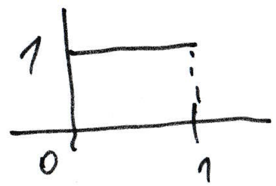
Př. 2. $X \sim Ro(0,1)$, $Y \sim Ro(0,1)$, $X \perp\!\!\!\perp Y$, $T = X + Y$

$$f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$



$$f_T(t) = (f_X * f_Y)(t) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(t-x)\, dx = \int_0^1 1 \cdot f_Y(t-x)\, dx =$$

$0 \le t-x \le 1$ $\quad t \in (0,2)$

$t-1 \le x \le t$ $\quad x \in (0,1)$

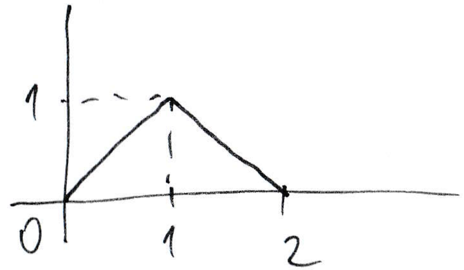
$t-1 \le x$ $\quad x \le t$

$t \in (0,1)$ $\quad t \in (1,2)$

$\Rightarrow$ 1) $t \in (0,1)$: $f_T(t) = \int_0^t 1\,dx = t$

2) $t \in (1,2)$: $f_T(t) = \int_{t-1}^1 1\,dx = 1-(t-1) = 2-t$

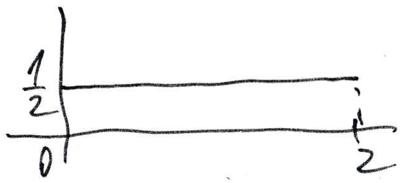
$$f_T(t) = \begin{cases} t & t \in (0,1) \\ 2-t & t \in \langle 1,2) \\ 0 & t \notin (0,2) \end{cases}$$



Neplést! $T = X + X = 2X$ $\quad t \in (0,2)$

$F_T(t) = P(2X \le t) = P(X \le \frac{t}{2}) = F_X(\frac{t}{2})$

$f_T(t) = f_X(\frac{t}{2}) \cdot \frac{1}{2}$



Př. 1 $X \sim A(\theta)$, $Y \sim A(\theta)$, $X \perp\!\!\!\perp Y$, $T = X + Y$

$$p_X(x) = \begin{cases} \theta^x (1-\theta)^{1-x} & x \in \{0,1\} \\ 0 & x \notin \{0,1\} \end{cases} \qquad t \in \{0,1,2\}$$

$$p_T(t) = \sum_{x \in M} p_X(x)\, p_Y(t-x) = \sum_{x=0}^{1} \theta^x (1-\theta)^{1-x}\, \theta^{t-x} (1-\theta)^{1-t+x} =$$

$$= (1-\theta) \cdot \begin{cases} \theta^t (1-\theta)^{1-t} & \text{pro } t \in \{0,1\} \\ 0 & \text{pro } t = 2 \end{cases} + \theta \cdot \begin{cases} 0 & \text{pro } t = 0 \\ \theta^{t-1} (1-\theta)^{2-t} & \text{pro } t = 1,2 \end{cases}$$

$$= \begin{cases} \theta^t (1-\theta)^{2-t} & \text{pro } t = 0 \\ \theta^t (1-\theta)^{2-t} + \theta^t (1-\theta)^{2-t} & \text{pro } t = 1 \quad (= 2 \cdot \theta^t (1-\theta)^{2-t}) \\ \theta^t (1-\theta)^{2-t} & \text{pro } t = 2 \end{cases}$$

$$= \binom{2}{t} \theta^t (1-\theta)^{2-t} \quad \text{pro } t \in \{0,1,2\} \quad \sim Bi(2;\theta)$$

D.V.3.

$Y = h(X) \qquad y = a + bx \Rightarrow x = \underbrace{\frac{y-a}{b}}_{h^{-1}} \qquad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \left| \frac{dh^{-1}(y)}{dy} \right| \qquad \frac{dh^{-1}(y)}{dy} = \frac{1}{b}$$

