D. V. J. Uvaž. posl. $\{x_n\}_{n=1}^{\infty} \to \infty$; $x_1 < x_2 < \dots x_n < \dots$, $\lim\limits_{n\to\infty} x_n = \infty$

$$A_n = \{X \le x_n\} \cap \{Y \le y\}; \quad A_n \subseteq A_{n+1} \subseteq \dots \Rightarrow \exists \lim_{n\to\infty} A_n = A = \bigcup_{n=1}^{\infty} A_n$$

$$A = \bigcup_{n=1}^{\infty} \{X \le x_n\} \cap \{Y \le y\} = \Omega \cap \{Y \le y\} = \{Y \le y\}$$

$$P(A) = P(Y \le y) = F_Y(y)$$

$$F_Y(y) = P(Y \le y) = P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P(X \le x_n, Y \le y) = \lim_{n\to\infty} F(x_n, y)$$

D. V. S. • $(X,Y) \sim (M, p)$; $M = M_1 \times M_2$

$$P_X(x) = P(X = x) = P(X = x, Y \in M_2) = P\Big(\bigcup_{y \in M_2} \{X = x\} \cap \{Y = y\}\Big) = \sum_{y \in M_2} P(X = x, Y = y)$$

$$= \sum_{y \in M_2} p(x, y)$$

• $(X,Y) \sim f$; $F_X(x) = \lim\limits_{y\to\infty} F(x,y) = \lim\limits_{y\to\infty} \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$

$$= \int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f(u,v)\,dv}_{f_X(u)}\,du$$

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]} + \left(\rho\cdot\frac{x-\mu_1}{\sigma_1}\right)^2 - \left(\rho\cdot\frac{x-\mu_1}{\sigma_1}\right)^2$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left\{-\rho\cdot\frac{x-\mu_1}{\sigma_1} + \frac{y-\mu_2}{\sigma_2}\right\}^2\right]} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2(1-\rho^2)}\cdot(1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{-\rho\sigma_2 x + \rho\mu_1\sigma_2 + \sigma_1 y - \sigma_1\mu_2}{\sigma_1\sigma_2}\right\}^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left\{\frac{y-\mu_2+\rho\mu_1\frac{\sigma_2}{\sigma_1}-\rho x\frac{\sigma_2}{\sigma_1}}{\sigma_2\sqrt{1-\rho^2}}\right\}^2} dy$$

$$t = \frac{y-\mu_2+\rho\mu_1\frac{\sigma_2}{\sigma_1}-\rho x\frac{\sigma_2}{\sigma_1}}{\sigma_2\sqrt{1-\rho^2}}$$

$$dt = \frac{1}{\sigma_2\sqrt{1-\rho^2}}\,dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \cdot \underbrace{\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}}_{f_U \sim N(0,1)} dt}_{1} \sim N(\mu_1, \sigma_1^2)$$

$\forall (x,y,z) \in \mathbb{R}^3 \quad \underline{F(x,y,z) = F_X(x) \cdot F_Y(y) \cdot F_Z(z)} \iff X, Y, Z$ jsou nezávislé

"$\Leftarrow$" X, Y, Z nezáv. $\quad \forall (x,y,z) \in \mathbb{R}^3 \quad \underline{\{X \le x\}} \; \underline{\{Y \le y\}} \; \underline{\{Z \le z\}}$ jsou nezáv.

$$\underline{P(X \le x, Y \le y, Z \le z)} = \underline{P(X \le x) \cdot P(Y \le y) \cdot P(Z \le z)}$$

$$F(x,y,z) = F_X(x) \cdot F_Y(y) \cdot F_Z(y)$$

"$\Rightarrow$" $\forall (x,y) \in \mathbb{R}^2 \quad \cancel{P(x,y)} = P(X \le x, Y \le y) \overset{?}{=} P(X \le x) \cdot P(Y \le y)$

$$F^*(x,y) \overset{?}{=} F_X(x) \cdot F_Y(y)$$

$$\underline{F^*_{(X,Y)}(x,y)} = \lim_{z \to \infty} F(x,y,z) = \lim_{z \to \infty} F_X(x) \cdot F_Y(y) \cdot F_Z(z) = \underline{F_X(x) \cdot F_Y(y)}$$

D.V.8. $(n=3)$ • $(X,Y,Z) \sim (M, p)$; X, Y, Z nezávislé $\iff \underline{p(x,y,z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)} \quad \forall (x,y,z) \in \mathbb{R}^3$

"$\Rightarrow$" $p(x,y,z) = P(X = x, Y = y, Z = z) = P(X = x) \cdot P(Y = y) \cdot P(Z = z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)$

"$\Leftarrow$" $$F(x,y,z) = \sum_{\substack{t \le x, t \in M_1 \\ u \le y, u \in M_2 \\ v \le z, v \in M_3}} p(t,u,v) = \sum_{\%} p_X(t) \cdot p_Y(u) \cdot p_Z(z) = \underbrace{\sum_{t \le x, t \in M_1} p_X(t)}_{F_X(x)} \cdot \; \ldots \quad F_Y(y) \quad F_Z(z)$$

$(X, Y, Z) \sim f(x,y,z)$ $\quad$ X, Y, Z jsou nezávislé $\Leftrightarrow$ $f(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z) \quad \forall (x,y,z) \in \mathbb{R}^3$

"$\Rightarrow$" $\quad F(x,y,z) = F_X(x) \cdot F_Y(y) \cdot F_Z(z)$; $\quad f(x,y,z) = \dfrac{\partial^3 F(x,y,z)}{\partial x\, \partial y\, \partial z} = \dfrac{\partial^3}{\partial x \partial y \partial z} F_X(x) \cdot F_Y(y) \cdot F_Z(z)$

$$= f_X(x) \cdot f_Y(y) \cdot f_Z(z)$$

"$\Leftarrow$" $\quad f(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z)$

$$F(x,y,z) = \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{z} f(t,u,v)\, dv\, du\, dt = \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{z} f_X(t) \cdot f_Y(u) \cdot f_Z(v)\, dv\, du\, dt$$

$$= \underbrace{\int_{-\infty}^{x} f_X(t)\, dt}_{F_X(x)} \cdot \underbrace{\int_{-\infty}^{y} f_Y(u)\, du}_{F_Y(y)} \cdot \underbrace{\int_{-\infty}^{z} f_Z(v)\, dv}_{F_Z(z)}$$