Biholomorphisms

Tuesday, December 7, 2021 8:03 AM

Def: A map $F: \Omega_1 \to \Omega_2$, $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ is a **Biholomorphism**, if:

(i) $F \in \mathcal{O}(\Omega_1)$

(ii) F is a Bijection $\Omega_1 \leftrightarrow \Omega_2$

(iii) $\mathrm{Jacob}\, F \neq 0$ in Ω_1 (can be in fact dropped...)

$\exists$ always $F^{-1}: \Omega_2 \to \Omega_1$ - inverse Bihol

Compos of Bihol-s is again a Bihol

Def: $\mathrm{Aut}(D)$, $D \subset \mathbb{C}^n$ - the group of Bihol: $D \circlearrowleft D$

Topology: (open-compact topology): the topology of normal convergence

H. Cartan Thm: if D - **Bounded** domain in $\mathbb{C}^n$, then $\mathrm{Aut}(D)$ - a **finite-dim** Lie group in the open-compact topology. Furthermore,

$S_p = \{F \in \mathrm{Aut}(D) : F(p) = p\} \subset \mathrm{Aut}(D)$ is a **compact** Lie subgroup

$p \in D$

Actually $S_p \hookrightarrow U(n)$, $\underset{\in S_p}{F} \to dF|_p$ - faithful representation in $U(n)$

stability group

stability group of a point $\qquad S_p \qquad$ in $U(n)$

$$\dim \operatorname{Aut}(D) \le \underset{\dim U(n)}{\underset{\|}{n^2}} + \underset{\dim D}{\underset{|}{2n}}$$

Remark: if D is unbounded, but $D \overset{\text{bihol}}{\rightsquigarrow} \Omega$ - bounded, then the same holds

D — a domain of bounded kind.

Big difference with $n=1$: $\operatorname{Aut}(\mathbb{C}^n)$ - ∞-dim, and is actually really weird and complicated:

Ex: $\mathbb{C}^2$; $F(z_1, z_2) \to (z_1, z_2 + \varphi(z_1))$, $\varphi \in \mathcal{O}(\mathbb{C}^1)$

$\forall$ such F is a bihol of $\mathbb{C}^2$, $\forall \varphi$!

$J \equiv 1$

Fact: $\exists$ domains in $\mathbb{C}^n$: $\mathbb{C}^n \setminus \Omega \neq \emptyset$ (and open)

and $\Omega \overset{\text{bihol}}{\rightsquigarrow} \mathbb{C}^n$

$\overline{\Omega} \neq \mathbb{C}^n$!

Ω: Fatou domain

In particular: Picard Thm and Sohotski-Wierstrass Thm both fail for $n > 1$!

$\operatorname{Aut}(B_1)$: there is large and transitive group of projective aut-s of B_1.

consider $B_1 \subset \mathbb{C}P^n$; $[\xi_0, \xi_1, \ldots, \xi_n]$; $z_j = \frac{\xi_j}{\xi_0}$

Consider $B_1 \subset \mathbb{C}^n$, $[\,\dots, z_n\,]$, ξ_0

B_1: $|z_1|^2 + \ldots + |z_n|^2 < 1 \iff -|\xi_0|^2 + |\xi_1|^2 + \ldots + |\xi_n|^2 < 0$

- „cone" in $\mathbb{C}P^n$

Now, take proj. trans-s generated by $U(n,1)$

Max possible!

$\xi \to A\xi$, $A \in U(n,1)$ $\quad \boxed{\dim = n^2 + 2n}$

The actual group is $SU(n,1)/$finite subgroup

the Lie algebra: $su(n,1)$

This group acts on the cone transitively:

$\exists \xi, \tilde{\xi}$ – in the cone $\exists A$: $\tilde{\xi} = A\xi$ (any two points can be mapped onto each other by an appr aut). – why?

rescale both to have $-|\xi_0|^2 + |\xi_1|^2 + \ldots + |\xi_n|^2 = -1$

$\exists A$: $\tilde{\xi} = A\xi$ (actually n unitary vectors can be transformed)

Conclusion: $\exists$ a proj group, acting on B_1 transitively

P_1: $P_1 = B_1 \times \ldots \times B_1$, $B_1 \subset \mathbb{C}^1$ (each factor). Make use of $G = \mathrm{Aut}(B_1) =$ ($\circlearrowleft$)

$$= \left\{ \frac{z - a}{1 - \bar{a} z} \cdot e^{i\varphi} \right\} \quad \varphi \in \mathbb{R},\ |a| < 1$$

