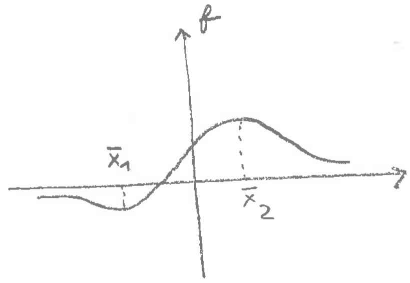
Odhad řešení úlohy $\qquad x' = \dfrac{t+x}{1+x^2}$, $x(0) = 0$



$f(t,x) = \dfrac{t+x}{1+x^2}$, t považujeme za parametr

$$\frac{df}{dx} = \frac{1+x^2-2(t+x)x}{(1+x^2)^2} = \frac{-x^2-2tx+1}{(1+x^2)^2}$$

stacionární body: $\bar{x}_{1,2} = \dfrac{2t \pm \sqrt{4t^2+4}}{-2} = -t \mp \sqrt{t^2+1}$

Extrémní hodnoty funkce f:

$$t + \bar{x}_{1,2} = \mp\sqrt{t^2+1}$$

$$(\bar{x}_{1,2})^2 = t^2 \pm t\sqrt{t^2+1} + t^2+1$$

$$1+(\bar{x}_{1,2})^2 = 2\left(t^2 \pm t\sqrt{t^2+1} + 1\right) = 2\sqrt{t^2+1}\left(\sqrt{t^2+1} \pm t\right)$$

$$f(\tau, \bar{x}_{1,2}) = \frac{\mp\sqrt{t^2+1}}{2\sqrt{t^2+1}\left(\sqrt{t^2+1} \pm t\right)} = \mp\frac{1}{2}\,\frac{\sqrt{t^2+1} \mp t}{t^2+1-t^2} = \frac{1}{2}\left(t \mp \sqrt{t^2+1}\right)$$

Tedy:

$$\frac{1}{2}\left(t - \sqrt{t^2+1}\right) \leq \frac{t+x}{1+x^2} \leq \frac{1}{2}\left(t + \sqrt{t^2+1}\right)$$

Maximální řešení x^*:

$$x^*(t) = \frac{1}{2}\int_0^t \left(\tau + \sqrt{\tau^2+1}\right) d\tau = \frac{1}{4}\left[\tau^2 + \tau\sqrt{\tau^2+1} + \ln\left(\tau + \sqrt{\tau^2+1}\right)\right]_{\tau=0}^{t} =$$

$$= \frac{1}{4}\left(t^2 + t\sqrt{t^2+1} + \ln\left(t + \sqrt{t^2+1}\right)\right)$$

Minimální řešení x_*:

$$x^*(t) = \frac{1}{2}\int_0^t \left(\tau - \sqrt{\tau^2+1}\right) d\tau = \frac{1}{4}\left(t^2 - t\sqrt{t^2+1} - \ln\left(t + \sqrt{t^2+1}\right)\right)$$