MUNI

M7180 Funkcionální analýza II

M7180 Functional analysis II

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0.1 Preliminaries and examples

Let X, Y be linear spaces.

Definition 0.1. Any map $F\colon X \to Y$ satisfying

$$F(\lambda x + \mu y) = \lambda F(x) + \mu F(y)$$

for all $x, y \in X$ and scalars λ, μ is called a linear operator.

Remark 0.2. For operator F, we can write Fx instead of $F(x)$.

Example 0.3. The operator $I\colon X \to X$ given by $x \mapsto x$ is called the identical operator. Obviously, I is linear. If X is normed, then I is continuous (by the Heine definition of continuity: $x_n \to x \implies Fx_n \to Fx$).

Example 0.4. Let H be a Hilbert space and H_1 its (closed) subspace. We know that $H = H_1 \oplus H_1^{\perp}$, i.e., all $x \in H$ can be uniquely expressed as $x = x_1 + x_2$, where $x_1 \in H_1$, $x_2 \in H_1^{\perp}$. We put $Px = x_1$ for $x \in H$. The operator P is called the (orthogonal) projection. Evidently, it is linear and continuous.

Example 0.5. Let us consider the linear space $C[a,b]$ and define the operator

$$T\colon C[a,b] \to C[a,b]$$

by the formula

$$Tf(t) = \int_a^b k(t,s) f(s)\,\mathrm{d}s, \qquad f \in C[a,b],\ t \in [a,b],$$

with the so-called core $k \in C([a,b] \times [a,b])$. We can see that T is linear. If we consider the norm

$$\|f\| = \max_{t \in [a,b]} |f(t)|, \qquad f \in C[a,b], \tag{0.1}$$

i.e., the norm of the uniform convergence, then T is continuous.

Example 0.6. Let us consider the linear space $C^1[a,b]$ of functions with continuous derivatives on $[a,b]$ and define the operator

$$D\colon C^1[a,b] \to C[a,b], \qquad Df(t) = f'(t), \qquad f \in C^1[a,b],\ t \in [a,b].$$

This operator is called the differential operator and it is linear. In spaces $C^1[a,b], C[a,b]$, let us consider the norm from (0.1). We prove that D is not continuous: the sequence of

$$f_n(t) = \frac{\sin(nt)}{n}, \qquad n \in \mathbb{N},$$

satisfies $\|f_n\| \to 0$ as $n \to \infty$, but the sequence of

$$Df_n(t) = f_n'(t) = \cos(nt)$$

does not converge to 0.

0.2 Continuity and boundedness

Throughout this section, let

$$X = (X, \|\cdot\|) = (X, \|\cdot\|_X), Y = (Y, \|\cdot\|) = (Y, \|\cdot\|_Y)$$

be normed linear spaces.

Definition 0.7. An operator $L\colon X \to Y$ is called bounded if it maps any bounded set into a bounded set.

Theorem 0.8. *Any continuous linear operator $L\colon X \to Y$ is bounded.*

Proof. We assume the opposite. Let the operator L not be bounded. Then, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $\|x_n\|_X \leq c$ for all $n \in \mathbb{N}$ and some $c \in \mathbb{R}$ and

$$\|Lx_n\|_Y > n, \qquad n \in \mathbb{N}. \tag{0.2}$$

Since L is continuous, there exists $\delta \in (0,1)$ such that, for all $x \in X$ satisfying $\|x\|_X < \delta$, we have $\|Lx\|_Y \leq 1$. Let us choose $n_0 \in \mathbb{N}$ such that $c < \delta n_0$. The inequality

$$\frac{\|x_n\|_X}{n} < \delta$$

is valid for all $n \geq n_0$. Hence, we get $\|Lx_n\|_Y \leq n$ for all $n \geq n_0$, which is a contradiction with (0.2). □

Theorem 0.9. *Any bounded linear operator $L\colon X \to Y$ is continuous at $0 \in X$.*

Proof. Let us assume the opposite. Let the operator L not be continuous at $0 \in X$. Then, there exist $\varepsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that

$$\|x_n\|_X < \frac{1}{n}, \qquad \|Lx_n\|_Y \geq \varepsilon, \qquad n \in \mathbb{N}.$$

Let us denote $y_n = nx_n$ for $n \in \mathbb{N}$. The sequence $\{y_n\}_{n=1}^{\infty} \subseteq X$ is bounded, but the sequence $\{Ly_n\}_{n=1}^{\infty} \subseteq Y$ is not bounded, because $\|Ly_n\|_Y \geq n\varepsilon$, $n \in \mathbb{N}$, which is a contradiction. □

Theorem 0.10. *If a linear operator $L\colon X \to Y$ is continuous at $x_0 \in X$, then L is continuous at any point (vector) of X.*

Proof. For all $\varepsilon > 0$, there exists $\delta > 0$ such that, for $x \in X$ satisfying

$$\|x - x_0\| < \delta,$$

we have

$$\|Lx - Lx_0\| < \varepsilon.$$

Let $x_1 \in X$ be arbitrarily given and $y \in X$ satisfy $\|y - x_1\| < \delta$. Then,

$$\|(y - x_1 + x_0) - x_0\| < \delta,$$

and thus

$$\|L(y - x_1 + x_0) - Lx_0\| < \varepsilon,$$

i.e., $\|Ly - Lx_1\| < \varepsilon$, which proves that L is continuous at x_1. □

Remark 0.11. The definition of bounded linear operators can be reformulated as follows. A linear operator $L\colon X \to Y$ is called bounded if there exists $c > 0$ such that

$$\|Lx\|_Y \le c\|x\|_X, \qquad x \in X.$$

Definition 0.12. Let $L\colon X \to Y$ be a bounded linear operator. The number

$$\inf\{c \in \mathbb{R};\ \|Lx\|_Y \le c\|x\|_X \text{ for } x \in X\}$$

is denoted by $\|L\|$ and called the norm of L.

Theorem 0.13. *Let $L\colon X \to Y$ be a bounded linear operator. Then,*

$$\|L\| = \sup_{\|x\|_X \le 1} \|Lx\|_Y = \sup_{x \ne 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

Proof. We denote

$$\lambda = \sup\{\|Lx\|_Y;\, x \in X, \|x\|_X \le 1\}.$$

Firstly, we can see that

$$\sup_{\|x\|_X \le 1} \|Lx\|_Y = \sup_{x \ne 0} \frac{\|Lx\|_Y}{\|x\|_X},$$

where it suffices to consider the linearity of L and $\|x\|_X = 1$. Therefore,

$$\frac{\|Lx\|_Y}{\|x\|_X} \le \lambda, \qquad x \ne 0,$$

i.e., $\|Lx\|_Y \le \lambda\|x\|_X$ for all $x \in X$. Considering Definition 0.12, we have $\|L\| \le \lambda$.

Let $\varepsilon > 0$ be arbitrary. Then, there exists $x_\varepsilon \in X$ such that $x_\varepsilon \ne 0$ and

$$\lambda - \varepsilon \le \frac{\|Lx_\varepsilon\|_Y}{\|x_\varepsilon\|_X}.$$

However, from Definition 0.12, we obtain $\|Lx\|_Y \le \|L\| \cdot \|x\|_X$ for all $x \in X$. Thus, $\lambda - \varepsilon \le \|L\|$. The arbitrariness of ε gives $\lambda = \|L\|$. □

Remark 0.14. The set of all continuous linear operators $L\colon X \to Y$ is denoted by $\mathscr{L}(X,Y)$. For $L_1, L_2 \in \mathscr{L}(X,Y)$ and a scalar k, we put

$$(L_1 + L_2)(x) = L_1x + L_2x, \qquad x \in X,$$
$$(kL_1)(x) = kL_1(x), \qquad x \in X.$$

Evidently, $\mathscr{L}(X,Y)$ forms a linear space.

This space is normed (with respect to the norm of operators introduced above). Indeed, for the triangular inequality, it is sufficient to consider

$$\|(L_1 + L_2)x\|_Y \le \|L_1x\|_Y + \|L_2x\|_Y \le \|L_1\| + \|L_2\|$$

for all $L_1, L_2 \in \mathscr{L}(X,Y)$ and $\|x\|_X \le 1$, $x \in X$. Moreover, from Definition 0.12, we have the inequality

$$\|Lx\|_Y \le \|L\| \cdot \|x\|_X, \qquad x \in X, L \in \mathscr{L}(X,Y).$$

In the case when $X = Y$, we can write only $\mathscr{L}(X)$ (instead of $\mathscr{L}(X,X)$).

Remark 0.15. Recall that norms $\|-\|_1$ and $\|-\|_2$ on X are called equivalent if there exist $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1, \qquad x \in X.$$

Two norm are equivalent if and only if the topologies generated by them are same. For any space with a finite dimension, all norms are equivalent.

Theorem 0.16. *Let X have a finite dimension. Any linear operator $L\colon X \to Y$ is continuous.*

Proof. According to Remark 0.15, we can choose a norm of X. Let $e_1, \ldots, e_n$ be a base of X. For $x \in X, x = \lambda_1 e_1 + \cdots + \lambda_n e_n$, we put

$$\|x\|_X = |\lambda_1| + \cdots + |\lambda_n|.$$

It is seen that

$$\|Lx\|_Y = \|\lambda_1 Le_1 + \cdots + \lambda_n Le_n\|_Y \leq \max\{\|Le_1\|_Y, \ldots, \|Le_n\|_Y\} \cdot \|x\|_X.$$

□

Remark 0.17. A basic example of continuous linear operators is given by functionals f on X with the norm

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

For details, we refer to the course Functional Analysis I. We add:

1. If X has a finite dimension, then $\|f\|$ is realized. In addition, the norm of any continuous linear functional is realized on the closed unit ball with the center 0 if and only if the Banach space X is reflexive (the so-called *James characteristic*).
2. If any linear functional on X is bounded, then the dimension of X is finite.
3. The so-called *Bishop–Phelps theorem* says that the set of all continuous linear functionals on a Banach space X, whose norms are realized on the closed unit ball with the center 0, is a dense subset of the dual space X'.

Example 0.18. We mention a series of examples.

a) Let us consider the space $C[-1,1]$ with the norm

$$\|f\| = \max_{t \in [-1,1]} |f(t)|$$

and the functional

$$Lf = 9f(-1) - 2f(0) + f\left(\frac{1}{4}\right).$$

If $\|f\| \leq 1$, then

$$|Lf| \leq 9\|f\| + 2\|f\| + \|f\| \leq 12.$$

Especially, $\|L\| \leq 12$. On the contrary, let us consider a function $g \in C[-1,1]$ for which

$$\max_{t \in [-1,1]} |g(t)| = 1$$

and

$$g(-1) = 1, \qquad g(0) = -1, \qquad g\left(\frac{1}{4}\right) = 1.$$

We get $Lg = 12$. Therefore, $\|L\| = 12$.

b) For the space from a), we consider the functional

$$Lf = \int_{-1}^{1} \operatorname{sgn}(t) f(t)\,\mathrm{d}t.$$

Since

$$|Lf| \le \int_{-1}^{1} |f(t)|\,\mathrm{d}t \le \|f\| \int_{-1}^{1} \mathrm{d}t = 2\|f\|,$$

we have $\|L\| \le 2$. Let choose $\varepsilon \in (0,1)$ and put $g_\varepsilon(t) = t/\varepsilon$ for $t \in (-\varepsilon, \varepsilon)$ and $g_\varepsilon(t) = \operatorname{sgn}\, t$ for others t. Then, $\|g_\varepsilon\| = 1$ and $|Lg_\varepsilon| = 2 - \varepsilon$. Now, we see that $\|L\| = 2$.

c) For the space l^2, we consider $L\{x_n\}_{n=1}^\infty = x_1 + x_2$. For $x = \{x_n\}_{n=1}^\infty \in l^2$ satisfying

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots} \le 1,$$

we get

$$|Lx|^2 \le (|x_1| + |x_2|)^2 \le 2\left(|x_1|^2 + |x_2|^2\right) \le 2\left(|x_1|^2 + |x_2|^2 + \cdots\right) \le 2.$$

Therefore, $\|L\| \le \sqrt{2}$. For

$$x = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \dots \right\},$$

we have $\|x\| = 1$ and $|Lx| = \sqrt{2}$. Thus, $\|L\| = \sqrt{2}$.

We also know (from the Riesz theorem) that there exists $h \in l^2$ such that

$$Lx = \langle x, h\rangle, \qquad x \in l^2.$$

We see that $h = \{1, 1, 0, 0, \dots\}$. In addition, we know that $\|L\| = \|h\|$, which gives $\|L\| = \sqrt{1+1} = \sqrt{2}$. Our knowledge of the dual space gives a powerful tool to compute the norm of L.

d) We compute the norm of the functional

$$L\colon \{x_n\}_{n=1}^\infty \mapsto \sum_{n=1}^\infty \frac{x_n}{n}$$

in l^1 and l^2.

For $x = \{x_n\}_{n=1}^\infty \in l^1$, we have

$$|Lx| = \left| \sum_{n=1}^\infty \frac{x_n}{n} \right| \le \sum_{n=1}^\infty |x_n| = \|x\|.$$

Therefore, $\|L\| \le 1$. For $x = \{1, 0, 0, \dots\}$, we have $\|x\| = 1$, $|Lx| = 1$, consequently, $\|L\| = 1$.

We can also use our knowledge of the dual space. The dual space of l^1 is l^∞. There exists just one $\{a_n\}_{n=1}^\infty \in l^\infty$ satisfying

$$Lx = \sum_{n=1}^\infty a_n x_n, \qquad x = \{x_n\}_{n=1}^\infty \in l^1,$$

where $\|L\| = \|\{a_n\}_{n=1}^\infty\|_{l^\infty}$. Obviously, $a_n = 1/n$, $n \in \mathbb{N}$. We get

$$\|L\| = \left\| \left\{ \frac{1}{n} \right\}_{n=1}^\infty \right\|_{l^\infty} = 1.$$

Now, we consider L in l^2. Since $\{1/n\}_{n=1}^\infty \in l^2$, for $x = \{x_n\}_{n=1}^\infty \in l^2$, we have

$$|Lx| = \left| \sum_{n=1}^\infty \frac{x_n}{n} \right| \le \sqrt{\sum_{n=1}^\infty \frac{1}{n^2}} \cdot \sqrt{\sum_{n=1}^\infty |x_n|^2} = \frac{\pi}{\sqrt{6}} \cdot \|x\|.$$

Thus, $\|L\| \le \pi/\sqrt{6}$. For $y = \left\{ \sqrt{6}/(\pi n) \right\}_{n=1}^\infty \in l^2$, we have $\|y\| = 1$ and

$$|Ly| = \frac{\sqrt{6}}{\pi} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi}{\sqrt{6}},$$

which yields that $\|L\| = \pi/\sqrt{6}$.

e) Let us consider $X = C^1[a,b]$, $Y = C[a,b]$, and the operator (see Example 0.6)

$$D\colon X \to Y, \qquad Df(t) = f'(t), \qquad f \in C^1[a,b],\, t \in [a,b].$$

We consider the standard norm (see (0.1)) in Y and the norm

$$\|f\|_{C^1} = \max_{t\in[a,b]} |f(t)| + \max_{t\in[a,b]} \left|f'(t)\right|$$

in X. Since

$$\|Df\| = \max_{t\in[a,b]} |f'(t)| \le \max_{t\in[a,b]} |f(t)| + \max_{t\in[a,b]} |f'(t)| = \|f\|_{C^1}, \qquad f \in C^1[a,b],$$

the operator D is continuous and $\|D\| \le 1$. To prove $\|D\| = 1$, it is sufficient to consider

$$f_n(t) = \frac{\sin(nt)}{n}, \qquad n \in \mathbb{N},$$

for which

$$\|f_n\|_{C^1} = 1 + \frac{1}{n}, \qquad \|Df_n\| = 1,$$

where n is sufficiently large.

f) Let us consider the operators $L_1, L_2 \in \mathscr{L}\left(l^2\right)$ given by

$$L_1 x = \left\{0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots \right\}, \qquad x = \{x_n\}_{n=1}^\infty \in l^2,$$

$$L_2 x = \{0, x_1, x_2, x_3, \dots\}, \qquad x = \{x_n\}_{n=1}^\infty \in l^2.$$

Since the operator L_2 is isometric, we see that $\|L_2\| = 1$.

For L_1, we consider arbitrary $x = \{x_n\}_{n=1}^\infty \in l^2$. We have

$$\|L_1 x\|^2 = \sum_{n=1}^\infty \left| \frac{x_n}{n} \right|^2 \le \sum_{n=1}^\infty |x_n|^2 = \|x\|^2,$$

which gives $\|L_1\| \le 1$. For $y = \{1, 0, 0, \dots\}$, we have $\|y\| = 1$ and $\|L_1 y\| = 1$. Altogether, $\|L_1\| = 1$.

0.3 Inverse operator

In this section, let X, Y be linear spaces.

Definition 0.19. Let $L : X \to Y$ be an arbitrary operator. We put

$$R(L) = \{y \in Y; \text{there exists } x \in X : Lx = y\}.$$

We say that the operator L has an inverse if, for all $y \in R(L)$, there exists just one $x \in X$ such that $Lx = y$. In this case, the map $R(L) \to X$ given by $y \mapsto x$ is called the inverse operator of L and it is denoted by L^{-1}.

Theorem 0.20. *Let $L : X \to Y$ be linear and have an inverse. Then, $L^{-1} : Y \to X$ is linear as well.*

Proof. Note that the range $R(L)$ of the operator L, i.e., the domain $D\left(L^{-1}\right)$ of the inverse operator L^{-1}, is a linear space. Let $y_1, y_2 \in R(L)$. It suffices to prove the identity

$$L^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 L^{-1} y_1 + \alpha_2 L^{-1} y_2 \tag{0.3}$$

for all scalars α_1, α_2. Put $Lx_1 = y_1$, $Lx_2 = y_2$. We know that

$$L(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2. \tag{0.4}$$

According to Definition 0.19, we see that $L^{-1} y_1 = x_1$ and $L^{-1} y_2 = x_2$. Thus, we have

$$\alpha_1 L^{-1} y_1 + \alpha_2 L^{-1} y_2 = \alpha_1 x_1 + \alpha_2 x_2.$$

At the same time, from Definition 0.19 and (0.4), we get

$$\alpha_1 x_1 + \alpha_2 x_2 = L^{-1}(\alpha_1 y_1 + \alpha_2 y_2),$$

which gives (0.3). □

Lemma 0.21. *Let M be a dense subset of a Banach space Y. Any $y \neq 0$, $y \in Y$, can be expressed in the form*

$$y = \sum_{n=1}^{\infty} y_n, \qquad \text{i.e.,} \qquad y = \lim_{n \to \infty} (y_1 + y_2 + \cdots + y_n),$$

where $y_n \in M$ and

$$\|y_n\| \leq \frac{3\|y\|}{2^n}, \qquad n \in \mathbb{N}.$$

Proof. At first, we choose $y_1 \in M$ in such a way that the inequality

$$\|y - y_1\| \leq \frac{\|y\|}{2}$$

is valid. Next, we choose $y_2 \in M$ so that

$$\|y - y_1 - y_2\| \leq \frac{\|y\|}{4}.$$

In general, we choose $y_n \in M$ so that

$$\|y - y_1 - \cdots - y_n\| \leq \frac{\|y\|}{2^n}, \qquad n \in \mathbb{N}.$$

Such a choice is possible, because M is dense in Y.

