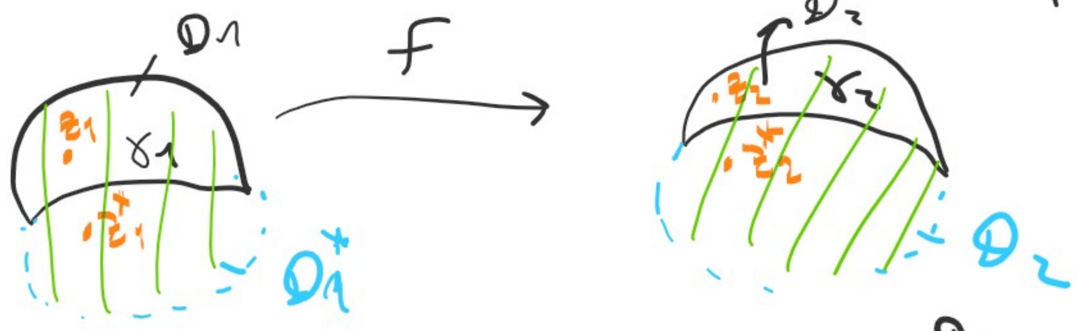
Schwarz Refl Principle



Proof: Main idea: let $\underset{\in \mathcal{D}_1}{z_1} \xrightarrow{f} \underset{\in \mathcal{D}_2}{z_2}$

Now, by defin, we set: $\boxed{F(z_1^*) = z_2^*}$

Refl map is contin and biject $\Rightarrow$ from our condit-s we conclude, that such an extended map is a homeom:

$\Omega_1 \to \Omega_2$; Now:

Q1: Why F is hol in $\mathcal{D}_1^*$?

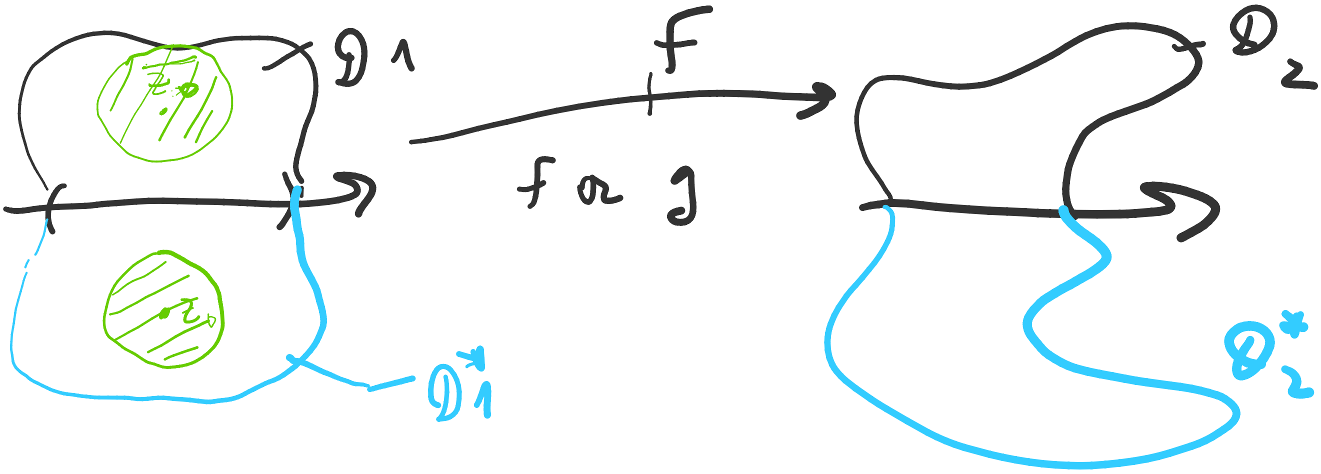
Q2: Why F is hol near γ_1?

