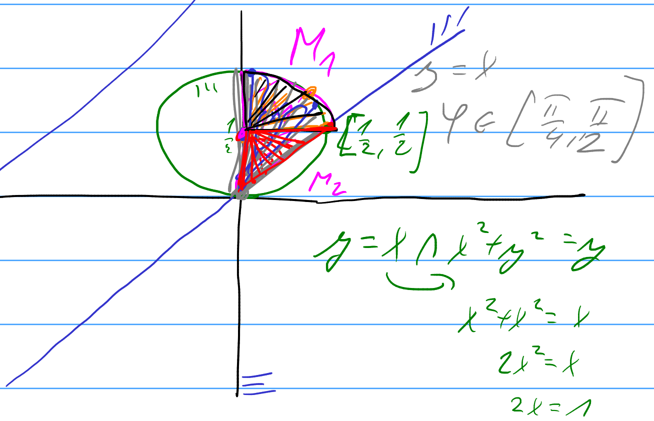
$\qquad\qquad y = \rho\sin\varphi \qquad \rho \ge 0$

U NÁS $\varphi\in\left[\frac{\pi}{4}, \frac{5}{4}\pi\right]$

$\rho\in[0,\sqrt{2}]$

$\rho\in[0,\sqrt{2}]\ ,\ \varphi\in\left[\frac{\pi}{4}, \frac{5}{4}\pi\right]$

② Transformujte $\iint_M f(x,y)\,dx\,dy$ $\qquad M: \underbrace{x^2+y^2 \le y}, \ \underline{y \ge x}, \ \underline{x \ge 0}$



$$\boxed{\begin{aligned} ①\ x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \end{aligned}} \qquad \boxed{\begin{aligned} ②\ x &= \rho\cos\varphi \\ y-\tfrac{1}{2} &= \rho\sin\varphi \\ J &= \rho \end{aligned}}$$

$y = x \cap x^2+y^2 = y$

$x^2+x^2 = x$

$2x^2 = x$

$2x = 1$

$x = \frac{1}{2}$

$$\rho^2 \le \rho\sin\varphi$$

$$\underline{\rho \le \sin\varphi}$$

$$\rho \in [0, \sin\varphi]$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sin\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\cdot\rho \, d\rho\, d\varphi$$

$x^2+y^2 \le y$

$x^2+y^2-y+\frac{1}{4}-\frac{1}{4} \le 0$

$y^2 - 2\cdot A\cdot y + A^2 - A^2$

$A = \frac{1}{2}$

$x^2 + \left(y-\frac{1}{2}\right)^2 \le \frac{1}{4}$ $\leftarrow$ KRUŽ $\quad S = \left[0, \frac{1}{2}\right]$, $r^2 = \frac{1}{4}$

$$M_1: \varphi \in \left[0, \frac{\pi}{2}\right], \ \rho \in \left[0, \frac{1}{2}\right]$$

$$M_2: \varphi \in \left[-\frac{\pi}{2}, 0\right], \ \rho \in \left[0, \frac{1}{2(\cos\varphi - \sin\varphi)}\right]$$

$y \ge x$

$\frac{1}{2} + \rho\sin\varphi \ge \rho\cos\varphi$

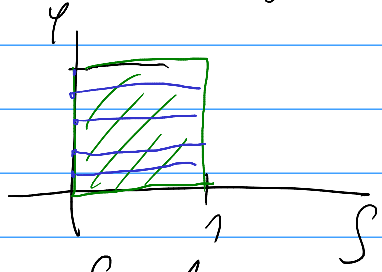
$\rho\,(\underbrace{\cos\varphi - \sin\varphi}_{\ge 0}) \le \frac{1}{2}$

$\rho \le \frac{1}{2(\cos\varphi - \sin\varphi)}$

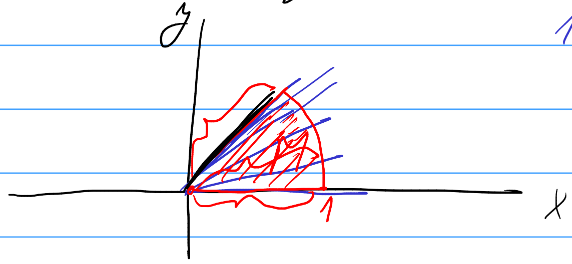
- . - -

③ po transf. $\rho \in [0,1]$, $\varphi \in [0,1]$ $S = ?$ původní množiny?