Now,

$$\left\| y - \sum_{n=1}^{m} y_n \right\| \to 0 \qquad \text{as} \qquad m \to \infty,$$

i.e.,

$$y = \sum_{n=1}^{\infty} y_n,$$

and we have

$$\begin{aligned}
&\|y_1\| = \|y_1 - y + y\| \le \|y_1 - y\| + \|y\| \le \frac{3\|y\|}{2},\\
&\|y_2\| = \|y_2 + y_1 - y + y - y_1\| \le \|y - y_1 - y_2\| + \|y - y_1\| \le \frac{3\|y\|}{4},\\
&\quad\vdots\\
&\|y_n\| = \|y_n + y_{n-1} + \cdots + y_1 - y + y - y_1 - \cdots - y_{n-1}\|\\
&\quad \le \|y - y_1 - \cdots - y_n\| + \|y - y_1 - \cdots - y_{n-1}\| \le \frac{\|y\|}{2^n} + \frac{\|y\|}{2^{n-1}} = \frac{3\|y\|}{2^n}.
\end{aligned}$$

□

Theorem 0.22 (Banach)**.** *Let X, Y be Banach spaces and let $L : X \to Y$ be a bounded, bijective, and linear operator. Then, the inverse operator L^{-1} is bounded as well.*

Proof. In Y, we consider the sets $M_k \subseteq Y$, $k \in \mathbb{N}$, of all elements $y \in Y$ for which

$$\left\|L^{-1}y\right\| \le k\|y\|.$$

All element $y \in Y$ belongs to some M_k. Thus,

$$Y = \bigcup_{k=1}^{\infty} M_k.$$

According to the Baire theorem, at least one of M_k, say M_n, is dense in some ball B. Inside B, we consider the set P of all elements z such that $\beta < \|z - y_0\| < \alpha$, where $0 < \beta < \alpha$, $y_0 \in M_n$. We move the set P so that the center is 0, i.e., let us consider the set

$$P_0 = \{z \in Y; \beta < \|z\| < \alpha\}.$$

We prove that some M_N is dense in P_0. If $z \in P \cap M_n$, then $z - y_0 \in P_0$ and

$$\begin{aligned}
\left\|L^{-1}(z - y_0)\right\| &\le \left\|L^{-1}z\right\| + \left\|L^{-1}y_0\right\|\\
&\le n(\|z\| + \|y_0\|) \le n(\|z - y_0\| + 2\|y_0\|)\\
&= n\|z - y_0\|\left(1 + \frac{2\|y_0\|}{\|z - y_0\|}\right) \le n\|z - y_0\|\left(1 + \frac{2\|y_0\|}{\beta}\right).
\end{aligned} \tag{0.5}$$

The term

$$n\left(1 + \frac{2\|y_0\|}{\beta}\right)$$

does not depend on z. So, we put

$$N = 1 + \left\lfloor n\left(1 + \frac{2\|y_0\|}{\beta}\right)\right\rfloor.$$

From (0.5), we have $z - y_0 \in M_N$. Since M_n is dense in P, we have that M_N is dense in P_0.

Let $y \neq 0$, $y \in Y$, be arbitrary. We can choose such a number λ that

$$\beta < \|\lambda y\| < \alpha,$$

i.e., $\lambda y \in P_0$. Since M_N is dense in P_0, we can construct a sequence of $y_k \in M_N$ which converges to λy. Then, y_k/λ converges to y. It is obvious that $\{x_k/\mu\}_{k=1}^{\infty} \subseteq M_N$ for arbitrary $\mu \neq 0$ if $\{x_k\}_{k=1}^{\infty} \subseteq M_N$. Therefore, the set M_N is dense in $Y \setminus \{0\}$ and, consequently, it is dense in Y.

We consider arbitrary $y \neq 0$, $y \in Y$. According to Lemma 0.21, we can expand y into the series $y = y_1 + y_2 + \cdots$, where $y_k \in M_N$ and

$$\|y_k\| \leq \frac{3\|y\|}{2^k}, \qquad k \in \mathbb{N}.$$

In X, we consider the series $L^{-1}y_1 + L^{-1}y_2 + \cdots$, where we put $x_k = L^{-1}y_k$. This series converges to some $x \in X$, because

$$\|x_k\| = \left\|L^{-1}y_k\right\| \leq N\|y_k\| \leq N\frac{3\|y\|}{2^k}, \qquad k \in \mathbb{N},$$

$$\|x\| \leq \sum_{k=1}^{\infty} \|x_k\| \leq 3N\|y\| \sum_{k=1}^{\infty} \frac{1}{2^k} = 3N\|y\|.$$

Since the series of x_k is convergent and the operator L is continuous, we have

$$Lx = Lx_1 + Lx_2 + \cdots = y_1 + y_2 + \cdots = y.$$

Therefore, $x = L^{-1}y$. We also know that

$$\left\|L^{-1}y\right\| = \|x\| \leq 3N\|y\|,$$

where N does not depend on y. This estimation is valid for arbitrary $y \neq 0$. Thus, the operator L^{-1} is bounded. □

Remark 0.23. Let X, Y be Banach spaces. The symbol $\tilde{\mathscr{L}}(X,Y)$ denotes the set of all bijective, continuous, and linear operators $X \to Y$.

Theorem 0.24. *Let X, Y be Banach spaces. Let $L_0 \in \tilde{\mathscr{L}}(X,Y)$ and $L \in \mathscr{L}(X,Y)$, where*

$$\|L\| < \frac{1}{\left\|L_0^{-1}\right\|}.$$

Then, the bounded operator $(L_0 + L)^{-1}$ exists on Y, i.e., $L_1 = L_0 + L \in \tilde{\mathscr{L}}(X,Y)$.

Proof. We choose $y \in Y$ and consider the map $B\colon X \to X$, $Bx = L_0^{-1}y - L_0^{-1}(Lx)$. From $\|L\| < \|L_0^{-1}\|^{-1}$, it follows that B is a contraction. Indeed,

$$\|Bx_1 - Bx_2\| \leq \left\|L_0^{-1}\right\| \cdot \|L\| \cdot \|x_1 - x_2\|, \qquad x_1, x_2 \in X.$$

Since X is complete, according to the Banach theorem, there exists just one $x \in X$ such that

$$x = Bx = L_0^{-1}y - L_0^{-1}(Lx),$$

i.e.,

$$L_1 x = L_0 x + Lx = y.$$

If $L_1 \bar{x} = y$ for some $\bar{x} \in X$, then $\bar{x}$ is also a fixed point of B, and therefore $\bar{x} = x$. Thus, for all $y \in Y$, there exists only one solution of the equation $L_1 x = y$ in X, i.e., for L_1, there exists the inverse operator L_1^{-1} on Y. Considering Theorem 0.22, L_1^{-1} is bounded. □

Remark 0.25. According to the previous theorem, $\tilde{\mathscr{L}}(X,Y)$ forms an open subset in $\mathscr{L}(X,Y)$, where X, Y are Banach spaces.

Theorem 0.26 (Neumann). *Let X be a Banach space, let I be the identical operator on X, and let $L\colon X \to X$ be a bounded linear operator, where $\|L\| < 1$. Then, the operator $(I-L)^{-1}$ exists on X, it is bounded, and it can be expressed in the form*

$$(I-L)^{-1} = \sum_{k=0}^{\infty} L^k,$$

where

$$L^k = \underbrace{L \circ L \circ \cdots \circ L}_{k}.$$

Proof. The existence on X and the boundedness of $(I-L)^{-1}$ come from Theorem 0.24 (also from treatments below). Because of $\|L\| < 1$, we have

$$\sum_{k=0}^{\infty} \left\|L^k\right\| \le \sum_{k=0}^{\infty} \|L\|^k < \infty. \tag{0.6}$$

The space X is complete. Thus, considering (0.6), the treated infinite sum of L^k is a bounded linear operator. For arbitrary $n \in \mathbb{N} \cup \{0\}$, we have

$$(I-L)\sum_{k=0}^{n} L^k = \sum_{k=0}^{n} L^k (I-L) = I - L^{n+1}.$$

Taking into account to $\left\|L^{n+1}\right\| \le \|L\|^{n+1} \to 0$ as $n \to \infty$, we get

$$(I-L)\sum_{k=0}^{\infty} L^k = \sum_{k=0}^{\infty} L^k (I-L) = I,$$

which yields

$$(I-L)^{-1} = \sum_{k=0}^{\infty} L^k.$$

□

0.4 Adjoint operator

Throughout this section, let X,Y be normed linear spaces.

Definition 0.27. Let $L \in \mathscr{L}(X,Y)$ and let $g \in Y'$, i.e., let g be a continuous linear functional on Y. Let us consider the continuous linear functional $f = g \circ L \in X'$ and the map $g \in Y' \mapsto f \in X'$. This map $L' : Y' \to X'$ is called the adjoint operator of L.

Remark 0.28. We also denote $f(x) = \langle f,x\rangle$. Thus, we can write

$$\langle f,x\rangle = \langle g,Lx\rangle, \qquad \text{i.e.,} \qquad \langle L'g,x\rangle = \langle g,Lx\rangle.$$

Remark 0.29. Let $L,L_1,L_2 \in \mathscr{L}(X,Y)$ and let k be a scalar. Immediately, from Definition 0.27, we see:

1. L' is linear;
2. $(L_1+L_2)' = L_1' + L_2'$;
3. $(kL)' = kL'$;
4. L' is continuous.

Example 0.30. Let us consider a linear (continuous) operator $L\colon \mathbb{R}^n \to \mathbb{R}^m$ given by a matrix (l_{ij}). The map $y = Lx$ can be expressed as the system

$$y_i = \sum_{j=1}^{n} l_{ij}x_j, \qquad i \in \{1,\dots,m\},$$

and any functional $f\colon \mathbb{R}^n \to \mathbb{R}$ as

$$f(x) = \sum_{j=1}^{n} f_j x_j,$$

where $f_j = f(e_j)$ for the standard base $e_1,\dots,e_n$ of $\mathbb{R}^n$. From

$$f(x) = g(Lx) = \sum_{i=1}^{m} g_i y_i = \sum_{i=1}^{m}\sum_{j=1}^{n} g_i l_{ij} x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} g_i l_{ij},$$

we obtain

$$f_j = f(e_j) = \sum_{i=1}^{m} g_i l_{ij}, \qquad j \in \{1,\dots,n\}.$$

Since $f = L'g$, the operator L' is given by the transpose matrix.

Theorem 0.31. *If $L \in \mathscr{L}(X,Y)$, then*

$$\|L\| = \left\|L'\right\|.$$

Proof. Obviously, it holds

$$\left|\langle L'g,x\rangle\right| = |\langle g,Lx\rangle| \le \|g\|\cdot\|Lx\| \le \|g\|\cdot\|L\|\cdot\|x\|$$

for all $x \in X$ and $g \in Y'$. Thus,

$$\left\|L'g\right\| \le \|g\|\cdot\|L\|, \qquad \text{i.e.,} \qquad \left\|L'\right\| \le \|L\|.$$

Now, we prove the opposite inequality. Let $x_0 \in X$, $Lx_0 \neq 0$. We put

$$y_0 = \frac{Lx_0}{\|Lx_0\|} \in Y.$$

It is seen that $\|y_0\| = 1$. Due to a well-known corollary of the Hahn–Banach theorem, there exists a functional g such that $\|g\| = 1$ and $\langle g, y_0\rangle = 1$, i.e.,

$$\langle g, Lx_0\rangle = \|Lx_0\|.$$

From

$$\begin{aligned}\|Lx_0\| = \langle g, Lx_0\rangle = |\langle L'g, x_0\rangle| &\leq \left\|L'g\right\| \cdot \|x_0\| \\ &\leq \left\|L'\right\| \cdot \|g\| \cdot \|x_0\| = \left\|L'\right\| \cdot \|x_0\|,\end{aligned}$$

we get that $\|L\| \leq \left\|L'\right\|$. □

Let H be a Hilbert space and let $L : H \to H$ be a bounded linear operator. We know that there exists a map τ which assigns to any element $y \in H$ the continuous linear functional $(\tau y)(x) = \langle x, y\rangle \in H'$. Moreover, this map is an isometry. For the operator L', we consider the map $\tilde{L}' = \tau^{-1}L'\tau$, which is a bounded linear operator on H. One can easily show that

$$\langle Lx, y\rangle = \left\langle x, \tilde{L}'y\right\rangle, \qquad x, y \in H.$$

Since $\left\|L'\right\| = \|L\|$ and the maps τ and τ^{-1} are isometries, we have the identity

$$\left\|\tilde{L}'\right\| = \|L\|.$$

Definition 0.32. In a Hilbert space H, the above mentioned operator $\tilde{L}' \colon H \to H$ is called the adjoint operator of $L \colon H \to H$.

Remark 0.33. It should be emphasized that Definition 0.32 differs from Definition 0.27. For a general Banach space X and a bounded linear operator $L \colon X \to X$, the adjoint operator of L is defined on X'.

The operator $\tilde{L}'$ is sometimes called the Hermitian adjoint. We write only L' (instead of $\tilde{L}'$) and speak about the adjoint operator of L. It should be remembered that, in Hilbert spaces, the concept of adjoint operators differs from the one in general Banach spaces. For H, it is seen that the adjoint operator of a bounded linear operator $L \colon H \to H$ can be defined as the operator $L' \colon H \to H$ which satisfies

$$\langle Lx, y\rangle = \left\langle x, L'y\right\rangle, \qquad x, y \in H.$$

Definition 0.34. Let H be a Hilbert space. A bounded linear operator $L \colon H \to H$ is called self-adjoint if

$$\langle Lx, y\rangle = \langle x, Ly\rangle, \qquad x, y \in H.$$

Definition 0.35. Let H be a Hilbert space and let $L \colon H \to H$ be a linear operator. A (closed) subspace H_1 of H is called invariant with respect to $L \colon H \to H$ if $x \in H_1$ implies $Lx \in H_1$.

Remark 0.36. Let H be a Hilbert space and let $L \colon H \to H$ be a bounded linear operator. If H_1 is a (closed) subspace of H, which is invariant with respect to L, then its orthogonal complement $H_1^\perp$ is invariant with respect to L'. Indeed, if $y \in H_1^\perp$, then

$$\left\langle x, L'y\right\rangle = \langle Lx, y\rangle = 0, \qquad x \in H_1,$$

because $Lx \in H_1$. Especially, if L is self-adjoint, then the orthogonal complement of any invariant subspace is invariant with respect to L as well.

0.5 Spectrum of operator

Let $L\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of L if $Lx = \lambda x$ for some non-zero $x \in \mathbb{C}^n$. Any such solution x is called an eigenvector of L. The set of all eigenvalues is called the spectrum of the operator L and all other values λ are called regular; i.e., λ is a regular value if the operator $L - \lambda I$ has an inverse. In this case, the operator $(L - \lambda I)^{-1}$ is defined on the entire space $\mathbb{C}^n$ and (as well as any linear operator on a space with a finite dimension) it is bounded (continuous).

In the space with a finite dimension, there are two possibilities:

1. the equation $Lx = \lambda x$ has a non-zero solution, i.e., λ is an eigenvalue of L—the operator $(L - \lambda I)^{-1}$ does not exist;

2. the bounded operator $(L - \lambda I)^{-1}$ is defined in the whole space, i.e., λ is a regular value.

If the operator L is defined on a space whose dimension is infinite, then we have the third possibility:

3. the operator $(L - \lambda I)^{-1}$ exists, i.e., the equation $Lx = \lambda x$ has only the zero solution, but this operator is not defined on the whole space (and it is not necessarily bounded).

Let X be a complex Banach space.

Definition 0.37. A number $\lambda \in \mathbb{C}$ is called a regular value of a bounded linear operator $L\colon X \to X$ if the operator $R_\lambda = (L - \lambda I)^{-1}$, called the resolvent of L, is defined in the whole space X. The set of all non-regular values is called the spectrum of L and it is denoted by $\sigma(L)$. The spectrum includes all eigenvalues of the operator L. Indeed, if $(L - \lambda I)x = 0$ for some $x \neq 0$, then $(L - \lambda I)^{-1}$ does not exist. The set of all eigenvalues of L is called the point (or discrete) spectrum and the corresponding $x \neq 0$ are called eigenvectors. The remaining part of the spectrum, i.e., the set of all λ, for which the inverse operator $(L - \lambda I)^{-1}$ exists, but is not defined in the whole space X, is called the continuous spectrum.

Theorem 0.38. *The set of all regular values of a bounded linear operator $L\colon X \to X$ is open, i.e., the spectrum is a closed set.*

Proof. Let λ be a regular value of L. Then,

$$L - \lambda I \in \tilde{\mathscr{L}}(X) = \tilde{\mathscr{L}}(X, X).$$

Let $\delta \in \mathbb{C}$ satisfy

$$|\delta| < \frac{1}{||\,(L - \lambda I)^{-1}\,||}.$$

From Theorem 0.24, we have $L - (\lambda + \delta)I \in \tilde{\mathscr{L}}(X)$. Hence, $\lambda + \delta$ is a regular value of L and the set of all regular values is open. □

Theorem 0.39. *Let $L\colon X \to X$ be a bounded linear operator and let $|\lambda| > \|L\|$. Then, λ is a regular value of L.*

Proof. Because of $\|L\| < |\lambda|$, applying Theorem 0.26, we know that the operator

$$R_\lambda = (L - \lambda I)^{-1} = -\frac{1}{\lambda}\left(I - \frac{L}{\lambda}\right)^{-1}$$

exists on X and it is bounded. Therefore, λ is a regular value. □

Remark 0.40. Theorem 0.39 can be specified as follows. Let

$$r = \lim_{n\to\infty} \sqrt[n]{\|L^n\|},$$

where it is possible to show that this limit exists for any bounded linear operator $L\colon X \to X$. The spectrum of L is in the closed circle with the radius r and the centre in 0. The value r is called the spectral radius of L and $r \le \|L\|$.

For example, the operator $L\colon C[0,1] \to C[0,1]$ (we consider the norm from (0.1)) given by

$$L\colon f(t) \mapsto t\int_0^1 f(x)\,\mathrm{d}x, \qquad t \in [0,1],\, f \in C[0,1],$$

has the norm $\|L\| = 1$ and

$$r = \lim_{n\to\infty} \sqrt[n]{||L^n||} = \frac{1}{2}.$$

We add that, for any bounded linear operator $L\colon X \to X$, it holds

$$r = \sup\{|\lambda|;\, \lambda \in \sigma(L)\}.$$

Example 0.41. We define the operator $T\colon C[0,1] \to C[0,1]$ by the formula

$$T\colon f(t) \mapsto f\left(t^2\right), \qquad t \in [0,1],\, f \in C[0,1],$$

where we consider the norm from (0.1). At first, we determine $\|T\|$, which is easy. We have

$$\|T\| = \sup\{\|Tf\|;\, \|f\| \le 1\} \le 1.$$

For $f_0 \equiv 1$, we obtain $\|f_0\| = 1$ and also $\|Tf_0\| = 1$. Thus, $\|T\| = 1$.