For Q1: let $\gamma_1 \xrightarrow{\varphi} (\alpha_1, \beta_1) \subset \mathbb{R}$
$\gamma_2 \xrightarrow{\Psi} (\alpha_2, \beta_2) \subset \mathbb{R}$

By switching to the map $g := \Psi \circ f \circ \varphi^{-1}$

By switching to the map $g := \varphi^{-1} \circ f \circ \psi$ ($f = \varphi \circ g \circ \psi^{-1}$), we arrive to all the same picture, but $\gamma_1, \gamma_2 \subset \mathbb{R}$

$\varphi, \psi \in \mathrm{Aut}(\overline{\mathbb{C}}) \Rightarrow$ prove for g = prove for f



Now, reflection is just linear: $z^* = \bar{z}$

This means that we extended g like that: $g(\bar{z}) := \overline{g(z)}$

(for $z \in \mathcal{D}_1$, $\overline{g(z)} = g(\bar{z})$)

Take $z_0 \in \mathcal{D}_1^*$, near $\bar{z}_0$, we have: $\to z \in B_\varepsilon(\bar{z}_0)$

$$g(z) = \sum_{n=0}^{\infty} c_n (z - \bar{z}_0)^n \Rightarrow$$

$$\overline{g(\bar{z})} = \sum_{n=0}^{\infty} \bar{c}_n (z - z_0)^n$$ — but this power series has the

$g(z) = \sum_{n=0} c_n (z - z_0)^n$ ... $(z \in B_2(z_0))$

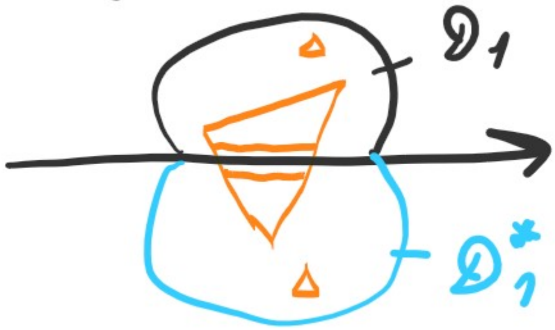
series has the same rad of conv

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$$

$$\Rightarrow \overline{g(\bar z)} \in \mathcal{O}(B_2(z_0)) \Rightarrow$$

the extended g is hol in $D_1^* \Rightarrow Q1$ ✓

For Q_2:



$g \in \mathcal{O}(D_1)$
$g \in \mathcal{O}(D_1^*)$

$g \in C(\underbrace{D_1 \cup D_1^* \cup \gamma_1}_{\Omega_1})$

We will prove that from here $g \in \mathcal{O}(\Omega_1)$

It's enough to prove that $\forall \bar\Delta \subset \Omega_1$, $\int_{\partial\Delta} g(\xi)d\xi = 0$; if $\bar\Delta \subset D_1$ or $\bar\Delta \subset D_1^* \Rightarrow$

$\int_{\partial\Delta} g(\xi)d\xi = 0$ from $g \in \mathcal{O}(D_1)$, $g \in \mathcal{O}(D_1^*)$

Finally, let $\bar\Delta \cap \gamma_1 \neq \emptyset$;

$$\int_{\partial\Delta} g(\xi)d\xi = \int_{\substack{\text{quadrangle}\\ \cap \\ D_1}} + \int_{\substack{\text{triangle}\\ \cap \\ D_1^*}} + \int_{\substack{\text{quadrangle}\\ \text{,,small''}}}$$

$\partial D \qquad \ldots \qquad \ldots \qquad$ quadrangle

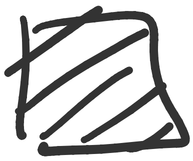
$\qquad \wedge \qquad\qquad \wedge$

$\qquad \partial_1 \qquad\qquad D_1^* \qquad\qquad$ „small"

$= 0 =$ from $g \in \mathcal{O}(D_1),\ g \in \mathcal{O}(D_1^*)$

$$\Rightarrow \int_{\partial D} g\,d\xi = \int_{\text{„small" quadrangle}} \longrightarrow 0 \Rightarrow$$