$$S = \iint_M 1 \, dx\, dy = \int_0^1\int_0^1 \rho \, d\rho \, d\varphi = \left[\frac{\rho^2}{2}\right]_0^1 = \frac{1}{2}$$



$S_p = 1$



$1 \approx \frac{\pi}{3}$

$S_K = \pi r^2 = \pi$

$S = \pi \cdot \alpha = \pi \frac{1}{2\pi} = \frac{1}{2}$

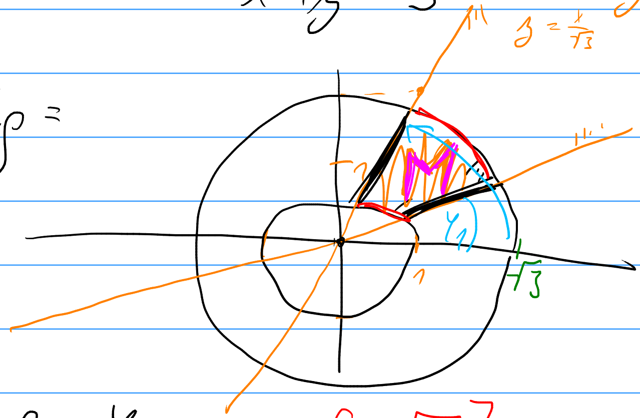
(4) $\iint\limits_M \operatorname{arctg}\frac{y}{x}\,dx\,dy =$

$M: x^2+y^2 \geq 1, \quad \frac{x}{\sqrt{3}} \leq y \leq \sqrt{3}x$

$x^2+y^2 \leq 3$

$y = \sqrt{3}x$

$y = \frac{x}{\sqrt{3}}$



$$= \int_1^{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \Big[\underbrace{\operatorname{arctg}\overbrace{\frac{\rho\sin\varphi}{\rho\cos\varphi}}^{\operatorname{tg}\varphi}}_{\varphi}\Big]\rho\, d\varphi\, d\rho =$$

$$= \int_1^{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \varphi\cdot\rho\, d\varphi\, d\rho =$$

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\rho \in [1, \sqrt{3}]$

$\varphi \in [\frac{\pi}{6}, \frac{\pi}{3}]$

$$= \int_1^{\sqrt{3}}\rho\, d\rho \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\varphi\, d\varphi =$$

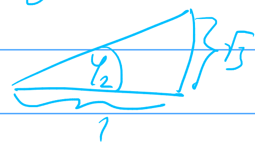
$y = \frac{x}{\sqrt{3}}$

$\Downarrow$

$\frac{\sin\varphi_1}{\cos\varphi_1} = \operatorname{tg}\varphi_1 = \frac{1}{\sqrt{3}}$

$\varphi_1 = \frac{\pi}{6}$

$y = \sqrt{3}$



$\operatorname{tg}\varphi_2 = \sqrt{3}$

$\varphi_2 = \frac{\pi}{3}$

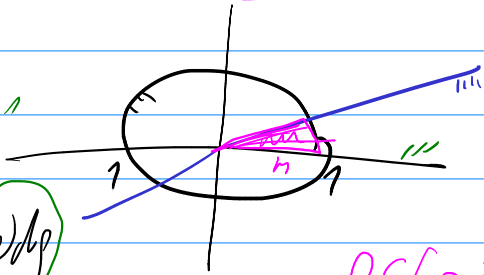
$$= \left[\frac{\rho^2}{2}\right]_1^{\sqrt{3}} \left[\frac{\varphi^2}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{4}\left(\overbrace{3-1}^{2}\right)\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right) =$$

$$= \frac{1}{2}\cdot\frac{\pi^2}{9}\Big(\underbrace{1-\frac{1}{4}}_{\frac{3}{4}}\Big) = \frac{\pi^2}{8\cdot 3} = \frac{\pi^2}{24}$$