We find eigenvalues of T. We find such λ that the equation $Tf = \lambda f$ has non-zero solutions. We know that

$$\sigma(T) \subseteq \{z \in \mathbb{C};\, |z| \le \|T\| = 1\}.$$

For $\lambda = 0$ and $Tf = \lambda f = 0$, we have $f \equiv 0$ and, therefore, 0 is not an eigenvalue. For $\lambda = 1$ and $Tf = \lambda f = f$, all constant functions are solutions of the equation

$$f\left(t^2\right) = f(t), \qquad t \in [0,1].$$

It remains to investigate such λ that $|\lambda| \in (0,1], \lambda \ne 1$. From

$$Tf(t) = f\left(t^2\right) = \lambda f(t), \qquad t \in [0,1],$$

we obtain

$$f(t) = \frac{1}{\lambda} f\left(t^2\right) = \frac{1}{\lambda^2} f\left(t^4\right) = \cdots = \frac{1}{\lambda^n} f\left(t^{2^n}\right)$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. We see that

$$f(t) = \lambda^n f\left(t^{2^{-n}}\right).$$

If $|\lambda| < 1$, then $f \equiv 0$, because $\lambda^n \to 0$ as $n \to \infty$ and f is bounded on $[0,1]$.

Let us consider the last case when $|\lambda| = 1$ and $\lambda \neq 1$. In this case, we get

$$f(0) = \lambda f(0), \qquad f(1) = \lambda f(1)$$

and, consequently, $f(0) = f(1) = 0$. Let $t \in (0,1)$. From

$$f(t) = \frac{1}{\lambda^n} f\left(t^{2^n}\right)$$

and $t^{2^n} \to 0$ as $n \to \infty$, it follows that $f \equiv 0$. Therefore, the set of all eigenvalues is $\{1\}$.

It remains to find out other values belonging to the spectrum of T. Thus, we analyse solutions of the equation $Tf - \lambda f = g$ in $C[0,1]$. We know, when the operator $T - \lambda I$ is not injective. So, we are interested in the case, when it is not surjective. For $t \in [0,1]$, we obtain

$$f\left(t^2\right) = g(t) + \lambda f(t)$$

and

$$f(t) = g\left(t^{1/2}\right) + \lambda f\left(t^{1/2}\right) = g\left(t^{1/2}\right) + \lambda g\left(t^{1/4}\right) + \lambda^2 f\left(t^{1/4}\right).$$

By induction, one can obtain

$$f(t) = \sum_{j=0}^{n-1} \lambda^j g\left(t^{2^{-1-j}}\right) + \lambda^n f\left(t^{2^{-n}}\right), \qquad t \in [0,1].$$

Because of the boundedness of g, for $|\lambda| < 1$, the series above converges and

$$\lambda^n f\left(t^{2^{-n}}\right) \to 0 \qquad \text{as} \qquad n \to \infty$$

for all f. Therefore, the continuous function

$$f(t) = \sum_{j=0}^{\infty} \lambda^j g\left(t^{2^{-1-j}}\right), \qquad t \in [0,1],$$

is a solution of the equation $Tf - \lambda f = g$. If $|\lambda| = 1, \lambda \neq 1$, then the equation

$$Tf - \lambda f = g$$

has no solution for some $g \in C[0,1]$. For example, for $\lambda = -1$, there exists a continuous function $g \in C[0,1]$ such that

$$g\left(t^{2^{-1-j}}\right) = \frac{(-1)^j}{j}$$

for given $t \in (0,1), j \in \mathbb{N}$.

Altogether, we have

$$\sigma(T) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}.$$

Example 0.42. Now, on the space $C[0,1]$ with the norm from (0.1), we consider the operator

$$\tilde{T}\colon f(t) \mapsto \int_0^t f(x)\,\mathrm{d}x, \qquad t \in [0,1],\, f \in C[0,1].$$

Its norm is $\|\tilde{T}\| = 1$.

Let us identify the point spectrum of the operator $\tilde{T}$. We know that λ is an eigenvalue if there exists a non-zero solution of the equation $\tilde{T}f = \lambda f$. So, we are looking for a non-trivial continuous function $f \in C[0,1]$ with the property that

$$\int_0^t f(x)\,\mathrm{d}x = \lambda f(t), \qquad t \in [0,1].$$

We see that $\lambda f(0) = 0$ and that the function f has to have a continuous derivative if $\lambda \neq 0$. If $\lambda = 0$, then

$$\int_0^t f(x)\,\mathrm{d}x = 0, \qquad t \in [0,1], \qquad \text{i.e.,} \qquad f \equiv 0.$$

Therefore, 0 is not an eigenvalue of $\tilde{T}$. If $\lambda \neq 0$, from the mentioned identity, we obtain

$$f(t) = \lambda f'(t), \qquad t \in [0,1].$$

Solutions of this equation are functions $f(t) = K\mathrm{e}^{t/\lambda}$. Since $f(0) = 0$, we get $K = 0$. Thus, the operator $\tilde{T}$ has no eigenvalues.

Now, we determine the whole spectrum of $\tilde{T}$. We need to find the values λ for which the considered operator is surjective, i.e., we need to determine when the equation $\tilde{T}f - \lambda f = g$ has solutions for all $g \in C[0,1]$. Let $\lambda = 0$. Since $\tilde{T}f(0) = 0$, for a function g such that $g(0) \neq 0$, any solution does not exist. Hence, $0 \in \sigma(\tilde{T})$. In the case when $\lambda \neq 0$, we are looking for solutions of the equation

$$\int_0^t f(x)\,\mathrm{d}x - \lambda f(t) = g(t), \qquad t \in [0,1],$$

where $g \in C[0,1]$ is a given function. Let h be the appropriate primitive function of f. The aim is to solve the differential equation $h - \lambda h' = g$. Of course, this equation has solutions. Thus, we get

$$\sigma\left(\tilde{T}\right) = \{0\}.$$

Example 0.43. On the space l^∞, we consider the operator

$$R\colon \{x_1, x_2, x_3, \dots\} \mapsto \{x_2, x_3, \dots\}.$$

It is easy to verify that $R \in \mathscr{L}(l^\infty)$ and that $\|R\| = 1$. We know that

$$\sigma(R) \subseteq \{\lambda \in \mathbb{C};\, |\lambda| \leq 1\}.$$

If $|\lambda| \leq 1$, the equation $Rx = \lambda x$ has the non-trivial solution

$$x_\lambda = \left\{1, \lambda, \lambda^2, \dots\right\}.$$

Since $x_\lambda \in l^\infty$ for $|\lambda| \leq 1$, we have $\sigma(R) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$.

Let us consider the same operator R, but on l^1. Of course, $R \in \mathscr{L}\left(l^1\right)$ and $\|R\| = 1$. The equation $Rx = \lambda x$ has (again) the non-trivial solution x_λ. But, for $|\lambda| = 1$, this element is not in l^1. However, $x_\lambda \in l^1$ for all λ satisfying $|\lambda| < 1$. Therefore, we get that all such λ are in the point spectrum. Since the spectrum $\sigma(R)$ is a closed set which contains the point spectrum, we have again

$$\sigma(R) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}.$$

1. Completely continuous operators

1.1 Preliminaries and examples

Throughout this section, let X, Y be Banach spaces.

Definition 1.1. An operator $L\colon X \to Y$ is called completely continuous if it maps any bounded set into a precompact set.

Remark 1.2. If X has a finite dimension, then any linear operator $L\colon X \to Y$ is completely continuous (see also Theorem 0.16). For spaces whose dimension is infinite, the complete continuity differs from the continuity.

Theorem 1.3. *Let $x_1, x_2, \dots$ be linearly independent vectors in X a let X_n be the subspace of X generated by $x_1, \dots, x_n$. Then, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that*

$$\|y_n\| = 1, \qquad y_n \in X_n, \qquad n \in \mathbb{N},$$

and

$$\inf_{x \in X_{n-1}} \|y_n - x\| > \frac{1}{2}, \qquad n \geq 2, n \in \mathbb{N}.$$

Proof. Since $x_1, x_2, \dots$ are linearly independent, $x_n \notin X_{n-1}$ and the distance between x_n and X_{n-1} is positive. Let us denote it by α and let x^* be an element of X_{n-1} for which $\|x_n - x^*\| < 2\alpha$. Then,

$$y_n = \frac{x_n - x^*}{\|x_n - x^*\|}, \qquad n \geq 2, n \in \mathbb{N},$$

because

$$\alpha = \inf_{x \in X_{n-1}} \|x_n - x\| = \inf_{x \in X_{n-1}} \big\| x_n - x^* - x \|x_n - x^*\| \big\|.$$

We add that we can easily put

$$y_1 = \frac{x_1}{\|x_1\|}.$$

□

Example 1.4. Let the dimension of X be infinite and let us consider the identical operator I on X. Using Theorem 1.3, in $B[0,1]$, one can construct a sequence $\{y_n\}_{n=1}^{\infty}$ such that

$$\|y_i - y_n\| > \frac{1}{2}, \qquad i \in \{1, 2, \dots, n-1\}, n \geq 2, n \in \mathbb{N}.$$

Obviously, such a sequence cannot have a convergent subsequence. Therefore, $B[0,1]$ is not (pre)compact and I is not completely continuous.

Example 1.5. Let L be a continuous linear operator which maps X into a subspace of X with a finite dimension. The operator L is evidently completely continuous. Especially, in a Hilbert space, the (orthogonal) projection is completely continuous if and only if the considered subspace has a finite dimension. Note that an operator, which maps X into a subspace of X having a finite dimension, is called degenerated.

Example 1.6. In the space l^2, we consider the operator $L\colon l^2 \to l^2$ given by

$$x = \{x_1, x_2, \ldots, x_n, \ldots\} \mapsto Lx = \left\{x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{2^{n-1}}, \ldots\right\}.$$

This operator is completely continuous. It suffices to consider that the image of the unit ball is precompact and to use the linearity.

Example 1.7. We consider $C[a,b]$ with the norm

$$\|f\| = \max_{t\in[a,b]} |f(t)|, \qquad f \in C[a,b],$$

and the operator $L\colon C[a,b] \to C[a,b]$ defined by

$$Lx = y(s) = \int_a^b k(s,t)x(t)\,\mathrm{d}t, \qquad x \in C[a,b],\, s \in [a,b]. \tag{1.1}$$

It is possible to prove the following implication. If k is bounded for $s \in [a,b]$, $t \in [a,b]$ and all points of the discontinuity of k are on finitely many curves $t = \varphi_k(s)$, $k = \{1,\ldots,n\}$, where φ_k are continuous functions on $[a,b]$, then the operator L given by (1.1) is completely continuous. We remark that this operator L is called the Fredholm operator.

The requirement, that the points of the discontinuity of k are only on finitely many curves which intersect the lines $s =$ const. in only one point, is essential. For example, for the function

$$k(s,t) = \begin{cases} 1, & s < \dfrac{1}{2}; \\ 0, & s \geq \dfrac{1}{2}, \end{cases}$$

the operator L maps $x \equiv 1$ into a discontinuous function.

We prove the complete continuity of the operator L only in the case, when the function k is continuous on $[a,b] \times [a,b]$. It is easy to see that Lx is defined correctly, $Lx \in C[a,b]$, and that L is a linear and bounded operator (see also Example 0.5). We consider $B[0,1] \subseteq C[a,b]$. It suffices to show that the set $L(B[0,1])$ is precompact. We apply the Arzelà–Ascoli theorem. Of course, $L(B[0,1])$ is a bounded set, because L is a bounded operator. It remains to show that $L(B[0,1])$ is a set of equicontinuous functions. For an arbitrarily given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|k(s_1,t) - k(s_2,t)| < \varepsilon$$

if

$$t \in [a,b], \qquad |s_1 - s_2| < \delta,\, s_1, s_2 \in [a,b].$$

Then,

$$|Lx(s_1) - Lx(s_2)| \leq (b-a) \max_{t\in[a,b]} |k(s_1,t) - k(s_2,t)| \cdot \|x\| \leq \varepsilon(b-a)$$

for all $x \in B[0,1]$ and $s_1, s_2 \in [a,b]$ satisfying $|s_1 - s_2| < \delta$.

If we put $k(s,t) = 0$ for $t > s$, then L takes the form

$$Lx = y(s) = \int_a^s k(s,t)x(t)\,\mathrm{d}t, \qquad x \in C[a,b],\, s \in [a,b]. \tag{1.2}$$

If the function k is continuous, then the operator defined in (1.2) is completely continuous. This operator is called the Volterra operator.

Remark 1.8. For a completely continuous operator, the image of the closed unit ball $B[0,1]$ does not need to be compact, although it is precompact. As at the end of Example 1.7, on $C[-1,1]$ with the uniform norm, we consider the completely continuous operator

$$Jx(s) = \int_{-1}^{s} x(t)\,\mathrm{d}t, \qquad s \in [-1,1], x \in C[-1,1].$$

For $n \in \mathbb{N}$, we put

$$x_n(t) = \begin{cases} 0, & -1 \le t \le 0; \\ nt, & 0 < t \le \dfrac{1}{n}; \\ 1, & \dfrac{1}{n} < t \le 1. \end{cases}$$

For $n \in \mathbb{N}$, it is seen that $x_n \in C[-1,1]$, $\|x_n\| = 1$, and that

$$y_n(s) = Jx_n(s) = \begin{cases} 0, & -1 \le s \le 0; \\ \dfrac{ns^2}{2}, & 0 < s \le \dfrac{1}{n}; \\ s - \dfrac{1}{2n}, & \dfrac{1}{n} < s \le 1. \end{cases}$$

We immediately see that, in $C[-1,1]$, $\{y_n\}_{n=1}^{\infty}$ converges to

$$y(s) = \begin{cases} 0, & -1 \le s \le 0; \\ s, & 0 < s \le 1. \end{cases}$$

But, for the operator J, the function y is not the image of any function from $C[-1,1]$, because y' is not continuous. However, it is possible to prove that, for any completely continuous linear operator, the image of $B[0,1]$ is compact if the considered space is reflexive.

1.2 Basic properties

Throughout this section, let X be a Banach space.

Theorem 1.9. *If $\{L_n\}_{n=1}^{\infty}$ is a sequence of linear completely continuous operators on X, which converges to an operator $L\colon X \to X$, i.e., $\|L_n - L\| \to 0$ as $n \to \infty$, then L is completely continuous as well.*

Proof. It is sufficient to prove that, for an arbitrarily given bounded sequence $\{x_k\}_{k=1}^{\infty} \subset X$, one can extract a convergent subsequence from $\{Lx_k\}_{k=1}^{\infty}$.

Since the operator L_1 is completely continuous, $\{L_1 x_k\}_{k=1}^{\infty}$ has a convergent subsequence. Let $\{x_k^1\}_{k=1}^{\infty}$ be a subsequence of $\{x_k\}_{k=1}^{\infty}$ such that $\{L_1 x_k^1\}_{k=1}^{\infty}$ converges. Now, let us consider $\{L_2 x_k^1\}_{k=1}^{\infty}$. From this sequence, we can extract a convergent subsequence as well. Let $\{x_k^2\}_{k=1}^{\infty}$ be a subsequence of $\{x_k^1\}_{k=1}^{\infty}$ such that $\{L_2 x_k^2\}_{k=1}^{\infty}$ converges. We proceed in the same way. An the end, we consider the diagonal sequence $\left\{x_k^k\right\}_{k=1}^{\infty}$. Any of the operators $L_1, L_2, L_3, \ldots$ transforms this sequence into a convergent one. We show that L also transforms it into a convergent sequence. Since X is complete, it suffices to show that $\left\{Lx_k^k\right\}_{k=1}^{\infty}$ satisfies the Cauchy criterion. For $n,k,l \in \mathbb{N}$, it holds

$$\left\|Lx_k^k - Lx_l^l\right\| \leq \left\|Lx_k^k - L_n x_k^k\right\| + \left\|L_n x_k^k - L_n x_l^l\right\| + \left\|L_n x_l^l - Lx_l^l\right\|. \tag{1.3}$$

Let $\varepsilon > 0$ be given and $c > 0$ be such that $\|x_k\| \leq c$, $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that

$$\|L - L_n\| < \frac{\varepsilon}{3c}.$$

Then, we consider so large $N \in \mathbb{N}$ that

$$\left\|L_n x_k^k - L_n x_l^l\right\| < \frac{\varepsilon}{3}$$

for all $k,l > N$, $k,l \in \mathbb{N}$. Now, from (1.3), it follows that

$$\left\|Lx_k^k - Lx_l^l\right\| < \varepsilon$$

for all sufficiently large k,l. □

Remark 1.10. Since any linear combination of completely continuous operators is a completely continuous operator, from Theorem 1.9, we know that completely continuous linear operators form a closed subspace of $\mathscr{L}(X)$.