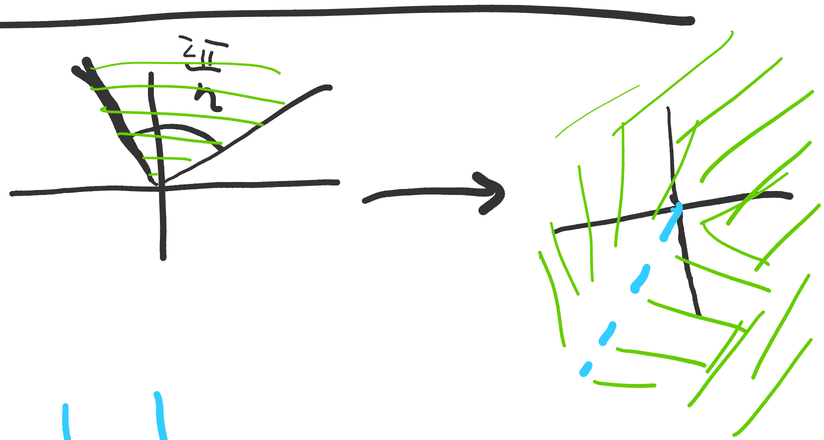
$$\int_{\partial D} g(\xi)\,d\xi = 0 \Rightarrow g \in \mathcal{O}(\Omega_1).$$