⑤ $\iint\limits_M \frac{dx\,dy}{\sqrt{2-x^2-y^2}} = \int\limits_0^{1}\int\limits_0^{\frac{\pi}{6}} \frac{\rho}{\sqrt{2-\rho^2}}\,d\varphi\,d\rho =$

$M: 0 \le y \le \frac{x}{\sqrt{3}}, \quad x^2+y^2 \le 1$

$y = \frac{x}{\sqrt{3}}$



$y = \frac{x}{\sqrt{3}}$

$\operatorname{tg}\varphi = \frac{1}{\sqrt{3}}$

$\rho \in [0,1]$

$\varphi \in [0, \frac{\pi}{6}]$

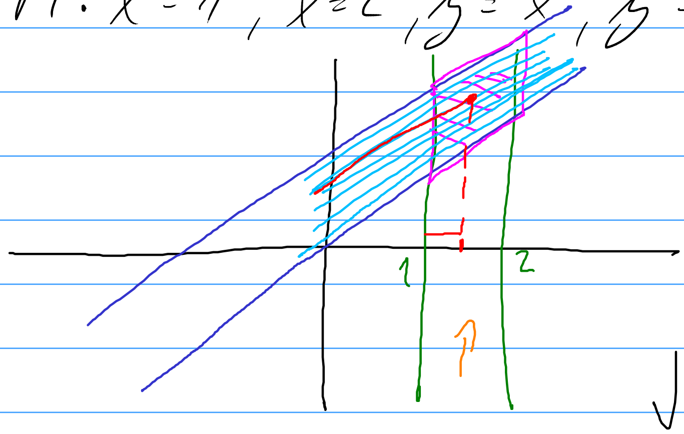
$= \frac{\pi}{6}\int\limits_0^1 \frac{\rho}{\sqrt{2-\rho^2}}\,d\rho = \left| \begin{array}{l} t = \sqrt{2-\rho^2} \\ dt = \frac{1}{\sqrt{2-\rho^2}}(-2\rho)\,d\rho \end{array} \right.$

$0 \rightsquigarrow \sqrt{2}$

$1 \rightsquigarrow \sqrt{1} = 1$

$= \frac{\pi}{6}\int\limits_{\sqrt{2}}^{1} \frac{dt}{-2} = \frac{\pi}{12}\int\limits_1^{\sqrt{2}} dt = \frac{\pi}{12}\left(\sqrt{2}-1\right)$

(6) $M: x = 1,\ x = 2,\ y = x,\ y = x + 2$ transformujte

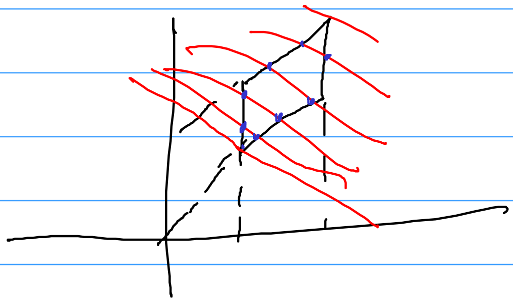


$u = x$ $\quad u \in [1, 2]$

$y = x + v$ $\quad v \in [0, 2]$

$v = y - x$ $\quad y = u + v$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = \underline{\underline{1}}$$



$y = x + v$

$y = -x + u$

7) M: $x \le y \le x+1$ & $2x-2 \le y \le 2x$ $\qquad J = ?$

$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$

$0 \le \underbrace{y-x}_{u} \le 1 \qquad -2 \le \underbrace{y-2x}_{v} \le 0$

$u = y - x \quad /\cdot(-2)$

$v = y - 2x$

$v - u = -x$

$x = u - v$

$v - 2u = -y$

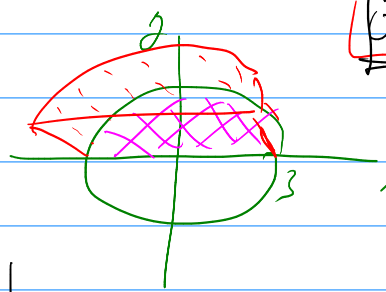
$y = 2u - v$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$J = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1 + 2 = \underline{\underline{1}}$$