Theorem 1.11. *Let $L_1, L_2 \in \mathscr{L}(X)$ and let L_1 be completely continuous. Then, the operators $L_1 \circ L_2$ and $L_2 \circ L_1$ are completely continuous as well.*

Proof. If a set $M \subseteq X$ is bounded, then

$$L_2(M) = \{y \in X;\, y = L_2 x, x \in M\}$$

is bounded as well. Thus, the set $L_1(L_2(M))$ is precompact and $L_1 \circ L_2$ is completely continuous. If $M \subseteq X$ is bounded, then $L_1(M)$ is precompact. Since L_2 is continuous, $L_2(L_1(M))$ is precompact and $L_2 \circ L_1$ is completely continuous. □

Corollary 1.12. *In the space X whose dimension is infinite, any linear completely continuous operator $L\colon X \to X$ does not have a bounded inverse L^{-1}.*

Proof. It is enough to consider Theorem 1.11 and the identical operator $I = L \circ L^{-1}$ (see Example 1.4). □

Theorem 1.13 (Schauder)**.** *The adjoint operator of a completely continuous operator $L \in \mathscr{L}(X)$ is completely continuous as well.*

Proof. We want to prove that $L'\colon X' \to X'$ maps any ball into a precompact set. Due to the linearity of L', it suffices to show that the image $L'\left(B'\right)$ of the closed unit ball with the centre in $0 \in X'$ is precompact. We point out that B' is the unit ball in X'. Elements of X' can be considered as functions defined on $\overline{L(B[0,1])}$. We show that the set Φ of all functions assigned to the functionals belonging to the ball B' is a set of uniformly bounded and equicontinuous functions. If a functional $\varphi \in X'$ satisfies $\|\varphi\| \leq 1$, then

$$\sup_{x \in \overline{L(B[0,1])}} |\varphi(x)| = \sup_{x \in L(B[0,1])} |\varphi(x)| \leq \|\varphi\| \sup_{x \in B[0,1]} \|Lx\| \leq \|L\|$$

and

$$\left|\varphi(x')-\varphi(x'')\right| \le \|\varphi\| \cdot \left\|x'-x''\right\| \le \left\|x'-x''\right\|.$$

Thus (consider the Arzelà–Ascoli theorem), the set Φ is precompact in

$$U = C\left[\overline{L(B[0,1])}\right].$$

But, the set Φ with the metric of the uniform convergence is isometric with the set $L'\left(B'\right)$ with the metric given by the norm of X', because, for $g_1, g_2 \in B'$, we have

$$\begin{aligned}\left\|L'g_1 - L'g_2\right\| &= \sup_{x\in B[0,1]} \left|\left\langle L'g_1 - L'g_2, x\right\rangle\right| = \sup_{x\in B[0,1]} \left|\left\langle g_1 - g_2, Lx\right\rangle\right| \\ &= \sup_{z\in L(B[0,1])} \left|\left\langle g_1 - g_2, z\right\rangle\right| = \sup_{z\in \overline{L(B[0,1])}} \left|\left\langle g_1 - g_2, z\right\rangle\right| = \|g_1 - g_2\|.\end{aligned}$$

Since Φ is precompact, it is totally bounded. Hence, the set $L'\left(B'\right)$, which is isometric with it, is totally bounded as well. Therefore, the set $L'\left(B'\right)$ is precompact in X'. □

Remark 1.14. One can prove that the set Φ (from the proof of Theorem 1.13) is closed in U. Hence, Φ is compact and, consequently, the set $L'\left(B'\right)$ is compact. By Remark 1.8, for a completely continuous linear operator, the image of the closed unit ball does not need to be compact. But, for any completely continuous linear operator on X', the image of the set B' is compact.

Theorem 1.15. *Let $L \in \mathscr{L}(X)$ be a completely continuous operator. For arbitrary $\delta > 0$, there exists only a finite number of linearly independent eigenvectors associated with eigenvalues of L whose absolute values are greater than δ.*

Proof. By contradiction, let us consider a sequence $\lambda_1, \lambda_2, \dots \lambda_n, \dots$ of eigenvalues of L such that $|\lambda_n| > \delta$, $n \in \mathbb{N}$, and a sequence $x_1, x_2, \dots x_n, \dots$ of associated linearly independent eigenvectors. According to Theorem 1.3, we can construct a sequence $y_1, y_2, \dots y_n, \dots$ such that

$$\|y_n\| = 1, \qquad y_n \in X_n, \qquad n \in \mathbb{N},$$

and that

$$\inf_{x\in X_{n-1}} \|y_n - x\| > \frac{1}{2}, \qquad n \ge 2, n \in \mathbb{N},$$

where X_n is the subspace generated by $x_1, \dots, x_n$. The sequence $\{y_n/\lambda_n\}_{n=1}^{\infty}$ is bounded, because $|\lambda_n| > \delta$, $n \in \mathbb{N}$. Now, we prove that, from the sequence of the images $\{L(y_n/\lambda_n)\}_{n=1}^{\infty}$, one cannot choose a convergent subsequence. If

$$y_n = \sum_{k=1}^{n} \alpha_k x_k,$$

then

$$L\left(\frac{y_n}{\lambda_n}\right) = \sum_{k=1}^{n-1} \frac{\alpha_k \lambda_k}{\lambda_n} x_k + \alpha_n x_n = y_n + z_n,$$

where

$$z_n = \sum_{k=1}^{n-1} \alpha_k \left(\frac{\lambda_k}{\lambda_n} - 1\right) x_k \in X_{n-1}.$$

Therefore, for any $p, q \in \mathbb{N}$, $q < p$, it holds

$$\left\| L\left(\frac{y_p}{\lambda_p}\right) - L\left(\frac{y_q}{\lambda_q}\right) \right\| = \|y_p + z_p - (y_q + z_q)\| = \|y_p - (y_q + z_q - z_p)\| > \frac{1}{2},$$

because $y_q + z_q - z_p \in X_{p-1}$. We have a contradiction. □

Remark 1.16. Especially, from Theorem 1.15, it follows that the number of linearly independent eigenvectors associated with an eigenvalue $\lambda \neq 0$ of a completely continuous linear operator is finite.

Remark 1.17. Let the dimension of X be infinite. For any completely continuous operator $L \in \mathscr{L}(X)$, we see that $0 \in \sigma(L)$. Indeed, if $0 \notin \sigma(L)$, then L^{-1} is bounded on X, which is a contradiction with Theorem 1.11 (Corollary 1.12). In the case of a linear completely continuous operator, the spectrum is non-empty. Using the so-called Fredholm alternative, one can prove that $\sigma(L)$ of a completely continuous operator $L \in \mathscr{L}(X)$ can contain only eigenvalues and 0. Thus, the spectrum of a completely continuous linear operator has a very simple structure. In particular, we recall that it is a closed set.

Example 1.18. We consider the following series of examples.

a) Let us consider the operator $T\colon C[0,1] \to C[0,1]$ given by

$$T\colon f(t) \mapsto f\left(t^2\right), \qquad t \in [0,1],\, f \in C[0,1].$$

See Example 0.41. Especially, we consider the norm from (0.1). The operator T is not completely continuous. The spectrum

$$\sigma(T) = \{\lambda \in \mathbb{C};\, |\lambda| = 1\}$$

is an uncountable set. In the case of a space whose dimension is infinite, we also know that $0 \in \sigma(L)$ for any completely continuous linear operator L. We can also use directly Definition 1.1 and consider the sequence $\{t^n\}_{n=1}^{\infty} \subseteq C[0,1]$. From the sequence

$$\{Tt^n\}_{n=1}^{\infty} = \left\{t^{2n}\right\}_{n=1}^{\infty},$$

one cannot choose a convergent subsequence.

b) Let us consider the operator $\tilde{T}\colon C[0,1] \to C[0,1]$ given by

$$\tilde{T}\colon f(t) \mapsto \int_0^t f(x)\,\mathrm{d}x, \qquad t \in [0,1],\, f \in C[0,1].$$

See Example 0.42. Especially, we consider the norm from (0.1). We show that $\tilde{T}$ is completely continuous. We consider an arbitrary bounded sequence $\{f_n\}_{n=1}^{\infty} \subseteq C[0,1]$ and, using the Arzelà–Ascoli theorem, we prove that $\left\{\tilde{T}f_n\right\}_{n=1}^{\infty}$ is precompact. Let $K > 0$ be such that

$$\|f_n\| \leq K, \qquad n \in \mathbb{N}.$$

We have

$$\left\|\tilde{T}f_n\right\| = \max_{t\in[0,1]} \left| \int_0^t f_n(x)\,\mathrm{d}x \right| \leq \max_{t\in[0,1]} \int_0^t |f_n(x)|\,\mathrm{d}x \leq \int_0^1 K\,\mathrm{d}x = K, \qquad n \in \mathbb{N}.$$

At the same time, for $t, s \in [0,1]$, $n \in \mathbb{N}$, we have

$$\left|\tilde{T} f_n(t) - \tilde{T} f_n(s)\right| \le \left| \int_t^s |f_n(x)| \,\mathrm{d}x \right| \le K\,|t-s|\,.$$

Thus, the sequence $\left\{\tilde{T} f_n\right\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous.

We remark that if we use the complete continuity of $\tilde{T}$, then it is easy to determine the spectrum $\sigma(\tilde{T}) = \{0\}$ (cf. Example 0.42; see Remark 1.17).

c) We consider $L_1, L_2 \in \mathscr{L}\left(l^2\right)$ from Example 0.18, f). For these operators, we determine $\sigma(L_1)$ and $\sigma(L_2)$ and we decide whether L_1, L_2 are completely continuous. We recall that

$$L_1 x = \left\{0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right\}, \qquad x = \{x_n\}_{n=1}^{\infty} \in l^2,$$

$$L_2 x = \{0, x_1, x_2, x_3, \dots\}, \qquad x = \{x_n\}_{n=1}^{\infty} \in l^2.$$

At first, we analyse L_1. For a constant λ, we want to find $x = \{x_n\}_{n=1}^{\infty} \in l^2$ such that $L_1 x = \lambda x$. We get

$$\left\{0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right\} = \{\lambda x_1, \lambda x_2, \lambda x_3, \dots\}\,.$$

For $\lambda \neq 0$, we have $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $\dots$; and, for $\lambda = 0$, we also have

$$x_1 = x_2 = x_3 = \cdots = 0.$$

The equation $L_1 x = \lambda x$ does not have any non-zero solution for any λ. Therefore, any complex number is not an eigenvalue of the operator L_1.

We show that L_1 is completely continuous. For all $k \in \mathbb{N}$, we define $L_1^k\colon l^2 \to l^2$ by

$$L_1^k\colon x = \{x_n\}_{n=1}^{\infty} \mapsto \left\{0, x_1, \frac{x_2}{2}, \dots, \frac{x_k}{k}, 0, 0, \dots\right\}.$$

Any of the operators L_1^k is linear, bounded, with a finite dimension of its range. Therefore, they are completely continuous. Since, for all $k \in \mathbb{N}$ and $x = \{x_n\}_{n=1}^{\infty} \in l^2$, $\|x\| \le 1$, it holds

$$\left\|L_1^k x - L_1 x\right\|^2 = \sum_{n=k+1}^{\infty} \left|\frac{x_n}{n}\right|^2 \le \frac{1}{k^2}\|x\|^2 \le \frac{1}{k^2},$$

we have

$$\left\|L_1^k - L_1\right\|^2 \le \frac{1}{k^2}.$$

Hence, the completely continuous operators L_1^k converge to L_1 (as $k \to \infty$), which proves that L_1 is completely continuous (see Theorem 1.9). From Remark 1.17, we get that $\sigma(L_1) = \{0\}$.

Now, we prove that any λ satisfying $|\lambda| \le 1$ lies in $\sigma(L_2)$. In this case, the operator $L_2 - \lambda I$ does not map l^2 into l^2, because there is no $z = \{z_n\}_{n=1}^{\infty} \in l^2$ such that

$$(L_2 - \lambda I)z = \{-\lambda z_1, z_1 - \lambda z_2, z_2 - \lambda z_3, \dots\} = \{-1, 0, 0, \dots\}\,.$$

If we exclude the trivial case $\lambda = 0$, we have

$$z = \left\{ \frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}, \dots \right\}.$$

Such z (for $|\lambda| \le 1$) is not in l^2. Therefore, the operator L_2 is not completely continuous. It follows from the fact that its spectrum is the uncountable set

$$\{\lambda \in \mathbb{C}; |\lambda| \le 1\}.$$

d) On the space $C[0,1]$ with the uniform norm, we consider the operator given by

$$F\colon f(t) \mapsto t^2 f(0), \qquad t \in [0,1], f \in C[0,1].$$

This operator is completely continuous. Obviously, it is bounded, linear, and its range is a one-dimensional subspace of $C[0,1]$. The estimation

$$|Ff(t)| = \left|t^2 f(0)\right| \le |f(0)| \le \|f\|, \qquad t \in [0,1], f \in C[0,1],$$

gives $\|F\| \le 1$. Then, the choice $f \equiv 1$ shows that $\|F\| = 1$.

The complete continuity of the operator F can be proved also directly. Let $\{f_n\}_{n=1}^{\infty} \subset B[0,1] \subset C[0,1]$. From the estimations

$$\|Ff_n\| \le \|F\| \cdot \|f_n\| \le \|f_n\| \le 1, \qquad n \in \mathbb{N},$$

$$\begin{aligned} |Ff_n(t) - Ff_n(s)| &= \left|\left(t^2 - s^2\right) f_n(0)\right| \\ &\le |(t-s)(t+s)| \le 2|t-s|, \qquad n \in \mathbb{N},\ t,s \in [0,1], \end{aligned}$$

and from the Arzelà–Ascoli theorem, it follows the complete continuity of F.

Now, we identify eigenvalues of the operator F. We look for non-trivial solutions of the equation

$$t^2 f(0) = \lambda f(t).$$

For $\lambda = 0$, e.g., $f(t) = t$ is a solution. Therefore, 0 is an eigenvalue. For $\lambda \ne 0$, we obtain

$$f(t) = \frac{1}{\lambda} f(0) t^2.$$

Hence, $f(0) = 0$ and, consequently, $f \equiv 0$. Since F is completely continuous, $\sigma(F) = \{0\}$.

e) On the complex space l^2, we define the operator $R\colon l^2 \to l^2$ by

$$Rx = \{\mathrm{i}^n x_n\}_{n=1}^{\infty}, \qquad x = \{x_n\}_{n=1}^{\infty} \in l^2,$$

where i is the imaginary unit. For $x = \{x_n\}_{n=1}^{\infty} \in l^2$, we have

$$\|Rx\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2,$$

which gives $\|R\| = 1$. If λ is an eigenvalue, then there exists a non-zero element $x = \{x_n\}_{n=1}^{\infty} \in l^2$ such that $Rx = \lambda x$, i.e., $\mathrm{i}^n x_n = \lambda x_n$, $n \in \mathbb{N}$. It is seen that the eigenvalues are $\mathrm{i}, \mathrm{i}^2, \mathrm{i}^3, \mathrm{i}^4$, i.e., $\mathrm{i}, -1, -\mathrm{i}, 1$. For example, for i, the corresponding eigenvector is $\{1,0,0,\ldots\}$. For $\lambda \notin \{\mathrm{i},-1,-\mathrm{i},1\}$, the operator $R-\lambda I$ has an inverse. The inverse is

$$Sx = \left\{ \frac{1}{\mathrm{i}^n - \lambda} x_n \right\}_{n=1}^{\infty}, \qquad x = \{x_n\}_{n=1}^{\infty} \in l^2,$$

where

$$\|Sx\| \le k\,\|x\|$$

for

$$k = \max\left\{ \frac{1}{|\mathrm{i}-\lambda|}, \frac{1}{|\mathrm{i}^2-\lambda|}, \frac{1}{|\mathrm{i}^3-\lambda|}, \frac{1}{|\mathrm{i}^4-\lambda|} \right\}.$$

Note that the operator R is not completely continuous, because $0 \notin \sigma(R)$ (the dimension of l^2 is infinite).

f) Let the operator $T_1 \in \mathscr{L}\left(L^2[0,1]\right)$ be defined by

$$T_1 f(x) = x \cdot \int_0^1 f(t)\,\mathrm{d}t, \qquad x \in [0,1],\, f \in L^2[0,1].$$

For $f \in L^2[0,1]$, we have

$$\begin{aligned}
\|T_1 f\| &= \left\| x \cdot \int_0^1 f(t)\,\mathrm{d}t \right\| = \left(\int_0^1 \left| x \cdot \int_0^1 f(t)\,\mathrm{d}t \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&= \left| \int_0^1 f(t)\,\mathrm{d}t \right| \cdot \left(\int_0^1 x^2\,\mathrm{d}x \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \left| \int_0^1 f(t)\,\mathrm{d}t \right| \\
&\le \frac{1}{\sqrt{3}} \left(\int_0^1 |f(t)|^2\,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^1 1\,\mathrm{d}t \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \|f\|.
\end{aligned}$$

Therefore, $\|T_1\| \le 1/\sqrt{3}$. For $f \equiv 1$, we see that $\|f\| = 1$, $\|T_1 f\| = 1/\sqrt{3}$. Thus,

$$\|T_1\| = \frac{1}{\sqrt{3}}.$$

We determine the discrete spectrum of T_1. We want to find a non-trivial function f such that $T_1 f(x) = \lambda f(x)$, $x \in [0,1]$. For $\lambda = 0$, it suffices to consider an arbitrary identically non-zero function $f \in C[0,1]$ for which

$$\int_0^1 f(t)\,\mathrm{d}t = 0.$$

For example, $f(x) = \sin(2\pi x)$. For $\lambda \neq 0$ and

$$T_1 f(x) = x \int_0^1 f(t)\,\mathrm{d}t = \lambda f(x),$$

we see that f has to be linear, i.e., $f(x) = kx$. Then,

$$x \int_0^1 kt \, \mathrm{d}t = \lambda kx, \qquad x \in [0,1],$$

i.e., $\lambda = 1/2$. Thus, the point spectrum is $\{0, 1/2\}$. Obviously, the operator T_1 is linear and continuous and its range has a finite dimension. Therefore, it is completely continuous and $\sigma(T_1) = \{0, 1/2\}$.

Let the operator $T_2 \in \mathscr{L}\left(L^2[-1,1]\right)$ be defined by

$$T_2\colon\, f(x) \mapsto \int_{-1}^1 x^2 t f(t) \, \mathrm{d}t, \qquad x \in [-1,1], f \in L^2[-1,1].$$

For any $f \in L^2[-1,1]$, we have

$$\begin{aligned}\|T_2 f\|^2 &= \int_{-1}^1 |T_2 f(x)|^2 \, \mathrm{d}x = \int_{-1}^1 x^4 \left(\int_{-1}^1 t f(t) \, \mathrm{d}t \right)^2 \mathrm{d}x \\ &= \frac{2}{5} \left(\int_{-1}^1 t f(t) \, \mathrm{d}t \right)^2 \le \frac{2}{5} \|t\|^2 \cdot \|f\|^2 = \frac{4}{15} \|f\|^2 .\end{aligned}$$

For $f(x) = x$, we have

$$\|f\| = \sqrt{\frac{2}{3}}, \qquad \|T_2 f\|^2 = \frac{2}{5} \left(\frac{2}{3}\right)^2 .$$

Thus,

$$\|T_2\| = \frac{2}{\sqrt{15}}.$$

Let us find eigenvalues of T_2. We look for non-trivial solutions of the equation $(T_2 - \lambda I)f = 0$. From the identity

$$x^2 \int_{-1}^1 t f(t) \, \mathrm{d}t = \lambda f(x),$$

we see that the function f has to be a multiple of the function x^2 for $\lambda \neq 0$, i.e., $f(x) = kx^2$. From

$$x^2 \int_{-1}^1 tkt^2 \, \mathrm{d}t = \lambda kx^2,$$

it follows

$$\lambda = \int_{-1}^1 t^3 \, \mathrm{d}t = 0.$$

The point spectrum is $\{0\}$. Since T_2 is completely continuous, $\sigma(T_2) = \{0\}$.