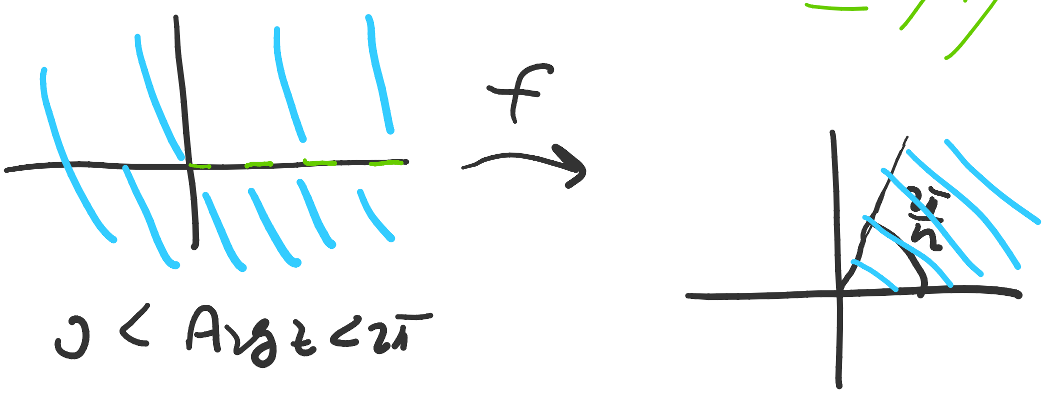
Q2 ✓

Rimender – standard conf maps

1) $z^n,\ n \geq 1$



2) $\sqrt[n]{z}$

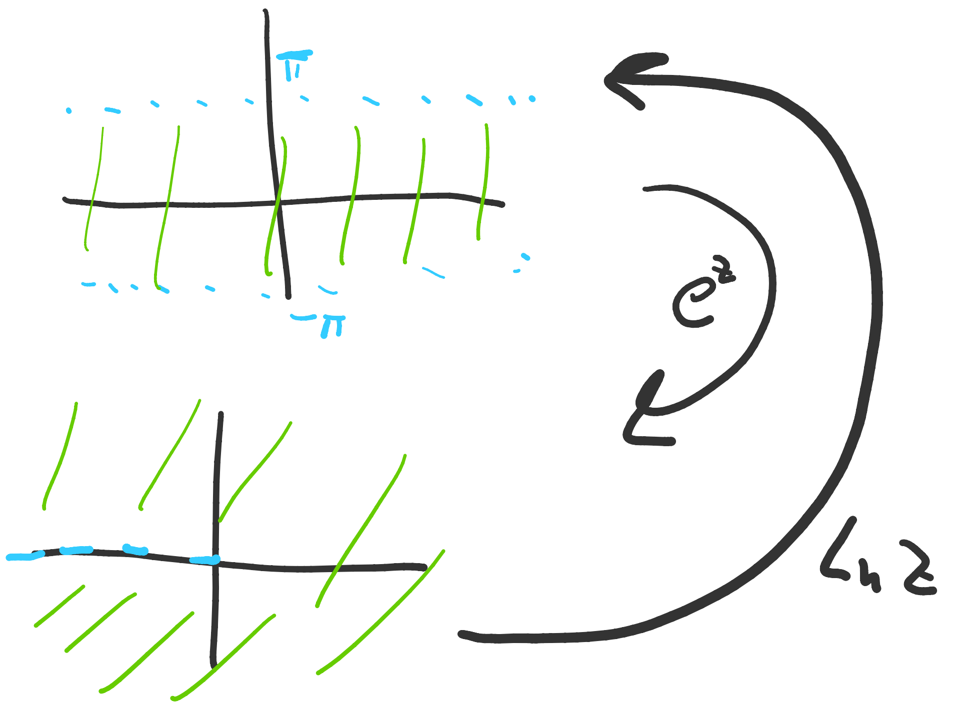


$0 < \operatorname{Arg} z < 2\pi$

3) $w = e^z$

$|e^z| = e^x$

$\operatorname{Arg} e^z = y + 2\pi k$



4) New conf map: Zhukovsky transform

$\text{Ж}(z) := \frac{1}{2}\left(z + \frac{1}{z}\right)$; $\text{Ж}(0) = \infty$, $\text{Ж}(\infty) = \infty$

$\text{Ж}\left(\frac{1}{z}\right) = \text{Ж}(z)$

$\exists$ Two classical „maximal" domains of conformality:

$\boxed{1}$ $\mathcal{D} = \mathbb{C} \setminus \overline{B_1} = \{z : |z| > 1\}$

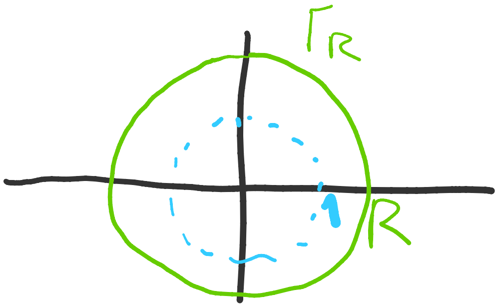
$\text{Ж} \in \mathcal{O}(\mathcal{D})$; it is injective in $\mathcal{D}$:

$\text{Ж}(z_1) = \text{Ж}(z_2) \Leftrightarrow z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2}$; $\Leftrightarrow$

$\text{Ж}(z_1)=\text{Ж}(z_2) \Leftrightarrow z_1+\frac{1}{z_1}=z_2+\frac{1}{z_2}$; $\Leftrightarrow$

$(z_1-z_2)=\frac{z_1-z_2}{z_1 z_2}$; if $z_1 \neq z_2 \Rightarrow z_1 z_2 = 1$ — impossible if $z_1, z_2 \in D$; so, $\underline{z_1 = z_2}$.

What is $\text{Ж}(D)$?



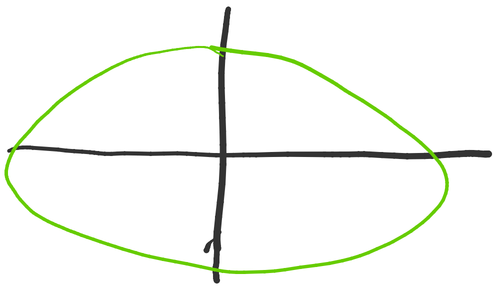
$\text{Ж}(\Gamma_R) = ?$

$z \in \Gamma_R \Rightarrow z = Re^{it}$;