(8) $I = \iiint\limits_V z\sqrt{x^2+y^2}\,dx\,dy\,dz$

$V:\ x^2+y^2 \le 9,\ y \ge 0,\ \boxed{0 \le z \le 2}$



$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$z = z$

$J = \rho$

$\varphi \in [0, \pi]$

$\rho \in [0, 3]$

$$I = \underbrace{\int_0^{\pi}\int_0^{3}\int_0^{2} z\cdot\rho^2\,dz\,d\rho}\,d\varphi = \pi\int_0^3 \rho^2\,d\rho \cdot \int_0^2 z\,dz =$$

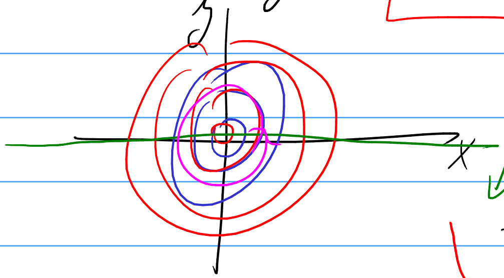
$$= \pi\cdot\left[\frac{\rho^3}{3}\right]_0^3 \cdot \left[\frac{z^2}{2}\right]_0^2 = \pi\cdot 9\cdot 2 = \underline{\underline{18\pi}}$$

(9) $\iiint_V 3z^2\,dx\,dy\,dz = \quad V: x^2+y^2 \le z \le 2-x^2-y^2$

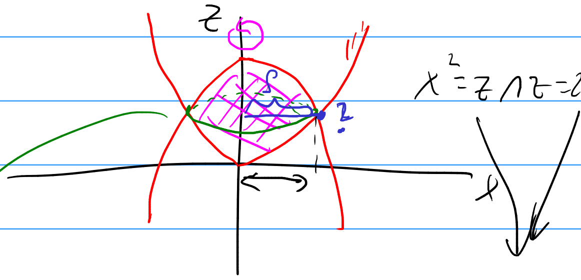
$\rho^2 \le z \le 2-\rho^2 \to \rho^2 = 2-\rho^2$

$\rho = 1$

$$= \int_0^{2\pi}\int_0^1\int_{\rho^2}^{2-\rho^2} 3z^2\rho\,dz\,d\rho\,d\varphi =$$



$y=0 \quad x^2 \le z \le 2-x^2$



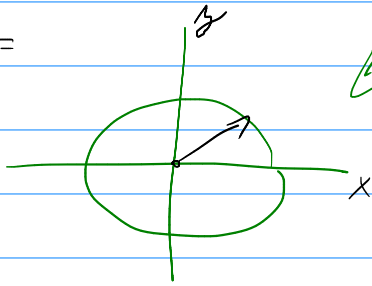
$x^2 = z \wedge z = 2-x^2$

$x^2 = 2-x^2$

$x^2 = 1$

$x = 1$

$$= 2\pi \int_0^1 \left[z^3\right]_{\rho^2}^{2-\rho^2}\rho\,d\rho =$$



$\varphi \in [0, 2\pi]$

$\rho \in [0, 1]$

$$= 2\pi \int_0^1 \left[(2-\rho^2)^3 - (\rho^2)^3\right]\boxed{\rho\,d\rho} = \left| \begin{array}{l} t = \rho^2 \\ dt = 2\rho\,d\rho \end{array} \right| =$$

$$= \frac{2\pi}{2}\int_0^1 (2-t)^3 - t^3\,dt = \pi\int_0^1 (2-t)^3\,dt - \pi\int_0^1 t^3\,dt =$$

$s = 2-t$

$ds = -dt$

$0 \to 2$

$1 \to 1$

$$= \pi\int_1^2 s^3\,ds - \pi\int_0^1 t^3\,dt =$$

$$= \frac{\pi}{4}\left[\left[s^4\right]_1^2 - \left[t^4\right]_0^1\right] = \frac{\pi}{4}\left[2^4 - 1 - 1 + 0\right] = \frac{14}{4}\pi$$