Let the operator $T_3 \in \mathscr{L}\left(L^2[0,1]\right)$ be defined by

$$T_3 f(x) = x \cdot f(x), \qquad x \in [0,1],\, f \in L^2[0,1].$$

Since

$$\|T_3 f\| = \left(\int_0^1 |x f(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \le \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} = \|f\|, \qquad f \in L^2[0,1],$$

we have the inequality $\|T_3\| \le 1$. Let us determine the discrete spectrum. We look for non-trivial solutions of the equation $(T_3 - \lambda I) f = 0$. As a solution of the equation $(x-\lambda) f(x) = 0$, $x \in [0,1]$, we get only $f = 0$. Thus, the discrete spectrum is empty. Now, we consider the continuous spectrum. We consider a function $g \in L^2[0,1]$ and find λ for which the equation $(T_3 - \lambda I) f = g$ has a solution in $L^2[0,1]$. If $\lambda = 0$, then the equation $x f(x) = g(x)$ has only the solution

$$f(x) = \frac{g(x)}{x}.$$

For example, for $g(x) = \sqrt{x} \in L^2[0,1]$, we have $f(x) = 1/\sqrt{x} \notin L^2[0,1]$. Thus, $0 \in \sigma(T_3)$. In the case when $\lambda \neq 0$, we get the solution

$$f(x) = \frac{g(x)}{x-\lambda}, \qquad x \in [0,1].$$

For $\lambda \in (0,1]$ and $g \equiv 1$, we obtain that $f \notin L^2[0,1]$. For others $\lambda \in \mathbb{C}$, i.e., $\lambda \notin [0,1]$, the function $1/(x-\lambda)$ is continuous on $[0,1]$. Therefore, the equation $T_3 f - \lambda f = g$ has a solution in $L^2[0,1]$ for any function $g \in L^2[0,1]$. The spectrum is $\sigma(T_3) = [0,1]$. The operator T_3 cannot be completely continuous, because its spectrum is an uncountable set. Moreover, $\|T_3\| = 1$ (see Theorem 0.39).

1.3 Self-adjoint operator in Hilbert space

For self-adjoint linear operators in Hilbert spaces with finite dimensions, we have the well-known theorem about the existence of an operator matrix in the diagonal form. Now, we extend this theorem to completely continuous self-adjoint operators in Hilbert spaces. Let H be a Hilbert space.

Theorem 1.19. *All eigenvalues λ of a self-adjoint operator $L\colon H \to H$ are real.*

Proof. Let $Lx = \lambda x$ for some non-zero $x \in H$. Then,

$$\lambda \langle x,x\rangle = \langle \lambda x,x\rangle = \langle Lx,x\rangle = \langle x,Lx\rangle = \langle x,\lambda x\rangle = \bar{\lambda} \langle x,x\rangle .$$

We see that $\lambda = \bar{\lambda}$. □

Theorem 1.20. *Eigenvectors of a self-adjoint operator $L\colon H \to H$, which correspond to different eigenvalues, are orthogonal.*

Proof. If $Lx = \lambda x$ and $Ly = \mu y$ for $\lambda \neq \mu$, then

$$\lambda \langle x,y\rangle = \langle \lambda x,y\rangle = \langle Lx,y\rangle = \langle x,Ly\rangle = \langle x,\mu y\rangle = \mu \langle x,y\rangle ,$$

which gives $\langle x,y\rangle = 0$. □

Lemma 1.21. *If a sequence* $\{\xi_n\}_{n=1}^{\infty} \subseteq H$ *and* $\xi \in H$ *satisfy*

$$\sup_{n\in\mathbb{N}} \|\xi_n\| < +\infty$$

and

$$\|L(\xi_n - \xi)\| \to 0 \qquad \text{as} \qquad n \to \infty$$

for a self-adjoint operator $L\colon H \to H$*, then*

$$Q(\xi_n) = \langle L\xi_n, \xi_n \rangle \to \langle L\xi, \xi \rangle = Q(\xi) \qquad \text{as} \qquad n \to \infty.$$

Proof. For all $n \in \mathbb{N}$, we have

$$|\langle L\xi_n, \xi_n\rangle - \langle L\xi, \xi\rangle| \le |\langle L\xi_n, \xi_n\rangle - \langle L\xi, \xi_n\rangle| + |\langle \xi, L\xi_n\rangle - \langle \xi, L\xi\rangle|$$

together with

$$|\langle L\xi_n, \xi_n\rangle - \langle L\xi, \xi_n\rangle| = |\langle L(\xi_n - \xi), \xi_n\rangle| \le \|\xi_n\| \cdot \|L(\xi_n - \xi)\|$$

and

$$|\langle \xi, L\xi_n\rangle - \langle \xi, L\xi\rangle| = |\langle \xi, L(\xi_n - \xi)\rangle| \le \|\xi\| \cdot \|L(\xi_n - \xi)\|.$$

Since the set of all numbers $\|\xi_n\|$ for $n \in \mathbb{N}$ is bounded and $\|L(\xi_n - \xi)\| \to 0$ as $n \to \infty$, we get

$$|\langle L\xi_n, \xi_n\rangle - \langle L\xi, \xi\rangle| \to 0 \qquad \text{as} \qquad n \to \infty.$$

□

Lemma 1.22. *If the functional*

$$\xi \mapsto |Q(\xi)| = |\langle L\xi, \xi\rangle|, \qquad \xi \in H,$$

where $L\colon H \to H$ *is a self-adjoint operator, assumes a maximum on* $B[0,1] \subseteq H$ *at an element* ξ_0*, then*

$$\langle \xi_0, \eta\rangle = 0$$

implies that

$$\langle L\xi_0, \eta\rangle = \langle \xi_0, L\eta\rangle = 0.$$

Proof. Obviously, $\|\xi_0\| = 1$. Let $\eta \ne 0$ and $\langle \xi_0, \eta\rangle = 0$. We set

$$\xi = \frac{\xi_0 + a\eta}{\sqrt{1 + |a|^2 \|\eta\|^2}},$$

where $|a| > 0$ is a sufficiently small number. From $\|\xi_0\| = 1$, we get $\|\xi\| = 1$. It holds

$$\begin{aligned} Q(\xi) &= \frac{1}{1 + |a|^2 \|\eta\|^2} \left[Q(\xi_0) + \bar{a}\langle L\xi_0, \eta\rangle + a\overline{\langle L\xi_0, \eta\rangle} + |a|^2 Q(\eta) \right] \\ &= \frac{1}{1 + |a|^2 \|\eta\|^2} \left[Q(\xi_0) + 2\bar{a}\langle L\xi_0, \eta\rangle + |a|^2 Q(\eta) \right] \end{aligned}$$

for such a number a that $\bar{a}\langle L\xi_0, \eta\rangle$ is real. From the expression above, we see the following implication. If $\langle L\xi_0, \eta\rangle \ne 0$, then (consider $|a| \approx 0^+$)

$$|Q(\xi)| > |Q(\xi_0)|,$$

which is a contradiction. □

Remark 1.23. From Lemma 1.22, it follows that ξ_0 is an eigenvector of the operator L if the functional $|Q(\xi)|$ assumes a maximum on $B[0,1]$ at $\xi = \xi_0$.

Theorem 1.24 (Hilbert–Schmidt)**.** *For every completely continuous self-adjoint operator $L\colon H \to H$, there exists an orthonormal system of eigenvectors $\varphi_1, \varphi_2, \dots$ corresponding to non-zero eigenvalues $\lambda_1, \lambda_2, \dots$ such that any element $\xi \in H$ can be uniquely written as*

$$\xi = \sum_{k=1}^{N} c_k \varphi_k + \bar{\xi},$$

where $N \in \mathbb{N} \cup \{\infty\}$, $\bar{\xi}$ satisfies $L\bar{\xi} = 0$, and

$$L\xi = \sum_{k=1}^{N} \lambda_k c_k \varphi_k.$$

If the system of φ_k is infinite, then

$$\lim_{k\to\infty} \lambda_k = 0.$$

Proof. By induction, we construct eigenvectors φ_n so that the absolute values of the corresponding eigenvalues satisfy

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq \cdots$$

In the construction of φ_1, we investigate the functional $|Q(\xi)| = |\langle L\xi, \xi\rangle|$ and we prove that it assumes a maximum on $B[0,1]$. We denote

$$S = \sup_{\|\xi\|\leq 1} |\langle L\xi, \xi\rangle|.$$

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence such that $\|\xi_k\| \leq 1$ for $k \in \mathbb{N}$ and

$$|\langle L\xi_k, \xi_k\rangle| \to S \qquad \text{as} \qquad k \to \infty.$$

By Remark 1.14 (or Remark 1.8), from the sequence $\{\xi_k\}_{k=1}^{\infty}$, one can extract a subsequence $\{\xi_k^1\}_{k=1}^{\infty}$ such that

$$\left\|L\xi_k^1 - L\eta\right\| \to 0 \qquad \text{as} \qquad k \to \infty$$

for some $\eta \in H$, where $\|\eta\| \leq 1$. By Lemma 1.21, it holds $|\langle L\eta, \eta\rangle| = S$. We put $\varphi_1 = \eta$. We add that $\|\eta\| = 1$. Indeed, for $\|\eta\| < 1$, the element $\eta_1 = \eta/\|\eta\|$ satisfies

$$\|\eta_1\| = 1, \qquad |\langle L\eta_1, \eta_1\rangle| > S.$$

We know that (see Remark 1.23)

$$L\varphi_1 = \lambda_1 \varphi_1,$$

where

$$|\lambda_1| = \frac{|\langle L\varphi_1, \varphi_1\rangle|}{\langle \varphi_1, \varphi_1\rangle} = |\langle L\varphi_1, \varphi_1\rangle| = S.$$

Let eigenvectors $\varphi_1, \varphi_2, \dots, \varphi_n$ correspond to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ from our construction. Let H_n be the subspace of H generated by $\varphi_1, \varphi_2, \dots, \varphi_n$. We consider the functional $|\langle L\xi, \xi\rangle|$ on the set $H_n^{\perp} \cap B[0,1] \subseteq H$. Since the subspace H_n is invariant and L is self-adjoint, the set $H_n^{\perp}$ is invariant with respect to L (see Remark 0.36). According to the considerations above for $H_n^{\perp}$, we get that, in $H_n^{\perp} \cap B[0,1]$, one can find the required element (denoted as φ_{n+1}) which is an eigenvector of the operator L. The following two cases are possible:

1. after a finite number of the steps, we get a subspace $H_{n_0}^{\perp}$, where $\langle L\xi, \xi\rangle \equiv 0$;

2. $\langle L\xi, \xi\rangle \not\equiv 0$ on $H_n^{\perp}$ for all $n \in \mathbb{N}$.

In the first case, from Lemma 1.22, it follows that L maps the subspace $H_{n_0}^{\perp}$ into $\{0\}$, i.e., $H_{n_0}^{\perp}$ is composed of eigenvectors corresponding to the eigenvalue $\lambda = 0$. In this case, the set $\{\varphi_k\}$ is finite. In the second case, we obtain a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of eigenvectors for which $\lambda_k \neq 0$, $k \in \mathbb{N}$. We know that $\lambda_k \to 0$ as $k \to \infty$ (see Theorem 1.15). Let

$$\bar{H} = \bigcap_{n=1}^{\infty} H_n^{\perp} \neq \{0\}.$$

If $\xi \in \bar{H}$, then

$$|\langle L\xi, \xi\rangle| \leq |\lambda_n| \cdot \|\xi\|^2, \qquad n \in \mathbb{N},$$

i.e., $\langle L\xi, \xi\rangle = 0$. Therefore, by Lemma 1.22 (for the subspace $\bar{H}$), the operator L maps $\bar{H}$ into $\{0\}$.

From the construction of $\{\varphi_k\}_{k=1}^{N}$, it follows that any element $\xi \in H$ can be uniquely expressed as

$$\xi = \sum_{k=1}^{N} c_k \varphi_k + \bar{\xi},$$

where $N \in \mathbb{N} \cup \{\infty\}$, $L\bar{\xi} = 0$, and

$$L\xi = \sum_{k=1}^{N} \lambda_k c_k \varphi_k.$$

□

Remark 1.25. Theorem 1.24 says that, for any completely continuous self-adjoint operator on H, there exists an orthogonal base of the space H which is composed of eigenvectors of this operator. To obtain such a base, it is enough to consider $\{\varphi_k\}_{k=1}^{N}$ with an arbitrary orthogonal base of $H_{n_0}^{\perp}$ or $\bar{H}$ (see the proof of Theorem 1.24). In other words, we get a result entirely analogous to the theorem about the existence of an operator matrix in the diagonal form for self-adjoint operators in a space with a finite dimension.

2. Derivative in Banach spaces

Let X, Y be real Banach spaces.

2.1 Weak and strong derivative

Let f be a map defined on an open set $G \subseteq X$ with values in Y.

Definition 2.1. For $x \in G$, we define the directional derivative of f in the direction of $h \in X$ as the limit

$$\lim_{\lambda \to 0} \frac{f(x+\lambda h) - f(x)}{\lambda}$$

if it exists. We denote it by $\mathrm{D}_h f(x)$.

Definition 2.2. If f has the directional derivative in the direction of any $h \in X$ at $x \in G$ and $\mathrm{d}f(x) \colon h \mapsto \mathrm{D}_h f(x)$ is a continuous linear map from X to Y, we say that f has the weak derivative $\mathrm{d}f(x)$ at x. The weak derivative is sometimes called the Gâteaux derivative.

Definition 2.3. If there exists a continuous linear map $L \colon X \to Y$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L(h)}{\|h\|} = 0, \tag{2.1}$$

we say that f has the strong derivative at $x \in G$. If f has the strong derivative at $x \in G$, the map L from (2.1), which is uniquely determined, is called the Fréchet derivative of f at x and is denoted by $f'(x)$.

Remark 2.4. If $f'(x)$ exists, f has also the weak derivative at x and $f'(x) = \mathrm{d}f(x)$. In this case, there exists the directional derivative of f in the direction of any h and

$$\mathrm{D}_h f(x) = f'(x)(h).$$

Remark 2.5. Let f be a real function on X (i.e., $G = X$ and f is a functional). The function f has the Gâteaux derivative at $x \in X$ if there exists $L \in X'$ such that

$$\lim_{\lambda \to 0} \frac{f(x+\lambda h) - f(x)}{\lambda} = Lh \tag{2.2}$$

for all $h \in X$.

Example 2.6. On the Banach space $X = C[0,1]$ (with the norm of the uniform convergence), we consider the functional

$$F : f \mapsto \int_0^1 f^2.$$

For any $\varphi \in X$, we compute the directional derivative of F in the direction of φ as (see (2.2))

$$\mathrm{D}_\varphi F(f) = \lim_{\lambda\to 0} \frac{F(f+\lambda\varphi)-F(f)}{\lambda} = \lim_{\lambda\to 0} \frac{1}{\lambda} \int_0^1 \left((f+\lambda\varphi)^2 - f^2\right)$$
$$= \lim_{\lambda\to 0} \int_0^1 \left(2f\varphi + \lambda\varphi^2\right) = \int_0^1 2f\varphi.$$

We introduce

$$L\colon\ \varphi \mapsto \int_0^1 2f\varphi.$$

It is immediately seen that L is a linear functional on X. From

$$|L(\varphi)| \le 2\int_0^1 |f\varphi| \le 2 \max_{t\in[0,1]} |\varphi(t)| \cdot \int_0^1 |f| = 2\,\|\varphi\| \int_0^1 |f|,$$

it follows that L is bounded. Thus, F has the Gâteaux derivative at f (which is given by L).

If the Fréchet derivative of F at f exists, then it is L (see Remark 2.4). Its existence is guaranteed by the limit

$$\lim_{\varphi\to 0} \frac{F(f+\varphi)-F(f)-L(\varphi)}{\|\varphi\|} = \lim_{\varphi\to 0} \frac{1}{\|\varphi\|} \int_0^1 \left((f+\varphi)^2 - f^2 - 2f\varphi\right)$$
$$= \lim_{\varphi\to 0} \frac{1}{\|\varphi\|} \int_0^1 \varphi^2 \le \lim_{\varphi\to 0} \|\varphi\| = 0.$$

We see that the functional F has also the strong derivative at f and its Fréchet derivative is equal to L.

Before the next example, we recall that continuous linear functionals on l^1 have the form

$$\sum_{n=1}^{\infty} a_n x_n$$

for $\{a_n\}_{n=1}^{\infty} \in l^\infty$. If $\{a_n\}_{n=1}^{\infty} \in l^\infty$ and

$$\Psi(x) = \sum_{n=1}^{\infty} a_n x_n$$

for $x = \{x_n\}_{n=1}^{\infty} \in l^1$, then $\Psi \in \left(l^1\right)' (= l^\infty)$.

Example 2.7. In the Banach space l^1, we consider the function $f\colon t \mapsto \|t\|$ on l^1 and $x = \{x_n\}_{n=1}^{\infty} \in l^1$. We show that f has the Gâteaux derivative at x if and only if $x_n \neq 0$ for all $n \in \mathbb{N}$. In this case,

$$\mathrm{d}f(x) = \{\operatorname{sgn} x_n\}_{n=1}^{\infty} \in l^\infty.$$

Then, we show that the Fréchet derivative of the norm in l^1 does not exist at any point.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in l^1 such that $x_k = 0$ for some $k \in \mathbb{N}$. We put

$$e_k = \{0, \ldots, 0, 1, 0, \ldots\},$$

where 1 is in the k-th position. The non-existence of the directional derivative of f in the direction of e_k comes from

$$\frac{\|x + \lambda e_k\| - \|x\|}{\lambda} = \frac{1}{\lambda}\left(\sum_{n=1}^{\infty} |x_n| + |\lambda| - \sum_{n=1}^{\infty} |x_n|\right) = \frac{|\lambda|}{\lambda}.$$

For $\lambda \to 0$, we see that the limit of the considered term does not exist. Therefore, the function f does not have the weak (and also the strong) derivative at any point with a zero element.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in l^1 such that $x_n \neq 0$ for all $n \in \mathbb{N}$. We consider $h = \{h_n\}_{n=1}^{\infty} \in l^1$ and $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that

$$\sum_{n=k+1}^{\infty} |h_n| < \varepsilon.$$

Obviously, there exists $\delta > 0$ such that

$$\operatorname{sgn}(x_n + \lambda h_n) = \operatorname{sgn} x_n, \qquad |\lambda| < \delta,\, n \in \{1, \ldots, k\}.$$

For $\lambda \in (-\delta, \delta)$, we obtain

$$\begin{aligned}
&\left| \frac{\|x + \lambda h\| - \|x\|}{\lambda} - \sum_{n=1}^{\infty} h_n \operatorname{sgn} x_n \right| \\
&= \left| \frac{1}{\lambda}\left(\sum_{n=1}^{k} |x_n + \lambda h_n| - |x_n| - \lambda h_n \operatorname{sgn} x_n + \sum_{n=k+1}^{\infty} |x_n + \lambda h_n| - |x_n| - \lambda h_n \operatorname{sgn} x_n\right)\right| \\
&\leq \frac{1}{|\lambda|} \sum_{n=k+1}^{\infty} (|x_n| + |\lambda| \cdot |h_n| - |x_n| + |\lambda| \cdot |h_n|) \leq \frac{1}{|\lambda|} \sum_{n=k+1}^{\infty} 2|\lambda| \cdot |h_n| < 2\varepsilon.
\end{aligned}$$

Due to the fact that $\varepsilon > 0$ is arbitrary, the function f has the weak derivative at x which is equal to

$$\{\operatorname{sgn} x_n\}_{n=1}^{\infty} \in l^{\infty},$$

because the map

$$h \mapsto \sum_{n=1}^{\infty} h_n \operatorname{sgn} x_n$$

is a continuous linear functional on l^1.