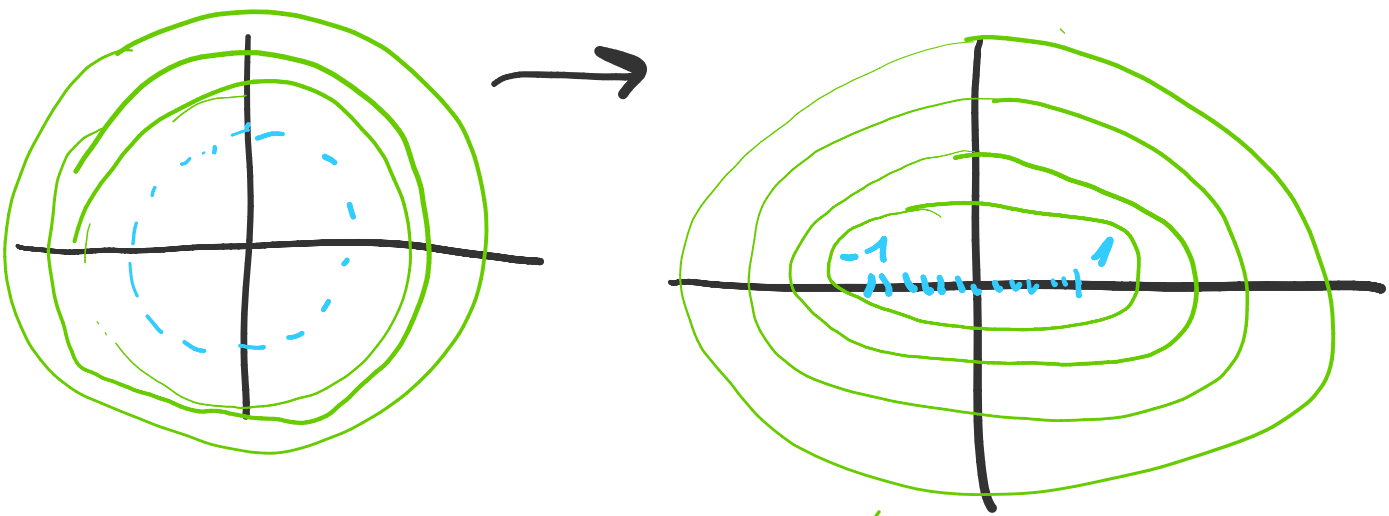
$$\text{Ж}(z) = \frac{1}{2}\left(R+\frac{1}{R}\right)\cos t + \frac{1}{2}\left(R-\frac{1}{R}\right)\sin t \cdot i$$

(R - fixed, $0 \le t < 2\pi$)

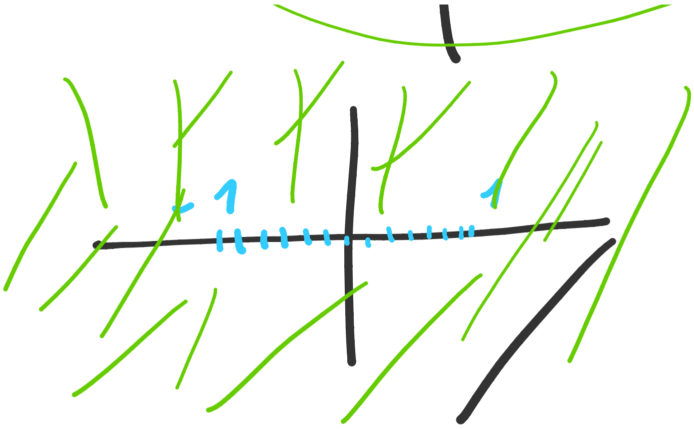
$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$



ellipse!