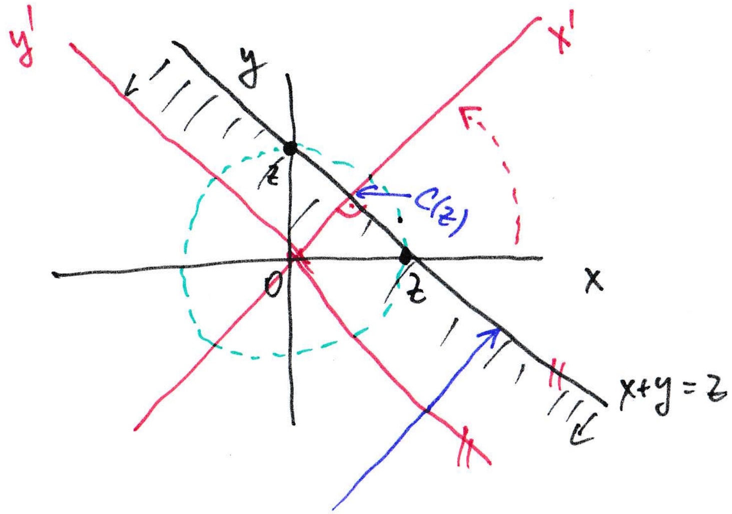
$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(y-a-\mu b)^2}{\sigma^2 b^2}} \cdot \left|\frac{1}{b}\right| = \frac{1}{\sqrt{2\pi}\,\sigma \cdot |b|} e^{-\frac{1}{2}\frac{(y-(a+b\mu))^2}{b^2\sigma^2}} \sim N(a+b\mu;\, b^2\sigma^2)$$

$$V = \frac{X-\mu}{\sigma} = \underbrace{-\frac{\mu}{\sigma}}_{a} + \underbrace{\frac{1}{\sigma}}_{b} X \sim N\Big(\underbrace{-\frac{\mu}{\sigma} + \frac{1}{\sigma}\cdot\mu}_{0} ;\, \underbrace{\frac{1}{\sigma^2}\sigma^2}_{1}\Big).$$

D. Důsl. 5. 1) $X \sim N(0,1)$, $Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$, $Z = X+Y$

$$F_Z(z) = P(Z \le z) = P(X+Y \le z) = \int_{M_z} f_{(X,Y)}(x,y)\,dx\,dy \overset{\text{nez.}}{=} \int_{M_z} f_X(x)\cdot f_Y(y)\,dx\,dy$$

$$M_z = \{(x,y) \in \mathbb{R}^2;\ x+y \le z\}$$

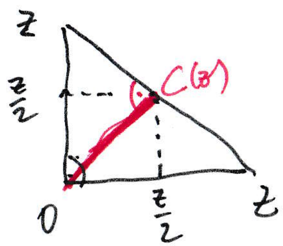


$$f_X(x)\cdot f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$$\underline{F_Z(z)} = \int_{M_z} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \int_{-\infty}^{C(z)}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x'^2+y'^2}{2}}\,dy'\,dx'$$

$$= \int_{-\infty}^{C(z)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}}\,dx' \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y'^2}{2}}\,dy'}_{1} = F_X(C(z)) \quad (\Phi(C(z)))$$

$$= \underline{F_X\left(\frac{z}{\sqrt{2}}\right)}$$

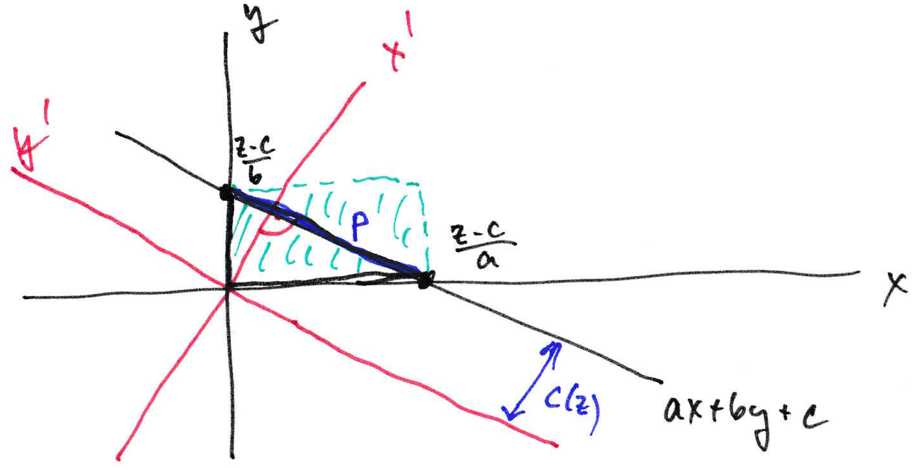


$$C^2(z) = \frac{z^2}{4} + \frac{z^2}{4}$$

$$C(z) = \frac{z}{\sqrt{2}}$$

$$f_Z(z) = f_X\left(\frac{z}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\cdot\frac{z^2}{2}} \sim N(0;2)$$