So, we take: $G \times \ldots \times G \times S_n$ – permutation group

$$(z_1, \ldots, z_n) \to \left(\frac{z_{\sigma(1)} - a_1}{1 - z_{\sigma(1)} \bar{a}_1} e^{i\varphi_1}, \ldots, \frac{z_{\sigma(n)} - a_n}{1 - z_{\sigma(n)} \bar{a}_n} e^{i\varphi_n} \right)$$

$\sigma \in S_n$

In, partic, $\sigma = \mathrm{Id}$ already provides transitivity

Let's call this group „the group of linear –

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-fract aut of P_1"

$\boxed{\dim = 3n}$

<u>Goal</u>: prove that these are the <u>whole</u>
$\mathrm{Aut}(B_1)$, $\mathrm{Aut}(P_1)$

<u>Schwartz Lemma</u>: Let $\|\cdot\|^1$, $\|\cdot\|^2$ – two norms
in $\mathbb{C}^n$, $\|\cdot\|^2$ – <u>strictly</u> convex, B^1, B^2 – unit balls.
Let $F: B^1 \to B^2$ – hol map, $F(0)=0$. Then:
$\|F(z)\|^2 \le \|z\|^1 \quad \forall z \in B_1$.

<u>Proof</u>: Pick a cx line $L \subset \mathbb{C}^n$, $L \ni 0$. $L = \{\xi \cdot a\}$,
$\xi \in \mathbb{C}^1$ – arbitrary, $\|a\|^1 = 1$; now,

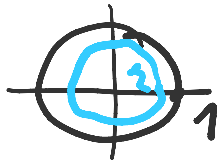
$$L \cap B^1 = \{|\xi| < 1,\ z = a\xi\}.$$

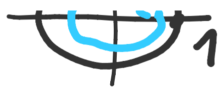
Now, $\underset{F(\xi)}{\underset{\|}{F|_L}} : \underset{\mathbb{C}^1}{\underset{\cap}{B_1}} \to B^2$; $F(0) = 0$.

$$F = \sum_{j=0}^{\infty} P_j(z), \quad P_j \text{ – homog of deg} = j; \quad P_0 = 0.$$

$$F|_L = \sum_{j=1}^{\infty} P_j(a)\xi^j, \quad \xi \in \mathbb{C}. \Rightarrow \text{consider}$$

$\frac{F(\xi)}{\xi} \in \mathcal{O}(B_1)$. Apply the max princ for
$\frac{F(\xi)}{\xi}$ in $B_r(0) \subset \mathbb{C}$, $r < 1$.



ξ in $B_r(0) \subset \mathbb{C}$, $r < 1$. 

$\|\cdot\|^2$ - strictly convex $\Rightarrow \max\limits_{|\xi|\le r} \dfrac{\|F(\xi)\|^2}{|\xi|} = \max\limits_{|\xi|=r} \dfrac{\|F(\xi)\|^2}{|\xi|} =$

$= \frac{1}{r} \max\limits_{|\xi|=r} \|F\|^2 \le \frac{1}{r}$ $\quad (F(B^1) \subset B^2)$.

$\Rightarrow$ make $r \to 1$: $\max\limits_{|\xi|<1} \dfrac{\|F(\xi)\|^2}{|\xi|} \le 1 \Rightarrow$

$\|F(z)\|^2 \le |z|^1$, $z = a\xi$ $(\Rightarrow \|z\|^1 = |\xi|)$ $\forall z \in L$.

Since L - arbitrary $\Rightarrow \|F(z)\|^2 \le \|z\|^1$, $\forall z \in B^1$ $\blacksquare$

__Thm 1__: $\mathrm{Aut}(B_1)$ is the above transitive group of its projective autom. $(B_1 \subset \mathbb{C}^n)$.

__Proof__: $F \in \mathrm{Aut}(B_1)$; $F(0) = a$; because of transit.,

$\exists \underset{\text{-proj aut}}{\psi}$: $\psi(a) = 0 \Rightarrow$ switch to $G = \psi \circ F$, $G(0) = 0$,

enough to prove that G is proj. $\qquad \|\cdot\| = \text{Eucl.}$

$G(0) = 0 \Rightarrow$ use the Schwarz Lemma:

$\|G(z)\| \le \|z\|$; then Schwarz Lemma for G^{-1}:

$\|G^{-1}(w)\| \le \|w\| \Leftrightarrow \|z\| \le \|G(z)\| \Rightarrow \|G(z)\| = \|z\|$

