M6140 Topology Exercises - 2nd Week (2020)

1 Subspaces

Exercise 1. Let A be a subspace of a topological space X. Prove that closed sets in A are precisely the intersections of A with closed sets in X.

Exercise 2. Let A be a subspace of a topological space X and let $\mathcal{B}$ be a basis of X. Show that $\{U \cap A \mid U \in \mathcal{B}\}$ is a basis of A.

Exercise 3. Let X be a topological space. Prove that a subset U of X is open iff each point of U has an open neighborhood V such that $U \cap V$ is open in V.

Exercise 4. Let A be a subspace of a topological space X and let $\iota \colon A \to X$ be the inclusion. Show that a mapping $f \colon Y \to A$ is continuous iff the composition $\iota \circ f \colon Y \to X$ is continuous.

2 Products

Exercise 5. Consider a product $\prod_{i \in I} X_i$ of topological spaces. Prove that each projection map $p_j \colon \prod_{i \in I} X_i \to X_j$ is an *open map*, i.e. sends open sets to open sets.

Exercise 6. Consider a product $\prod_{i \in I} X_i$ of topological spaces. Prove that if for each $i \in I$ we have a closed set $C_i \subseteq X_i$, then $\prod_{i \in I} C_i$ is closed in $\prod_{i \in I} X_i$.

Exercise 7. Let A be a subspace of a topological space X and let B be a subspace of a topological space Y. Show that the product topology on $A \times B$ coincides with the subspace topology when viewed as a subspace of $X \times Y$.

Exercise 8. Consider the product $\mathbb{R}^{\mathbb{N}} = \prod_{\mathbb{N}} \mathbb{R}$ of sets $\mathbb{R}$. Give each component $\mathbb{R}$ the Euclidean topology. Consider the diagonal mapping $f \colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, $f(x) = (x, x, x, \dots)$.

(a) Prove that if $\mathbb{R}^{\mathbb{N}}$ has the product topology, then f is continuous.

(b) Prove that if $\mathbb{R}^{\mathbb{N}}$ has the box topology, then f is not continuous.

3 Disjoint Unions

Exercise 9. Show that a disjoint union of discrete spaces is discrete.

Exercise 10. Consider a disjoint union $\coprod_{i \in I} X_i$ of topological spaces. Show that each "injection" $\iota_j \colon X_j \to \coprod_{i \in I} X_i$ is both an open map and a closed map.

4 Quotient Spaces

Exercise 11. Consider the relation on $\mathbb{R}$ given by $x \sim y$ iff $x - y$ is an integer. Prove that this is an equivalence relation and that the quotient space $X/\sim$ is homeomorphic to the one-dimensional sphere.

Exercise 12. Let $\sim$ be an equivalence relation on a topological space X and let $p\colon X \to X/\sim$ be the projection. Show that a mapping $f\colon X/\sim \to Y$ is continuous iff the composition $f \circ p\colon X \to Y$ is continuous.