M6140 Topology Exercises - 6th Week (2020)

1 Homotopy

Definition 1. For topological spaces X, Y we denote by $[X, Y]$ the set of homotopy classes of continuous maps $X \to Y$.

Exercise 1. Prove that $[*, \mathbb{R}^n \setminus \{0\}]$, where $n > 1$, is a singleton.

Exercise 2. Show that $[*, \mathbb{R} \setminus \{0\}]$ is a two-element set.

Exercise 3. Show that $[X, I]$ is a singleton for each topological space X.

Exercise 4. Prove that if Y is path-connected, then $[I, Y]$ is a singleton.

Definition 2. A subset A of $\mathbb{R}^n$ is called *star-shaped* if there exists a point $a \in A$ such that for each $x \in A$ the line segment joining a with x is contained in A.

Exercise 5. Prove that $[X, A]$ is a singleton for each topological space X and each star-shaped subset A of $\mathbb{R}^n$.

Exercise 6. Suppose that A is a star-shaped subset of $\mathbb{R}^n$ and Y is a path-connected topological space. Show that $[A, Y]$ is a singleton.

Exercise 7. Show that each non-surjective continuous map $X \to S^n$, where X is a topological space, is *null-homotopic*, i.e. homotopic to a constant.

Exercise 8. Let X be a topological space and let f, g be continuous maps $X \to \mathbb{C} \setminus \{0\}$ such that for all $x \in X$: $|f(x) - g(x)| < |f(x)|$. Prove that f and g are homotopic.

Exercise 9. Let X be a topological space and let f, g be continuous maps $X \to S^n$ such that for all $x \in X$: $|f(x) - g(x)| < 2$. Show that f and g are homotopic.