$G(z) = \sum\limits_{j=0}^{\infty} P_j(z)$, P_j - homog of deg $= j$; $P_0 = 0$

L - cx line through 0; $L = \{a\xi\}_{\xi \in \mathbb{C}}$, $\|a\| = 1$

$G|_L = G(\xi) = \sum\limits_{j=1}^{\infty} P_j(a)\xi^j$ $\qquad L \cap B_1 = \{|\xi| < 1\}$

$G : \underset{\mathbb{C}}{\underset{\cap}{\overset{\text{unit disc}}{\mathbb{D}}}} \to B_1$; $\quad \|G(z)\| = \|z\|$; for $z = a\xi$: $\|G(\xi)\| = |\xi| \Rightarrow$

$\Rightarrow \left\| \frac{G(\xi)}{\ldots} \right\| = \ldots$

$\ldots = 1$, $\|G(w)\| = \|z\|$, for $z = \ldots$: $\|G(\ldots)\| = |\ldots| \Rightarrow$ [illegible] $\mathbb{C}$

$\Rightarrow \left\| \frac{G(\xi)}{\xi} \right\| \equiv 1 \Rightarrow$ By the Max Princ ($\|\cdot\|$ – strictly convex): $\frac{G(\xi)}{\xi} = \text{const} \Rightarrow$

$\Rightarrow G(\xi) = \alpha \cdot \xi \Rightarrow P_j(a) = 0$, $j \geq 2$

So, $\forall a : \|a\| = 1$, $\forall j \geq 2$, $P_j(a) = 0$;

P_j – homog. $\Rightarrow$ $P_j(a) = 0$ $\forall a \in \mathbb{C}^n \Rightarrow$

$G(z) = P_1(z)$ – lin. map $\Rightarrow$ a proj. aut. ☒

Thm 2: $\mathrm{Aut}(P_1)$ is the above transitive group of lin-frac. aut-s.

Proof: similarly to Thm 1, we use transitivity and reduce to the case $F(0) = 0$

$F = (f_1, f_2, \ldots, f_n)$; $\forall f_j$ can be treated as a map $P_1 \to B_1 \subset \mathbb{C}$ $\quad P_1 = \{ |z_j| < 1 \ \forall j \}$

$f_j(0) = 0$; $\Rightarrow$ can apply the Schwarz lemma:

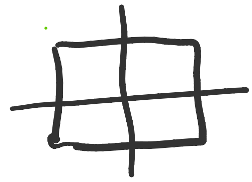
Let $|f_j(z)| \leq \|z\|_\infty$; $\forall j \Rightarrow \|F(z)\|_\infty \leq \|z\|_\infty$

Simil $\|F^{-1}(w)\|_\infty \leq \|w\|_\infty \Leftrightarrow \|z\|_\infty \leq \|F(z)\|_\infty$

$\Rightarrow \boxed{\|F(z)\|_\infty = \|z\|_\infty}$

$\ldots$

$\Rightarrow \boxed{\|F(z)\|_\infty = \|z\|_\infty}$



$\forall j$, $\exists$ open set wher $|z_j|$ realizes $\|z\|_\infty$

F - aut $\Rightarrow$ $\exists$ open U: on U, $\|F(z)\| = |f_j(z)|$

Within U, find open V: $\|z\|_\infty = |z_k|$

$\Rightarrow$ from $\|F(z)\|_\infty = \|z\|_\infty$, we have on V:

$|F_j(z)| = |z_k| \Rightarrow \left|\frac{f_j(z)}{z_k}\right| \equiv 1$ on $V \Rightarrow$

(Homework - use Cauchy-Riemann cond) hol Func in $\mathbb{C}^n$ with $|\cdot| \equiv$ const is a const $\Rightarrow \frac{f_j(z)}{z_k} = e^{i\varphi_{kj}} \Rightarrow \boxed{F_j(z) = e^{i\varphi_{kj}} \cdot z_k}$

linear!

j was arbitrary $\Rightarrow$ all comp-s of F are linear $\Rightarrow$ F is linear. $\blacksquare$

Thm (Anti-Riemann Thm)

$B_1 \not\simeq P_1 \quad (n>1)$

$\dim \mathrm{Aut}(B_1) = n^2+2n \qquad \dim \mathrm{Aut}(P_1) = 3n$

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$n^2+2n > 3n$, $n > 1$.

(even though both are bounded and topol-ly identical domain)

No Riem Map, $n > 1$.