2) $X \sim N(0,1)$, $Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$, $Z = aX + bY + c$, $a^2+b^2 > 0$



$$F_Z(z) = F_X\left(\frac{z-c}{\sqrt{a^2+b^2}}\right)$$

$$f_Z(z) = f_X\left(\frac{z-c}{\sqrt{a^2+b^2}}\right) \cdot \frac{1}{\sqrt{a^2+b^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{a^2+b^2}} e^{-\frac{1}{2}\frac{(z-c)^2}{a^2+b^2}}$$

$$\sim N(c; a^2+b^2)$$

$$S_\Delta = \frac{(z-c)^2}{2ab} = \frac{p}{2} c(z) = \frac{(z-c)\sqrt{a^2+b^2}}{2ab} c(z) \Rightarrow \boxed{c(z) = \frac{z-c}{\sqrt{a^2+b^2}}}$$

$$p^2 = \frac{(z-c)^2}{b^2} + \frac{(z-c)^2}{a^2} = \frac{(z-c)^2(a^2+b^2)}{a^2b^2}$$

$$p = \frac{(z-c)\sqrt{a^2+b^2}}{ab}$$

3) $$X + Y = \underbrace{\sigma_1}_{a}\frac{X-\mu_1}{\sigma_1} + \underbrace{\sigma_2}_{b}\frac{X-\mu_2}{\sigma_2} + \underbrace{\mu_1+\mu_2}_{c} \sim N(\mu_1+\mu_2; \sigma_1^2+\sigma_2^2)$$

D.V.7. $\mathbf{Y} = h(\mathbf{X}) \Rightarrow f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(h^{-1}(\mathbf{y})) \cdot |D_{h^{-1}}(\mathbf{y})|$; $\mathbf{X} \sim N_n(\boldsymbol{\mu}; \Sigma)$

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

$\mathbf{y} = a + B\mathbf{x}$

$\mathbf{x} = \underbrace{B^{-1}(\mathbf{y}-a)}_{h^{-1}(\mathbf{y})}$

$D_{h^{-1}}(\mathbf{y}) = |B^{-1}| = |B|^{-1}$

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} |B|^{-1} e^{-\frac{1}{2}[B^{-1}(\mathbf{y}-a)-\boldsymbol{\mu}]^T \Sigma^{-1} [B^{-1}(\mathbf{y}-a)-\boldsymbol{\mu}]}$$

$$= (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} |B^T|^{-\frac{1}{2}} |B|^{-\frac{1}{2}} e^{-\frac{1}{2}[B^{-1}(\mathbf{y}-a-B\boldsymbol{\mu})]^T \Sigma^{-1} [B^{-1}(\mathbf{y}-a-B\boldsymbol{\mu})]}$$

$$= (2\pi)^{-\frac{n}{2}} |B^T \Sigma B|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{y}-a-B\boldsymbol{\mu})^T \underbrace{B^{-1T} \Sigma^{-1} B^{-1}}_{(B\Sigma B^T)^{-1}} (\mathbf{y}-a-B\boldsymbol{\mu})} \sim N_n(a+B\boldsymbol{\mu}; B\Sigma B^T)$$

D.V.8. $X_1, \dots, X_n$ nez. $X_i \sim N(\mu_i, \sigma^2)$

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x_i-\mu_i)^2}{\sigma^2}} = (2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu_i}{\sigma}\right)^2} \sim N_n(\boldsymbol{\mu}; \Sigma) \text{ kde } \overbrace{\Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}}^{\sigma^2 I_n}$$

$B^T = B^{-1}$; $\mathbf{Y} = B^T(\mathbf{X}-\boldsymbol{\mu}) = \underbrace{-B^T\boldsymbol{\mu}}_{a} + \underbrace{B^T}_{B}\mathbf{X} \sim N_n(\underbrace{-B^T\boldsymbol{\mu} + B^T\boldsymbol{\mu}}_{0}; B^T\Sigma B) = N_n(\mathbf{0}; \sigma^2 I_n)$

kde $B^T \Sigma B = B^T \cdot \sigma^2 \cdot I_n \cdot B = \sigma^2 \cdot B^T B = \sigma^2 \cdot I_n$

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} \sigma^{-n} \cdot e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{y_i^2}{\sigma^2}} = \prod_{i=1}^{n} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{y_i^2}{\sigma^2}}}_{f_{Y_i}(y_i)} \Rightarrow Y_i \sim N(0; \sigma^2)$$

$Y_1, \dots, Y_n$ nezávisé!