$\Rightarrow \mathcal{X}(\mathcal{D}) =$
$= \mathbb{C} \setminus [-1, 1]$

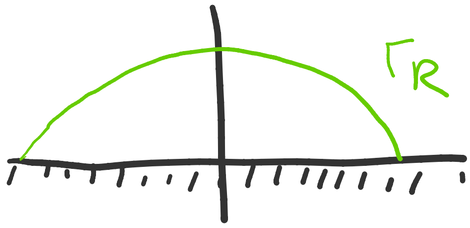


Alternatively, one takes $G = B_1 \setminus \{0\}$

$\boxed{2}$ $\mathcal{D} = \Pi^+ = \{ \operatorname{Im} z > 0 \}$

injectivity: similarly

Γ_R – semicircles

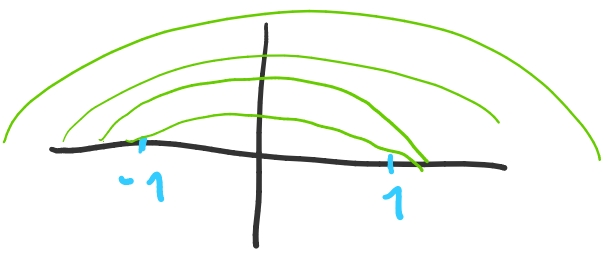


$$\mathcal{X}(\Gamma_R): \quad \begin{cases} x = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos t \\ y = \frac{1}{2}\left(R - \frac{1}{R}\right)\sin t \end{cases}$$

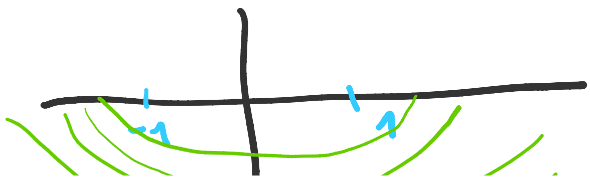
$0 < t < \pi$

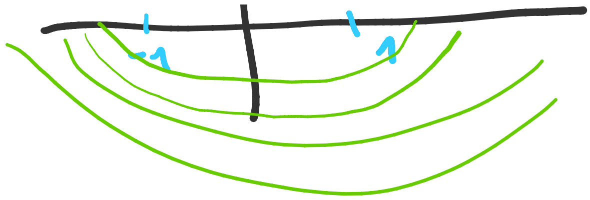
$R = 1$: $\mathcal{X}(\Gamma_R) = (-1, 1)$

$R > 1$:

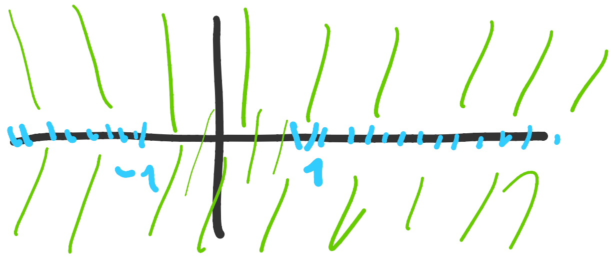


$R < 1$:





Union: $\text{Ж}(\mathcal{D}) = \mathbb{C} \setminus \big((-\infty, -1] \cup [1, +\infty) \big)$

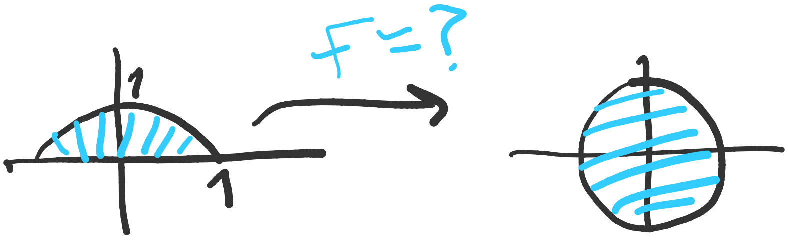


We as well consider $\text{Ж}^{-1}(z)$

in either $\left[\begin{array}{l} \mathbb{C} \setminus [-1, 1] \\ \mathbb{C} \setminus \big((-\infty, -1] \cup [1, +\infty) \big) \end{array} \right.$

$$\text{Ж}^{-1}(z) = z + \sqrt{z^2 - 1}$$

1) Construct a conf map of the half disc $\mathcal{D} = \{ |z| < 1, \operatorname{Im} z > 0 \}$ onto B_1.



