M6140 Topology Exercises - 5th Week (2020)

1 Topological Groups

Definition 1. A *topological group* $(G, \mathcal{G}, \cdot, 1, (-)^{-1})$ is a set G together with a topology $\mathcal{G}$ on G and a group structure $(\cdot, 1, (-)^{-1})$ on G such that the multiplication $\cdot \colon G \times G \to G$ and the inverse $(-)^{-1} \colon G \to G$ are continuous. A *morphism of topological groups* is a continuous group homomorphism.

Exercise 1. Find a topological group structure on $\mathbb{R}$.

Exercise 2. Find a topological group structure on $\mathbb{C} - \{0\}$.

Exercise 3. Prove that S^1 has a topological group structure.

Exercise 4. Show that $\mathrm{GL}(\mathbb{R}^n)$ has a topological group structure.

Exercise 5. Prove that each topological group G is *homogeneous*, i.e. for each pair of points $g, h \in G$ there is a homeomorphism $G \to G$ such that $g \mapsto h$.

Exercise 6. Show that a topological group is T_1 iff $\{1\}$ is a closed set.

Exercise 7. Prove that a set U in a topological group is open iff the set $U^{-1} := \{g \mid g^{-1} \in U\}$ is open.

Exercise 8. Suppose that K_1, K_2 are compact subspaces of a topological group. Show that the set $K_1 \cdot K_2 := \{g \cdot h \mid g \in K_1, h \in K_2\}$ is a compact subspace too.

Exercise 9. Show that an open subgroup of a topological group is clopen.

Exercise 10. Prove that a topological group is T_1 iff it is T_2.

Exercise 11. Prove that the closure of a subgroup of a topological group G is a subgroup of G.

Exercise 12. Show that in a topological group every neighborhood U of 1 contains an open neighborhood V of 1 such that $V \cdot V \subseteq U$ and $V = V^{-1}$.

Exercise 13. Prove that a topological group is T_1 iff it is $\mathrm{T}_{3\frac{1}{2}}$.