Now, we show that f does not have the strong derivative at any point. We assume that f has the derivative $f'(x)$ at $x = \{x_n\}_{n=1}^{\infty}$. If $f'(x)$ exists, it is $\{\operatorname{sgn} x_n\}_{n=1}^{\infty} \in l^{\infty}$. We consider

$$h^j = \{0, 0, \ldots, 0, -2x_j, -2x_{j+1}, -2x_{j+2}, \ldots\}, \qquad j \in \mathbb{N},$$

where the value $-2x_j$ is in the j-th position. Obviously,

$$\|h^j\| = 2\sum_{n=j}^{\infty} |x_n| \to 0 \qquad \text{as} \qquad j \to \infty$$

and (see (2.1))

$$\left|\left\|x+h^j\right\|-\|x\|-f'(x)\left(h^j\right)\right|=\left|\left\|x+h^j\right\|-\|x\|-\sum_{n=j}^{\infty}(-2x_n)\operatorname{sgn}x_n\right|$$
$$=\left|\sum_{n=1}^{\infty}|x_n|-\sum_{n=1}^{\infty}|x_n|+\sum_{n=j}^{\infty}2|x_n|\right|=\left\|h^j\right\|.$$

Now, it is seen that f cannot have the strong derivative at x.

2.2 Convex function

Now, we study derivatives of convex functions.

Definition 2.8. A real function $f\colon D\subseteq X\to\mathbb{R}$ is called convex on a convex set $D\subseteq X$ if

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y),\qquad x,y\in D,\ \lambda\in[0,1].$$

An elementary example of convex functions on a Banach space is the norm.

Theorem 2.9. *Let f be a real convex function on an open convex set $D\subseteq X$ which is continuous at $x_0\in D$. Then, there exist $K>0$ and $\delta>0$ such that*

$$|f(x)-f(y)|\le K\|x-y\|,\qquad x,y\in B(x_0,\delta),$$

where $B(x_0,\delta)$ is the open ball with the center in x_0 and the radius δ.

Proof. The continuity of f at x_0 guarantees the existence of $M>0$ and $\delta>0$ such that $|f(t)|\le M$ for $t\in B(x_0,2\delta)\subseteq D$. We consider $x,y\in B(x_0,\delta)$, where $x\ne y$. We denote

$$\alpha=\|x-y\|,\qquad z=y+\frac{\delta}{\alpha}(y-x).$$

We see that $z\in B(x_0,2\delta)$. Since y is a convex combination of z and x, where

$$y=\frac{\alpha}{\alpha+\delta}z+\frac{\delta}{\alpha+\delta}x,$$

using the convexity of f, we have

$$f(y)-f(x)\le\frac{\alpha}{\alpha+\delta}f(z)+\frac{\delta}{\alpha+\delta}f(x)-f(x)$$
$$=\frac{\alpha}{\alpha+\delta}(f(z)-f(x))\le\frac{\alpha}{\alpha+\delta}2M\le\frac{2M}{\delta}\|x-y\|.$$

Analogously,

$$f(x)-f(y)\le\frac{2M}{\delta}\|y-x\|.$$

□

Remark 2.10. On spaces whose dimension is infinite, there exist discontinuous linear functionals and all linear functional is a convex function. Thus, there exist discontinuous convex functions. Note that convex functions on open subsets of a space with a finite dimension are continuous.

Remark 2.11. From the proof of Theorem 2.9, it follows that it suffices to assume only the boundedness of f on some neighbourhood of x_0. Thus, we know that a convex function is continuous on an open convex set $D \subseteq X$ if and only if it is locally bounded on D.

Before the following theorem, we recall that a real function f defined on a metric space M is called upper semi-continuous on M if the set

$$\{x \in M; f(x) < \alpha\}$$

is open for all $\alpha \in \mathbb{R}$.

Theorem 2.12. *Let f be a convex function on an open convex set $D \subseteq X$. Then, the following conditions are equivalent:*

i) f is continuous on D;

ii) f is upper semi-continuous on D;

iii) f is upper bounded on some neighbourhood of a point of D;

iv) f is continuous at a point of D.

Proof. The implication $i) \Rightarrow ii)$ is obvious.

If a real function f is upper semi-continuous on D and $x \in D$, then the set

$$\{t \in D; f(t) < f(x) + 1\}$$

is a neighbourhood of x, where f is upper bounded. We have proved the implication $ii) \Rightarrow iii)$.

For the implication $iii) \Rightarrow iv)$, we consider that f is upper bounded on a neighbourhood of $x_0 \in D$. Let $f \le K$ on $B(x_0, \delta)$ and let $t \in B(x_0, \delta)$. We have $2x_0 - t \in B(x_0, \delta)$. Thus,

$$f(x_0) = f\left(\frac{2x_0 - t}{2} + \frac{t}{2}\right) \le \frac{1}{2} f(2x_0 - t) + \frac{1}{2} f(t) \le \frac{1}{2}(K + f(t)).$$

Then, we have

$$-f(t) \le K - 2f(x_0) \le K + 2|f(x_0)|.$$

Since also

$$f(t) \le K \le K + 2|f(x_0)|,$$

we obtain

$$|f(t)| \le K + 2|f(x_0)|, \qquad t \in B(x_0, \delta).$$

Therefore, f is bounded on $B(x_0, \delta)$ and it suffices to use Remark 2.11.

Now, we consider the implication $iv) \Rightarrow i)$. Let f be continuous at $x \in D$ and let $y \in D$ ($y \neq x$) be arbitrarily given. There exist $z \in D$ and $\lambda \in (0,1)$ such that $y = \lambda x + (1-\lambda) z$. We consider $\delta > 0$ and $K > 0$ such that $f \le K$ on $B(x, \delta) \subseteq D$. We show that f is upper bounded on the set $B(y, \lambda\delta)$ which implies that the function f is continuous at y. We consider $t \in B(y, \lambda\delta)$. Since

$$t = \lambda\left(x + \frac{t-y}{\lambda}\right) + (1-\lambda) z \in D,$$

where

$$x+\frac{t-y}{\lambda}\in B(x,\delta),$$

the convexity of f gives the estimation

$$f(t)\leq\lambda f\left(x+\frac{t-y}{\lambda}\right)+(1-\lambda)f(z)\leq\lambda K+(1-\lambda)f(z).$$

□

Remark 2.13. It is known that linear functionals are continuous if and only if they are continuous at 0, which is if and only if they are bounded. Continuous linear functionals are bounded on the unit ball, but continuous convex functions do not need to be bounded on the unit ball (although they are locally bounded). In addition, on any separable Banach space whose dimension is infinite, there exists continuous convex function, which is not bounded on the unit ball.

Before the following theorem, we recall that a real function p on X is called a convex functional on X if:

a) $p(\lambda x)=\lambda p(x),\ \lambda\geq 0,\ x\in X;$

b) $p(x+y)\leq p(x)+p(y),\ x,y\in X.$

Theorem 2.14. *Let $D\subseteq X$ be an open convex set, let f be a convex function on D, and let $x\in D$. Then,*

$$\mathrm{d}^{+}f(x)(h)=\lim_{t\to 0^{+}}\frac{f(x+th)-f(x)}{t}$$

exists for all $h\in X$ and the map $\mathrm{d}^{+}f(x)\colon h\mapsto\mathrm{d}^{+}f(x)(h)$ is a convex functional on X.

Proof. Let $h\in X$. The function

$$t\mapsto\frac{f(x+th)-f(x)}{t}$$

is non-decreasing on a right neighbourhood of 0. If t,s, where $0<t<s$, are sufficiently small (so that $x+sh\in D$), then we have

$$x+th=\frac{s-t}{s}x+\frac{t}{s}(x+sh)$$

and, consequently,

$$f(x+th)\leq\frac{s-t}{s}f(x)+\frac{t}{s}f(x+sh).$$

Thus,

$$\frac{1}{t}\left(f(x+th)-f(x)\right)\leq\frac{1}{s}\left(f(x+sh)-f(x)\right).$$

If we consider $t>0$ sufficiently small, then

$$-\frac{f(x-2th)-f(x)}{2t}\leq\frac{f(x+2th)-f(x)}{2t},\tag{2.3}$$

because

$$2f(x)=2f\left(\frac{x-2th+x+2th}{2}\right)\leq f(x-2th)+f(x+2th).\tag{2.4}$$

Therefore (see (2.3)), we have

$$-\mathrm{d}^+ f(x)(-h) \leq \mathrm{d}^+ f(x)(h).$$

Especially, the considered limit exists.

It remains to prove the convexity of the functional. If $\lambda > 0$, then

$$d^+ f(x)(\lambda h) = \lambda \lim_{t\to 0^+} \frac{f(x+t\lambda h) - f(x)}{\lambda t} = \lambda \mathrm{d}^+ f(x)(h).$$

For $h, k \in X$, it holds (see (2.4))

$$\begin{aligned} \mathrm{d}^+ f(x)(h+k) &= \lim_{t\to 0^+} \frac{f(x+(h+k)t) - f(x)}{t} \\ &\leq \lim_{t\to 0^+} \left[\frac{f(x+2th) - f(x)}{2t} + \frac{f(x+2tk) - f(x)}{2t} \right] \\ &= \mathrm{d}^+ f(x)(h) + \mathrm{d}^+ f(x)(k). \end{aligned}$$

□

Remark 2.15. In the proof of the previous theorem, we have obtained the inequality

$$-\mathrm{d}^+ f(x)(-h) \leq \mathrm{d}^+ f(x)(h).$$

It is easy to show that $\mathrm{D}_h f(x)$ exists for all $h \in X$ if and only if

$$-\mathrm{d}^+ f(x)(-h) = \mathrm{d}^+ f(x)(h), \qquad h \in X.$$

Theorem 2.16. *Let f be a convex function on an open convex set $D \subseteq X$, let it be continuous at $x \in D$, and let it have the derivative $\mathrm{D}_h f(x)$ linearly in the direction of all $h \in X$. Then, f has the Gâteaux derivative $\mathrm{d}f(x)$ at x.*

Proof. By Theorem 2.9, there exist $K > 0$ and $\delta > 0$ such that

$$|f(u) - f(v)| \leq K \|u - v\|$$

for all $u, v \in B(x, \delta) \subseteq D$. We consider $h \in X$. Let $\lambda > 0$ be such that $x + \lambda h \in B(x, \delta)$. Then,

$$|f(x+\lambda h) - f(x)| \leq K \|\lambda h\| = K\lambda \|h\|.$$

Therefore,

$$|\mathrm{d}^+ f(x)(h)| \leq K \|h\|$$

and, consequently, the derivative $\mathrm{d}^+ f(x)(h) = \mathrm{D}_h f(x)$ is continuous. □

2.3 Tangent functional

We know that the function $f\colon X \to \mathbb{R}$ given by $x \mapsto \|x\|$ is continuous and convex. We show that, under certain assumptions, the weak derivative of this function f is the so-called tangent functional. We recall the well-known corollary of the Hahn–Banach theorem which says that, for any non-zero element $x \in X$, there exists $g \in X'$ such that $\|g\| = 1$ and $g(x) = \|x\|$. This functional is called the tangent functional at x.

Theorem 2.17. *Let the function $f\colon x \mapsto \|x\|$ have the Gâteaux derivative at $x \neq 0$, $x \in X$. Then, the Gâteaux derivative $\mathrm{d}f(x) \in X'$ has the norm $\|\mathrm{d}f(x)\| = 1$ and $\mathrm{d}f(x)(x) = \|x\|$. At the same time, if $g \in X'$ satisfies $\|g\| = 1$ and $g(x) = \|x\|$, then $g = \mathrm{d}f(x)$.*

Proof. For $h \in X$ and sufficiently small $t \neq 0$, from the estimation

$$\frac{1}{t}\left[\|x+th\| - \|x\|\right] \leq \frac{1}{|t|}\|x+th-x\| = \|h\|,$$

it follows that $\|\mathrm{d}f(x)\| \leq 1$. Since

$$\begin{aligned}
\mathrm{d}f(x)\left(\frac{x}{\|x\|}\right) &= \lim_{t\to 0}\frac{1}{t}\left(\left\|x+t\frac{x}{\|x\|}\right\| - \|x\|\right)\\
&= \lim_{t\to 0}\frac{1}{t}\left(\left\|\left(1+\frac{t}{\|x\|}\right)x\right\| - \|x\|\right)\\
&= \lim_{t\to 0}\frac{\|x\|}{t}\left(1+\frac{t}{\|x\|}-1\right) = 1,
\end{aligned}$$

we have $\|\mathrm{d}f(x)\| = 1$ and $\mathrm{d}f(x)(x) = \|x\|$.

Let $g \in X'$ be such that $\|g\| = 1$ and $g(x) = \|x\|$. We consider $h \in X$ and we define

$$\varepsilon(t) = \mathrm{d}f(x)(h) - \frac{1}{t}\left[\|x+th\| - \|x\|\right]$$

for sufficiently small $t \neq 0$. Obviously, $\varepsilon(t) \to 0$ as $t \to 0$. For the considered t, we have

$$g(x+th) \leq \|g\|\cdot\|x+th\| = \|x\| + t\mathrm{d}f(x)(h) - t\varepsilon(t) = g(x) + t\mathrm{d}f(x)(h) - t\varepsilon(t).$$

Therefore,

$$tg(h) \leq t\mathrm{d}f(x)(h) - t\varepsilon(t)$$

which gives (as $t \to 0^+$)

$$g(h) \leq \mathrm{d}f(x)(h).$$

Since this inequality is valid for all $h \in X$ (also for $-h$), it is enough to consider the linearity of g and $\mathrm{d}f(x)$. Therefore, $g = \mathrm{d}f(x)$. □

The previous theorem relates to the geometry of a Banach space. According to this theorem, at all non-zero point in which the norm has the Gâteaux derivative, the space is “smooth” in the sense that there exists just one tangent functional. We discuss this topic in the next chapter.

3. Strictly and uniformly convex spaces

In this chapter, we consider a real Banach space X.

3.1 Strictly convex space

Definition 3.1. An extreme point of a convex set $C \subseteq X$ is a point $x \in C$ for which $x = (a+b)/2$, where $a, b \in C$, implies $a = b$. The set of all extreme points of the set C is denoted by $\operatorname{ext} C$.

Remark 3.2. Definition 3.1 says that $x \in C$ is an extreme point if there does not exist a non-degenerated line segment in C having the center x. It is seen that x is an extreme point of C if and only if the set $C \setminus \{x\}$ is convex.

Definition 3.3. A Banach space is called strictly convex if all point of the unit sphere $\partial B(0,1)$ is an extreme point of the closed unit ball $B[0,1]$, i.e., if $\operatorname{ext} B[0,1] = \partial B(0,1)$. Strictly convex spaces are also called rotund.

Remark 3.4. In strictly convex spaces, any line segment cannot lie on any sphere. Therefore, the following implication is valid. If C is a convex set in the strictly convex space X, $x \in X$, $a, b \in C$, and if $\|x-a\| = \|x-b\| = \operatorname{dist}(x,C)$, then $a = b$. Indeed, for $\lambda \in [0,1]$, it suffices to consider that

$$\|x - \lambda a - (1-\lambda)b\| \leq \|\lambda x - \lambda a\| + \|(1-\lambda)x - (1-\lambda)b\| = \operatorname{dist}(x,C).$$

Remark 3.5. Since extreme points of the closed unit ball $B[0,1]$ have to lie on the unit sphere $\partial B(0,1)$, we can say that X is strictly convex if and only if

$$\|x+y\| < 2, \qquad x, y \in B[0,1],\, x \neq y.$$

Let the identity

$$\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{3.1}$$

be valid for points x, y of a Hilbert space. Since, in any Hilbert space, the identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

is valid, we see that $x = y$. Geometrically, (3.1) says that, in the parallelogram with the sides x and y, the second diagonal $x - y$ is missing. Now, we consider points x, y of X for which (3.1) is valid. Since

$$\begin{aligned} 0 &= 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 \geq 2\|x\|^2 + 2\|y\|^2 - (\|x\| + \|y\|)^2 \\ &= 2\|x\|^2 + 2\|y\|^2 - \|x\|^2 - 2\|x\| \cdot \|y\| - \|y\|^2 = (\|x\| - \|y\|)^2 \geq 0, \end{aligned}$$

we get $\|x\| = \|y\|$, but not $x = y$ necessarily. This observation motivates the following theorem.

Theorem 3.6. *For the Banach space X, the following statements are equivalent:*

i) X is strictly convex;

ii) if $\|x+y\| = \|x\| + \|y\|$, $x \neq 0, y \neq 0$, then $x = \lambda y$ for some $\lambda > 0$;

iii) if $x, y \in X$ and $\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$, then $x = y$,

Proof. We begin with the implication $i) \Rightarrow ii)$. Let X be strictly convex and let x, y be non-zero points of X satisfying the equality $\|x+y\| = \|x\| + \|y\|$. For example, let $\|x\| \leq \|y\|$. We have

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \geq \left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\| - \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\|$$
$$= \frac{1}{\|x\|} (\|x\| + \|y\|) - \|y\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) = 2.$$

Therefore, we obtain (see Remark 3.5)

$$\frac{x}{\|x\|} = \frac{y}{\|y\|},$$

i.e.,

$$x = \frac{\|x\|}{\|y\|} y.$$

Let us consider the implication $ii) \Rightarrow iii)$. Let the implication in $ii)$ be true and let

$$\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Hence (see the text before Theorem 3.6), $\|x\| = \|y\|$. We do not consider the trivial case $x = 0, y = 0$. In the non-zero case, we obtain

$$\|x+y\|^2 = 4\|x\|^2.$$

Therefore,

$$\|x+y\| = 2\|x\| = \|x\| + \|y\|.$$

Based on the assumption, $x = \lambda y$ for some $\lambda > 0$. Considering $\|x\| = \|y\|$, we obtain that $\lambda = 1$. The implication has been proved.

In the last part of the proof, we can use, e.g., Remark 3.4 or Remark 3.5. If

$$\|x\| = \|y\| = \left\| \frac{1}{2}(x+y) \right\| = 1,$$

then

$$\|x+y\|^2 = 4 = 2\|x\|^2 + 2\|y\|^2.$$

□

3.2 Uniformly convex space

If points x, y of a strictly convex space have a constant distance and they are on the unit sphere $\partial B(0,1)$, then the midpoint of the line segment between x and y is in the open unit ball. But, in a space whose dimension is infinite, it is not clear, whether this midpoint can be arbitrarily close to the sphere $\partial B(0,1)$. This motivates the following definition.

Definition 3.7. A Banach space is called uniformly convex if, for all $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, if x, y are in the unit ball $B[0,1]$ and $\|x-y\| \geq \varepsilon$, then

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1-\delta.$$

The basic characteristics of uniformly convex spaces are mentioned in the following theorem.