$\mathcal{D} = \bigcup_{0 < R < 1} \Gamma_R$, $\Gamma_R = \{ |z| = R, \operatorname{Im} z > 0 \}$ $\Rightarrow$

... the proof of Ж in Π^+:

arguing like in the proof of $\mathcal{K}$ in Π^+:

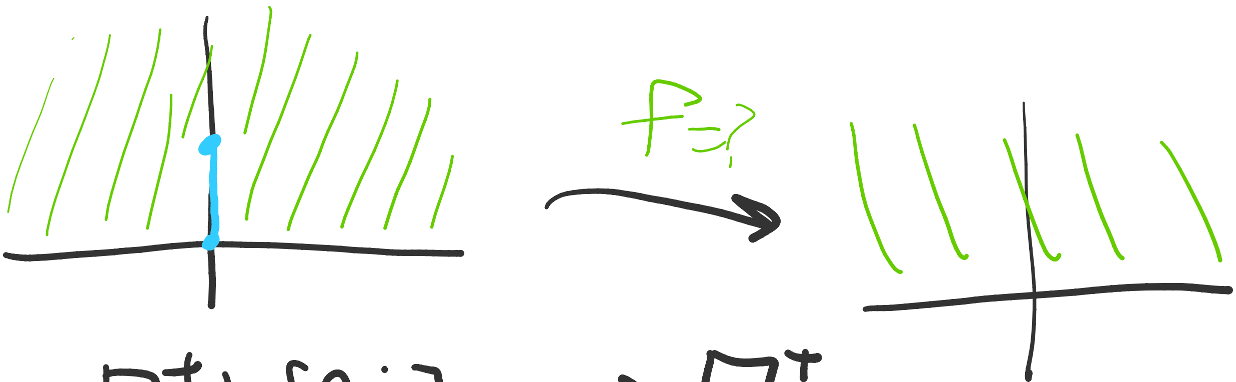
$\mathcal{K}(\mathcal{D}) = \Pi^-$ ($\mathcal{K}$ - conf in $\Pi^+ \supset \mathcal{D}$)

Then apply $z \to -z$; get Π^+;

finally apply $z \to \frac{z-i}{z+i}$

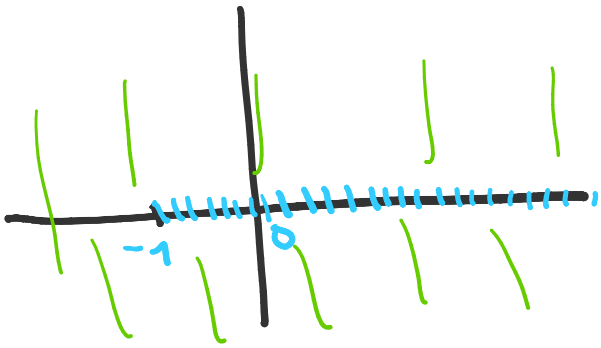
$$f(z) = \frac{-\mathcal{K}(z) - i}{-\mathcal{K}(z) + i}$$

2)



$f: \Pi^+ \setminus [0,i] \longrightarrow \Pi^+$

Step I: $z \to z^2$

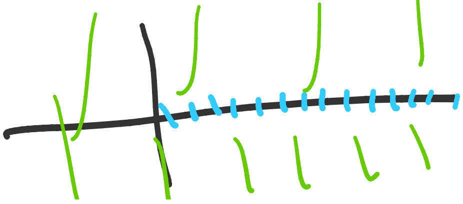


$\Pi^+ \xrightarrow{z^2} \mathbb{C} \setminus [0,+\infty)$

$[0,i] \xrightarrow{z^2} [-1,0]$

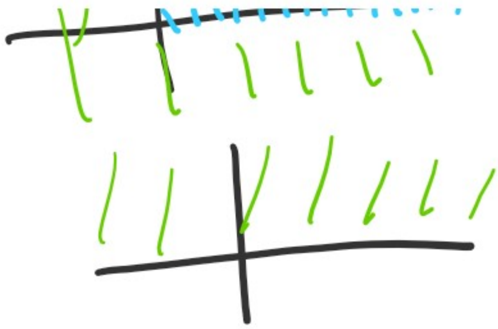
$\Rightarrow \mathcal{D} \xrightarrow{z^2} \mathbb{C} \setminus [-1,+\infty)$