Theorem 3.8. *For the Banach space X, the following statements are equivalent:*

i) X is uniformly convex;

ii) for any $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, if x, y are on the unit sphere $\partial B(0,1)$ and $\|x-y\| \geq \varepsilon$, then

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1-\delta;$$

iii) if $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \partial B(0,1)$ satisfy

$$\lim_{n\to\infty} \left\| \frac{x_n+y_n}{2} \right\| = 1,$$

then $x_n - y_n \to 0$ as $n \to \infty$.

Proof. If X is uniformly convex, then $ii)$ is true. The implication $ii) \Rightarrow iii)$ is trivial as well.

If X is not uniformly convex, then there exist $\varepsilon > 0$ and sequences

$$\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq B[0,1]$$

such that

$$\|x_n - y_n\| \geq \varepsilon, \qquad n \in \mathbb{N},$$

and

$$1-\frac{1}{n} \leq \frac{1}{2}\|x_n+y_n\| \leq \frac{1}{2}(\|x_n\|+\|y_n\|) \leq 1, \qquad n \in \mathbb{N}.$$

Therefore,

$$\|x_n\|+\|y_n\| \to 2 \qquad \text{as} \qquad n \to \infty.$$

Since $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1.$$

Especially, without loss of generality, we can assume that $\|x_n\| \cdot \|y_n\| > 0$ for all $n \in \mathbb{N}$. For

$$u_n = \frac{x_n}{\|x_n\|}, \qquad v_n = \frac{y_n}{\|y_n\|}, \qquad n \in \mathbb{N},$$

we obtain $\|u_n\| = 1 = \|v_n\|$, $n \in \mathbb{N}$, and

$$|\|u_n + v_n\| - \|x_n + y_n\|| \le \|u_n - x_n\| + \|v_n - y_n\|, \qquad n \in \mathbb{N}.$$

The right side converges to 0 as $n \to \infty$, because

$$\|u_n - x_n\| = \left\| \frac{x_n}{\|x_n\|} - x_n \right\| = \|x_n\| \cdot \left(\frac{1}{\|x_n\|} - 1 \right) = 1 - \|x_n\|, \qquad n \in \mathbb{N},$$

and, analogously,

$$\|v_n - y_n\| = 1 - \|y_n\|, \qquad n \in \mathbb{N}.$$

With respect to

$$\frac{\|x_n + y_n\|}{2} \to 1 \qquad \text{as} \qquad n \to \infty,$$

we get

$$\frac{\|u_n + v_n\|}{2} \to 1 \qquad \text{as} \qquad n \to \infty. \tag{3.2}$$

We have two sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty} \subseteq \partial B(0,1)$ for which (3.2) is valid and, at the same time, for which

$$\liminf_{n \to \infty} \|u_n - v_n\| \ge \varepsilon > 0,$$

because

$$0 < \varepsilon \le \|x_n - y_n\| \le \underbrace{\|x_n - u_n\|}_{\to 0} + \|u_n - v_n\| + \underbrace{\|v_n - y_n\|}_{\to 0}, \qquad n \in \mathbb{N}.$$

□

Remark 3.9. Many other equivalences can be mentioned in Theorems 3.6 and 3.8. The statements in Theorem 3.6 are also equivalent to:

iv) if $p \in (1, \infty)$ and $x, y \in X$, $x \neq y$, then

$$\left\| \frac{x+y}{2} \right\|^p < \|x\|^p + \|y\|^p;$$

v) if

$$\|x - y\| = \|x - z\| + \|z - y\|,$$

then there exists $\lambda \in [0,1]$ such that $z = \lambda x + (1 - \lambda) y$.

Similarly, the statements in Theorem 3.8 are equivalent to:

iv) for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\|x\| < 1 + \delta$, $\|y\| < 1 + \delta$, and if

$$\left\| \frac{1}{2}(x+y) \right\| \ge 1,$$

then $\|x - y\| < \varepsilon$.

Remark 3.10. Obviously, any uniformly convex space is strictly convex. In spaces whose dimension is finite, these notions are same. It follows from the compactness of the closed unit ball $B[0,1]$ in spaces with finite dimensions and the continuity of the function

$$(x, y) \mapsto \frac{x+y}{2}.$$

Remark 3.11. We consider $X = C[0,1]$ with the norm

$$\|f\| = \max\{|f(t)|; t \in [0,1]\} + \left(\int_0^1 f^2\right)^{\frac{1}{2}}, \qquad f \in X.$$

One can show that this space X is strictly convex, but it is not uniformly convex.

Example 3.12. Any Hilbert space H is uniformly convex. It is enough to consider that, for $x, y \in H$, $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$, we have

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = \frac{\|x\|^2}{2} + \frac{\|y\|^2}{2}$$

and, consequently,

$$\left\|\frac{x+y}{2}\right\|^2 \leq \frac{1}{2} + \frac{1}{2} - \left(\frac{\varepsilon}{2}\right)^2,$$

i.e.,

$$\left\|\frac{x+y}{2}\right\| \leq \sqrt{1 - \frac{\varepsilon^2}{4}} < 1.$$

Example 3.13. The spaces

$$Z_p = \left(L^p(\Omega), \|\cdot\|_p\right)$$

are strictly convex for all $p \in (1,\infty)$ and any measurable set $\Omega \subseteq \mathbb{R}$. This fact is possible to easily show taking into account the case, when the Minkowski inequality becomes the equality. But, we prove the stronger result that the space Z_p is uniformly convex for all $p > 1$. It is enough to prove that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $u, v \in L^p(\Omega)$, $\|u\|_p = \|v\|_p = 1$, and if

$$\left\|\frac{u+v}{2}\right\|_p^p > 1 - \delta,$$

then

$$\left\|\frac{u-v}{2}\right\|_p^p \leq 2\varepsilon^p.$$

We consider an arbitrary number $\varepsilon > 0$. For simplicity, we denote

$$s = \frac{1}{2}(u+v), \qquad t = \frac{1}{2}(u-v),$$

where $u = s + t$, $v = s - t$. We put

$$\begin{aligned}
S &= \{\omega \in \Omega; |t(\omega)| \leq \varepsilon|s(\omega)|\}, \\
S_0 &= \{\omega \in \Omega; t(\omega) = 0\}, \\
S_+ &= \{\omega \in \Omega; 0 < |t(\omega)| \leq \varepsilon|s(\omega)|\}, \\
T &= \{\omega \in \Omega; |t(\omega)| > \varepsilon|s(\omega)|\}.
\end{aligned}$$

Evidently,

$$\int_S |t(\omega)|^p \, \mathrm{d}\omega \leq \varepsilon^p \int_\Omega |s(\omega)|^p \, \mathrm{d}\omega \leq \varepsilon^p. \tag{3.3}$$

It is well-known that the function $\lambda \mapsto |\lambda|^p$ is strictly convex and continuous on $\mathbb{R}$. Therefore,

$$\frac{|\lambda+1|^p+|\lambda-1|^p}{2} > |\lambda|^p = \left|\frac{\lambda+1}{2}+\frac{\lambda-1}{2}\right|^p, \qquad \lambda \in \mathbb{R},$$

and there exists $\gamma > 0$ such that

$$\frac{1}{2}\left[|\lambda+1|^p+|\lambda-1|^p\right]-|\lambda|^p \geq \gamma, \qquad \lambda \in \left[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right]. \tag{3.4}$$

In (3.4), we consider

$$\lambda = \frac{s(\omega)}{t(\omega)}, \qquad \omega \in \Omega \setminus S_0.$$

For $\omega \in T$, we obtain

$$\frac{1}{2}\left[|s(\omega)+t(\omega)|^p+|s(\omega)-t(\omega)|^p\right] \geq \gamma|t(\omega)|^p+|s(\omega)|^p$$

and, for $\omega \in S = S_0 \cup S_+$, it holds

$$\frac{1}{2}\left[|s(\omega)+t(\omega)|^p+|s(\omega)-t(\omega)|^p\right] \geq |s(\omega)|^p.$$

Thus,

$$\begin{aligned} 1 &= \int_\Omega \frac{1}{2}\left[|s(\omega)+t(\omega)|^p+|s(\omega)-t(\omega)|^p\right] \mathrm{d}\omega \\ &\geq \int_\Omega |s(\omega)|^p \,\mathrm{d}\omega + \int_T \gamma|t(\omega)|^p \,\mathrm{d}\omega. \end{aligned} \tag{3.5}$$

If

$$\int_\Omega |s(\omega)|^p \,\mathrm{d}\omega > 1-\delta,$$

then, using (3.5), we obtain

$$\int_T |t(\omega)|^p \,\mathrm{d}\omega \leq \frac{\delta}{\gamma}. \tag{3.6}$$

The choice $\delta = \gamma\varepsilon^p$, (3.3), and (3.6) give

$$\int_\Omega |t(\omega)|^p \,\mathrm{d}\omega = \int_T |t(\omega)|^p \,\mathrm{d}\omega + \int_S |t(\omega)|^p \,\mathrm{d}\omega \leq \varepsilon^p+\varepsilon^p = 2\varepsilon^p.$$

3.3 Projection

Now, we focus on projections in uniformly convex Banach spaces. Let X be a uniformly convex Banach space.

Theorem 3.14. *Let $C(\neq \emptyset)$ be a closed convex subset of X. Then, for any $y \in X$, there exists just one $c \in C$ such that $\|y-c\| = \operatorname{dist}(y,C)$.*

Proof. The uniqueness is presented in Remark 3.4. Hence, it suffices to prove the existence of an element $x \in C$ with the minimal norm, because, without loss of generality, we can assume that $y = 0$. We put

$$d = \inf\{\|c\|\,;\, c \in C\}\,.$$

If $d = 0$, then we see that $0 \in C$ (the set C is closed). Therefore, without loss of generality, let $d = 1$. In this case, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ such that

$$\lim_{n\to\infty} \|x_n\| = 1.$$

If $\{x_n\}_{n=1}^{\infty}$ is Cauchy, then there exists the limit

$$x = \lim_{n\to\infty} x_n, \qquad \text{where} \qquad x \in C,\ \|x\| = 1.$$

The convexity of C guarantees that

$$\left\| \frac{1}{2}(x_n + x_k) \right\| \geq 1, \qquad n,k \in \mathbb{N}.$$

We consider an arbitrary number $\varepsilon > 0$. For the given ε, there exists $n_0 \in \mathbb{N}$ such that $\|x_n\| < 1 + \varepsilon$ for all $n \geq n_0$, $n \in \mathbb{N}$. Therefore (using the uniform convexity), we have $\|x_n - x_k\| < \xi(\varepsilon)$ for all $n,k \geq n_0$, $n,k \in \mathbb{N}$, where $\xi(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. The theorem is proved. □

Definition 3.15. Let C be a closed convex subset of X. For any $x \in X$, the uniquely determined $P_C(x) \in C$ satisfying $\|x - P_C(x)\| = \operatorname{dist}(x, C)$ is called the projection (of x on C).

Theorem 3.16. *Let C be a closed convex subset of X. The projection P_C is continuous.*

Proof. For simplicity, we consider $\operatorname{dist}(0, C) = 1$. Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ be a sequence such that $x_n \to 0$ as $n \to \infty$. We want to show that $P_C(x_n) \to P_C(0)$ as $n \to \infty$. Since

$$\begin{aligned} |\|x_n - P_C(x_n)\| - 1| &= |\operatorname{dist}(x_n, C) - \operatorname{dist}(0, C)| \\ &\leq \|x_n - 0\| = \|x_n\| \to 0 \qquad \text{as} \qquad n \to \infty, \end{aligned}$$

we obtain that $\|x_n - P_C(x_n)\| \to 1$ as $n \to \infty$. From

$$1 \leftarrow |\|x_n - P_C(x_n)\| - \|x_n\|| \leq \|P_C(x_n)\| \leq \|x_n - P_C(x_n)\| + \|x_n\| \to 1$$

as $n \to \infty$, we have $\|P_C(x_n)\| \to 1$ as $n \to \infty$. The convexity of C gives

$$1 \leq \left\| \frac{1}{2}\left(P_C(0) + P_C(x_n)\right) \right\| \leq \frac{1}{2}\|P_C(0)\| + \frac{1}{2}\|P_C(x_n)\| \to 1 \qquad \text{as} \qquad n \to \infty.$$

We use *iii*) from Theorem 3.8 for the sequences

$$\{P_C(0)\}_{n=1}^{\infty}, \qquad \left\{ \frac{P_C(x_n)}{\|P_C(x_n)\|} \right\}_{n=1}^{\infty}.$$

Thus, $P_C(x_n) - P_C(0) \to 0$ as $n \to \infty$. It suffices to consider that

$$\begin{aligned} &\left| \left\| \frac{1}{2}\left(P_C(0) + P_C(x_n)\right) \right\| - \left\| \frac{1}{2}\left(P_C(0) + \frac{P_C(x_n)}{\|P_C(x_n)\|}\right) \right\| \right| \\ &\leq \left\| \frac{1}{2}\left(P_C(x_n) - \frac{P_C(x_n)}{\|P_C(x_n)\|}\right) \right\| \to 0 \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

□

Remark 3.17. The projection on closed convex sets does not need to be linear (even in Hilbert spaces). It is known that the projection on (closed) subspaces of Hilbert spaces is linear. But, it is not true in uniformly convex spaces.

For a better understanding of Theorem 3.20 below, we mention the following result.

Theorem 3.18. *Any uniformly convex space is reflexive.*

Remark 3.19. Strictly convex spaces do not need to be reflexive (see the example in Remark 3.11).

Now, without a proof, we mention a generalization of Theorem 3.14.

Theorem 3.20. *Let C be a closed convex subset of a strictly convex reflexive Banach space Y and $y \in Y$. Then, there exists just one $c \in C$ such that $\|y - c\| = \operatorname{dist}(y, C)$.*

Remark 3.21. We add that there exist strictly convex reflexive spaces, which are not uniformly convex.

Now, we generalize the concept of the projection to the concept of the so-called metric projections in the following definition.

Definition 3.22. Let M be a subset of a Banach space Y. For $x \in Y$, we denote

$$\mathscr{P}_M(x) = \{m \in M;\, \|x - m\| = \operatorname{dist}(x, M)\}\,.$$

The set M is called:

- proximinal if $\mathscr{P}_M(x) \neq \emptyset$ for all $x \in Y$;
- semi-Chebyshev if $\mathscr{P}_M(x)$ has at most one element for all $x \in Y$;
- Chebyshev if $\mathscr{P}_M(x)$ has just one element for all $x \in Y$.

We repeat that, in strictly convex spaces, any closed convex set is semi-Chebyshev and that any closed convex subset of a strictly convex reflexive Banach space is Chebyshev (see Remark 3.4 and Theorem 3.20). We add that any compact set is proximinal.

3.4 Smooth space

Now, let us define smooth spaces explicitly.

Definition 3.23. The space X is called smooth at $x \in \partial B(0,1)$ if there exists just one functional $\varphi \in X'$ such that $\|\varphi\| = 1$, $\varphi(x) = 1$. We say that X is smooth if it is smooth at any point of the unit sphere $\partial B(0,1)$.

We emphasize that, according to the above mentioned corollary of the Hahn–Banach theorem, the tangent functional exists at any point $x \in \partial B(0,1)$. Concerning Definition 3.23, we point out the uniqueness.

The following reinforcement of Theorem 2.17 is valid.

Theorem 3.24 (Šmuljan)**.** *The space X is smooth at $x \in \partial B(0,1)$ if and only if the function $f\colon t \mapsto \|t\|$ has the Gâteaux derivative at x.*

The most important connection between Chapters 2 and 3 is presented in the following theorem.

Theorem 3.25 (Klee)**.** *If X' is strictly convex, then X is smooth. If X' is smooth, then X is strictly convex.*

Proof. We assume that X is not smooth at $x \in \partial B(0,1)$. Thus, there exist $\varphi, \psi \in X'$ such that

$$\varphi \neq \psi, \qquad \|\varphi\| = \|\psi\| = 1 = \varphi(x) = \psi(x).$$

Since

$$\left\| \frac{\varphi + \psi}{2} \right\| = 1,$$

the space X' cannot be strictly convex.

If X is not strictly convex, then there exist $x, y \in \partial B(0,1)$ such that $x \neq y$ and

$$\frac{x+y}{2} \in \partial B(0,1).$$

By the recalled corollary of the Hahn–Banach theorem, there exists a functional φ such that

$$\|\varphi\| = 1, \qquad \varphi\left(\frac{x+y}{2}\right) = 1.$$

Since

$$1 = \varphi\left(\frac{x+y}{2}\right) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) \leq \frac{1}{2} + \frac{1}{2} = 1,$$

we see

$$\varphi(x) = \varphi(y) = 1.$$

For the elements of the unit sphere in X'' given by x and y (denoted by f_x and f_y), we have

$$f_x(\varphi) = f_y(\varphi) = 1.$$

Since $f_x \neq f_y$, the space X' cannot be smooth at φ. □

Corollary 3.26. *Let X be reflexive. The space X is smooth if and only if X' is strictly convex; and X is strictly convex if and only if X' is smooth.*

Remark 3.27. Note that there exist smooth spaces, whose dual spaces are not strictly convex. Similarly, there exist strictly convex spaces, whose dual spaces are not smooth. We repeat that smooth Banach spaces are the spaces whose norms have the Gâteaux derivative at any point of the unit sphere.

Definition 3.28. The space X is called uniformly smooth if there exists the limit

$$\lim_{\tau \to 0} \frac{\|x + \tau y\| - \|x\|}{\tau}$$

uniformly for $x, y \in \partial B(0,1)$.

We end this chapter with the analogy of Corollary 3.26.

Theorem 3.29. *The space X is uniformly smooth if and only if X' is uniformly convex; and X is uniformly convex if and only if X' is uniformly smooth.*

We add that uniformly smooth spaces are reflexive.

4. Fixed point theorems

In this chapter, we consider a real Banach space X.

Definition 4.1. Let Y be a metric space. A point y is called a fixed point of a map $f\colon D \subseteq Y \to Y$ if $f(y) = y$.

At first, we recall the most important fixed point theorem.

Theorem 4.2 (Banach)**.** *If Y is a complete metric space and $f\colon Y \to Y$ is a contraction, then there exists just one fixed point of f.*

Remark 4.3. Theorem 4.2 is well-known, e.g., from the theory of ODEs.

Definition 4.4. We say that a subset D of X has the fixed point property if any continuous map $f : D \to D$ has a fixed point.

Remark 4.5. For example, for $(\mathbb{R}, |\cdot|)$, it is seen that any closed interval $[a,b]$ has the fixed point property.

4.1 Topological degree

Let $f : G \to X$ be a map, where G is an open set in X. Our goal is to define the number $\deg(f, G, p)$, i.e., the so-called topological degree of f, which means "the number of solutions of the equation $f(x) = p$ on G". This number depends on f and on p continuously in a certain sense. Thus, for some perturbations of f, the number $\deg(f, G, p)$ has to be constant in a sufficiently small neighbourhood of p.

Let us consider the Euclidean space $\mathbb{R}^n$.

Definition 4.6. To any (f, G, p), where G is from the system of all bounded open subsets of $\mathbb{R}^n$, $f\colon \overline{G} \to \mathbb{R}^n$ is a continuous map, and $p \in \mathbb{R}^n \setminus f(\partial G)$, we assign the integer $\deg(f, G, p)$ satisfying the following conditions:

i) if f is the identity on G and $p \in G$, then $\deg(f, G, p) = 1$;

ii) if $G_1, G_2 \subseteq G$ are open sets satisfying $G_1 \cap G_2 = \emptyset$ and $p \notin f\left(\overline{G} \setminus (G_1 \cup G_2)\right)$, then
$$\deg(f, G, p) = \deg(f, G_1, p) + \deg(f, G_2, p);$$

iii) if $H\colon [0,1] \times \overline{G} \to \mathbb{R}^n$ is a continuous map, $f_0(x) = H(0,x)$, $f_1(x) = H(1,x)$, and if $H(t,x) \neq p$ for $t \in [0,1]$, $x \in \partial G$, then
$$\deg(f_0, G, p) = \deg(f_1, G, p);$$

iv) if $\deg(f, G, p) \neq 0$, then there exists $x \in G$ such that $f(x) = p$.

This map is called the topological degree in $\mathbb{R}^n$.

Remark 4.7. In $\mathbb{R}^n$, the map from the previous definition exists and it is determined by the conditions i)–iv) uniquely.

Before Remark 4.9 mentioned below, we recall the notion of the so-called homotopy.

Definition 4.8. Let Z and Y be metric spaces and let $f, g\colon Z \to Y$ be continuous maps. We say that f is homotopic with g if there exists a continuous map $H\colon [0,1] \times Z \to Y$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for $x \in Z$. The map H is called the homotopy between f, g.

Remark 4.9. Now, we comment the conditions i)–iv) from Definition 4.6.

The condition i) says that the equation $\mathrm{id}x = p$ has one solution $x = p$.

The condition ii) says that, if the equation $f(x) = p$ has just n_1 solutions on G_1, just n_2 solutions on G_2, and no solution on $\overline{G} \setminus (G_1 \cup G_2)$, then this equation has $n_1 + n_2$ solutions on G.

The condition iii) expresses the invariance of the topological degree with respect to homotopies.

The condition iv) says when the equation $f(x) = p$ has solutions on G.

Remark 4.10. Now, using a simple example, we explain why we cannot consider also $p \in f(\partial G)$ in Definition 4.6. Let us consider the function $f(t) = t$ on $G = (0,1) \subseteq \mathbb{R}$. If $p \in (-\infty, 0) \cup (1, \infty)$, then $\deg(f, G, p) = 0$, because the equation $f(x) = p$ has no solution from the interval $(0,1)$ for this p. Next, $\deg(f, G, p) = 1$ for $p \in (0,1)$. In any neighbourhood of $p = 0$ or $p = 1$, the topological degree $\deg(f, G, p)$ takes the both values 0 and 1. Hence, for $p \in f(\partial G) = \{0, 1\}$, $\deg(f, G, p)$ cannot be defined if the topological degree depends on p continuously.

Remark 4.11. In spaces whose dimension is infinite, one can construct the theory of the topological degree as well. It is called the Leray–Schauder degree and it is introduced for maps of the type $I - T$, where T is, e.g., a completely continuous linear operator.

Theorem 4.12. *The topological degree in $\mathbb{R}^n$ has the following properties:*

1. *if $f, g\colon \overline{G} \to \mathbb{R}^n$ are continuous maps, $f = g$ on ∂G, and $p \in \mathbb{R}^n \setminus f(\partial G)$, then*
$$\deg(f, G, p) = \deg(g, G, p);$$
2. *the map $\deg(f, G, -)$ is constant on any connected component of the open set $\mathbb{R}^n \setminus f(\partial G)$.*

4.2 Brouwer and 1. and 2. Schauder theorem

Now, we use the topological degree in the Euclidean space $\mathbb{R}^n$.

Theorem 4.13 (Brouwer)**.** *The closed unit ball $B[0,1] \subseteq \mathbb{R}^n$ has the fixed point property.*

Proof. By contradiction, we consider a continuous map $f\colon B[0,1] \to B[0,1]$ with the property that $f(x) \neq x$ for all $x \in B[0,1]$. The map

$$H(t,x) = x - tf(x), \qquad x \in \mathbb{R}^n,\ t \in [0,1],$$

is a homotopy. We show that $H(t,x) \neq 0$ for $t \in [0,1]$ and $\|x\| = 1$. For $t = 1$, it follows from the assumption. If $t \in [0,1)$, then we have the inequality $\|tf(x)\| \leq t < 1$ for $\|x\| = 1$. Therefore, $x \neq tf(x)$. We denote

$$g_0(x) = H(0,x) = x, \qquad g_1(x) = H(1,x) = x - f(x), \qquad x \in B(0,1).$$

From the condition iii) in Definition 4.6, we obtain that

$$\deg(g_0, B(0,1), 0) = \deg(g_1, B(0,1), 0).$$

But, g_0 is the identity, which gives $\deg(g_0, B(0,1), 0) = 1$. Thus,

$$\deg(g_1, B(0,1), 0) = 1$$

and the condition iv) from Definition 4.6 gives the existence of $x \in B(0,1)$ for which $g_1(x) = 0$. Of course, this is a contradiction. □

Remark 4.14. From Theorem 4.13, it follows that any compact convex subset of a Banach space with a finite dimension has the fixed point property.

Naturally, we obtain the question, whether the Brouwer theorem, i.e., Theorem 4.13, is valid also for the closed unit balls in spaces whose dimension is infinite. But, it is enough to consider, e.g., the map

$$h\colon x = \{x_1, x_2, \dots\} \mapsto \left\{ \sqrt{1 - \|x\|^2}, x_1, x_2, \dots \right\},$$

which does not have any fixed point on the closed unit ball of the space l^2 .

For spaces whose dimension is infinite, we have the following result.

Theorem 4.15 (Schauder)**.** *Let K be a (non-empty) compact convex subset of X and let $f\colon K \to K$ be a continuous map. Then, there exists $x \in K$ such that $f(x) = x$.*

Proof. We consider $\varepsilon > 0$. There exist $x_1, \dots, x_n \in K$ such that

$$K \subseteq \bigcup_{j=1}^{n} B(x_j, \varepsilon).$$

We define

$$\varphi_j(x) = \max\left\{0, \varepsilon - \left\|x - x_j\right\|\right\}, \qquad x \in K,\ j \in \{1, \dots, n\}.$$

The functions φ_j are non-negative on K and the function

$$\sum_{j=1}^{n} \varphi_j$$

is positive on K. Thus, we can define the function φ on K by

$$\varphi\colon x \mapsto \left(\sum_{j=1}^{n} \varphi_j(x) x_j\right) \left(\sum_{j=1}^{n} \varphi_j(x)\right)^{-1}.$$

Obviously, φ is a continuous function on K which maps K to the set

$$K_\varepsilon = \operatorname{conv}\{x_1, \dots, x_n\} \subseteq K,$$

where $\operatorname{conv}\{x_1, \dots, x_n\}$ is the convex hull of $x_1, \dots, x_n$. We have

$$\|\varphi(x) - x\| \le \varepsilon, \qquad x \in K.$$

The composition $\varphi \circ f$ maps K_ε into K_ε. According to Theorem 4.13 (see Remark 4.14), it has a fixed point $x_\varepsilon \in K_\varepsilon$. Since

$$\begin{aligned}\|x_\varepsilon - f(x_\varepsilon)\| &\le \|x_\varepsilon - \varphi(f(x_\varepsilon))\| + \|\varphi(f(x_\varepsilon)) - f(x_\varepsilon)\| \\ &= \|\varphi(f(x_\varepsilon)) - f(x_\varepsilon)\| \le \varepsilon,\end{aligned}$$

we have

$$\inf\{\|x - f(x)\|\,; x \in K\} = 0.$$

Considering that f is a continuous map on a compact set, we know that there exists $x \in K$ such that $f(x) = x$. □

Definition 4.16. A map $f\colon D \subseteq X \to X$ is called compact if it is continuous and if it maps any bounded subset of D into a set whose closure is a compact set.

Remark 4.17. We repeat that a linear map $f\colon X \to X$, which maps bounded sets into sets with compact closures, is continuous.

Theorem 4.18 (Schauder)**.** *Let K be a closed bounded convex subset of X and let $f\colon K \to K$ be compact. Then, f has a fixed point.*

5. Integration in Banach spaces

In this chapter, we consider a real Banach space X.

5.1 Preliminaries and basic definitions

At first, we recall basic definitions.

Definition 5.1. A system S of subsets of a given set Ω is called a σ-algebra if

a) $\Omega \in S$;

b) $A \in S \Rightarrow \Omega \setminus A \in S$;

c) $A_n \in S, n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in S$.

Then, (Ω, S) is called a measurable space.

Definition 5.2. Let S be a system of subsets of a set Ω. A map $\mu\colon S \to [0, +\infty]$ is called a measure if:

a) S is σ-algebra;

b) $\mu(\emptyset) = 0$;

c) for any sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets from S, it holds

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

We say that a measure μ is finite if $\mu(\Omega) < \infty$, and it is probability if $\mu(\Omega) = 1$. A measure μ is called complete if the implication

$$A, B \subseteq \Omega, A \subseteq B, B \in S, \mu(B) = 0 \qquad \Longrightarrow \qquad A \in S$$

is valid.

Let a measurable space (Ω, S) be given with a measure μ, where μ is a probability complete measure.

Definition 5.3. Let Y be an arbitrary set and let $A \subseteq Y$. We define the map $\chi_A\colon Y \to \mathbb{R}$ by

$$\chi_A(y) = \begin{cases} 1 & \text{for } y \in A; \\ 0 & \text{for } y \notin A. \end{cases}$$

Definition 5.4. A map $f\colon \Omega \to X$ is called

- simple if there exist $x_1,\dots,x_n \in X$ and $E_1,\dots,E_n \in S$ such that

$$f \equiv \sum_{i=1}^{n} x_i \chi_{E_i};$$

- measurable if there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of simple maps such that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega)$$

  for all $\omega \in \Omega$ up to a set whose measure is zero;

- weakly measurable if $\varphi \circ f$ is a measurable function for all $\varphi \in X'$.

Remark 5.5. It is easy to show that $f_1 + f_2$ and λf_1 are measurable if f_1, f_2 are measurable and $\lambda \in \mathbb{R}$ (analogously, in the weakly measurable case).

Theorem 5.6 (Pettis)**.** *A map $f\colon \Omega \to X$ is measurable if and only if f is weakly measurable and there exists a set $E \in S$ with $\mu(E) = 0$ such that $f(\Omega \setminus E)$ is a separable subset of X. Especially, for separable Banach spaces, the notions of measurable and weakly measurable maps are same.*

Definition 5.7. If x_n (for $n \in \mathbb{N}$) are elements of the Banach space X, then we say that the series $\sum_{i=1}^{\infty} x_i$

- converges if there exists

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i;$$

- converges absolutely if

$$\sum_{i=1}^{\infty} \|x_i\| < +\infty;$$

- converges unconditionally to $x \in X$ if

$$\sum_{i=1}^{\infty} x_{p(i)} = x$$

  for all permutation p of $\mathbb{N}$.

Remark 5.8. Due to the completeness of X, any absolutely convergent series is also unconditionally convergent. If the dimension of X is finite, then any unconditionally convergent series converges absolutely as well. In spaces whose dimension is infinite, it is not valid. It suffices to consider

$$x_n = \left\{0,\dots,0,\frac{1}{n},0,\dots\right\} \in c_0, \qquad n \in \mathbb{N},$$

where c_0 is the space of all sequences $\{x_n\}_{n=1}^{\infty}$ of real numbers which converge to zero with the norm

$$\|x\| = \max_{n\in\mathbb{N}} |x_n|.$$

Definition 5.9. Let $F\colon S \to X$. We say that F is an additive or σ-additive vector measure if $F(\emptyset) = 0$ and

$$F\left(\bigcup_n E_n\right) = \sum_n F(E_n)$$

for all finite or countable sequence of disjoint sets $E_n \in S$, respectively.

Remark 5.10. The convergence in Definition 5.9 is the convergence of series in X (in the case of the considered σ-additivity). This convergence is unconditional.

Definition 5.11. We say that a vector measure $F\colon S \to X$ is absolutely continuous with respect to the measure μ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|F(E)\| < \varepsilon$ if $\mu(E) < \delta$.

Theorem 5.12 (Pettis)**.** *Let F be a σ-additive vector measure. Then, F is absolutely continuous with respect to the measure μ if and only if $F(E) = 0$ for $\mu(E) = 0$.*

Definition 5.13. Let $F\colon S \to X$ be a vector measure. If

$$|F(\Omega)| = \sup\left\{\sum_{k=1}^{n} \|F(A_k)\|\,; A_k \in S \text{ are pairwise disjoint}, \bigcup_{k=1}^{n} A_k = \Omega\right\} < +\infty,$$

then we say that F has a bounded variation.

5.2 Bochner integral

Definition 5.14. A map $f\colon \Omega \to X$ is called integrable in the Bochner sense if f is measurable and if there exist simple maps f_n for $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \int_{\Omega} \|f - f_n\| \,\mathrm{d}\mu = 0.$$

One can show the following implication. If

$$\lim_{n\to\infty} \int_{B} \|f - f_n\| \,\mathrm{d}\mu = 0 = \lim_{n\to\infty} \int_{B} \|f - g_n\| \,\mathrm{d}\mu$$

for a measurable map f, simple maps f_n, g_n, $n \in \mathbb{N}$, and $B \in S$, then

$$\lim_{n\to\infty} \int_{B} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{B} g_n \,\mathrm{d}\mu \in X,$$

where we put

$$\int_{B} \varphi \,\mathrm{d}\mu = \sum_{i=1}^{p} x_i \mu\left(E_i \cap B\right)$$

for a simple map

$$\varphi \equiv \sum_{i=1}^{p} x_i \chi_{E_i},$$

The previous implication guarantees the correctness of the following definition.

Definition 5.15. Let f be integrable in the Bochner sense, let $B \in S$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of simple maps satisfying

$$\lim_{n\to\infty} \int_B \|f - f_n\| \, d\mu = 0.$$

The limit

$$\lim_{n\to\infty} \int_B f_n \, d\mu \in X$$

is called the Bochner integral of f over B and it is denoted by

$$\int_B f \, d\mu.$$

Theorem 5.16 (Bochner)**.** *Let $f : \Omega \to X$ be measurable. Then, f is integrable in the Bochner sense if and only if*

$$\int_\Omega \|f\| \, d\mu < +\infty.$$

As L_X^1 (or $L_X^1(\Omega, S, \mu)$), we denote the space of all maps which are integrable in the Bochner sense.

Theorem 5.17. *For $f \in L_X^1$ and $E \in S$, the inequality*

$$\left\| \int_E f \, d\mu \right\| \leq \int_E \|f\| \, d\mu$$

is valid.

Theorem 5.18. *Let us consider the identification of maps which are different on sets with zero measures. Then, L_X^1 is a Banach space with the norm*

$$\|f\|_{L_X^1} = \int_\Omega \|f\| \, d\mu.$$

Definition 5.19. The indefinite Bochner integral $F : S \to X$ is defined by

$$F(E) = \int_E f \, d\mu,$$

where $E \in S$.

Theorem 5.20. *Let a map $f : \Omega \to X$ be integrable in the Bochner sense. The indefinite Bochner integral is a σ-additive vector measure, which is absolutely continuous with respect to μ. If $E_n \in S$ are pairwise disjoint, then the series*

$$\sum_n F(E_n)$$

converges absolutely.

5.3 Gelfand, Pettis, and Darboux integral

Lemma 5.21. *Let $f\colon \Omega \to X$. If $\varphi \circ f \in L^1_{\mathbb{R}}$ for all $\varphi \in X'$, then, for any $E \in S$, there exists $L_E \in X''$ such that*

$$L_E(\varphi) = \int_E \varphi \circ f \, \mathrm{d}\mu$$

for all $\varphi \in X'$.

Definition 5.22. Let $f\colon \Omega \to X$. If $\varphi \circ f \in L^1_{\mathbb{R}}$ for all $\varphi \in X'$, then we say that f is weakly integrable. The element $L_E \in X''$, whose existence is guaranteed by Lemma 5.21, is called the Gelfand integral of f over E. Similarly as for the Bochner integral, one can introduce the indefinite Gelfand integral, which maps $E \in S$ into $L_E \in X''$.

In addition, if $L_E \in X$ for all $E \in S$, i.e., there exists $p_E \in X$ such that $L_E(\varphi) = \varphi(p_E)$ for all $\varphi \in X'$, then we say that f is integrable in the Pettis sense. The element $p_E \in X$ is called the Pettis integral of f over E. The indefinite Pettis integral maps $E \in S$ into $p_E \in X$.

Remark 5.23. If X is reflexive, then the Gelfand integral and the Pettis integral are identical.

Theorem 5.24. *Let a map $f\colon \Omega \to X$ be weakly integrable. Then, the following statements are equivalent:*

i) f is integrable in the Pettis sense;

ii) the indefinite Gelfand integral of f is a σ-additive vector measure;

iii) the indefinite Gelfand integral of f is absolutely continuous with respect to μ.

Remark 5.25. If a map f is integrable in the Bochner sense, then f is integrable in the Pettis sense as well. More specifically, the relationship between the “strong” integral and the “weak” integral is described in the following theorem.

Theorem 5.26. *Let $f\colon \Omega \to X$ be measurable and integrable in the Pettis sense. Then, f is integrable in the Bochner sense if and only if the indefinite Pettis integral of f is a vector measure having a bounded variation.*

Theorem 5.27. *Let K be a compact subset of X and let μ be a probability complete measure on K. If $f\colon K \to X$ is continuous, then there exists the Pettis integral of f over K.*

Definition 5.28. Let a map $f\colon [0,1] \to X$ be given. Let

$$D = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$$

be a partition of the interval $[0,1]$ and let

$$\mathscr{D}(f,D) = \sum_{i=1}^{n} \sup\{\|f(s) - f(t)\|\,; s,t \in [x_{i-1},x_i]\} \cdot (x_i - x_{i-1})\,.$$

We say that f is integrable in the Darboux sense if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathscr{D}(f,D) < \varepsilon$ if the norm of the partition D is less than δ.

Theorem 5.29. *A map $f\colon [0,1] \to X$ is integrable in the Darboux sense if and only if f is a bounded map which is continuous on $[0,1]$ up to a set whose Lebesgue measure is zero.*

Theorem 5.30. *Any map $f\colon [0,1] \to X$ integrable in the Darboux sense is integrable in the Bochner sense.*

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