Step II: $z \to z+1 \Rightarrow$



Step III: $z \to \sqrt{z}$

$0 < \operatorname{Arg} z < 2\pi$



Done! $f(z) = \sqrt{z^2+1}$

Def: Let (a, F_a) be an element. Let $\gamma(t): [0,1] \to \mathbb{C}$ be a path, $\gamma(0) = a$.

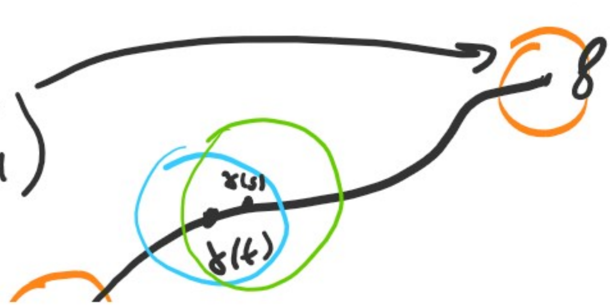
$(B_r(a), F_a)$

We say that the element $(B_r(a), F_a)$ *extends along* γ, if $\forall t$, there is an element $(B_{r(t)}(\gamma(t)), F_t)$ and:

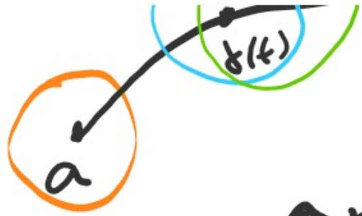
1) $F^0 = F_a$

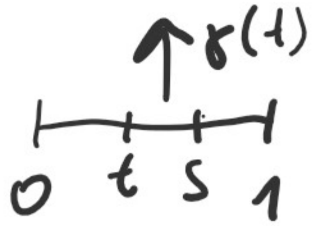
2) $\forall \varepsilon > 0$, $\exists \delta > 0$: if $|t-s| < \delta$, then $\gamma(s) \in B_\varepsilon(\gamma(t))$, and furthermore, the s-element is a direct anal contin. of the t-element.

The element $(B_{r(1)}(b), F_1)$ is called the $^{(a,\gamma)}$ anal



is called the (an) anal extension of the original element along γ.





Theorem: analytic extension of an element along a path is unique, i.e independent of the choice of $\{(B_{r(t)}, F_t)\}_{t\in[0,1]}$ (actually the family itself is unique)

Proof: By contrad, assume $\exists (B_{r(t)}, F_t)$, $(B_{R(t)}, G_t)$, $F_1 \neq G_1$. ($F_0 = G_0$ by def)

First note that for t close to 0, both F_t and G_t are direct anal cont of $F_0 = G_0$ ($|t|<\delta$)

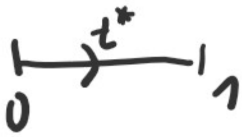
($\varepsilon = R(0) = r(0)$, take δ, if t is δ-close:

$F_t = F_0 = G_0$ in the inters, same for $G_t \Rightarrow F_t = G_t$ as direct extensions.

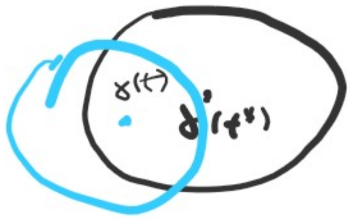
Take $\sup\{t: F_s = G_s \ \forall s \leq t\} = t^* > 0$

as the above ... $F \neq G$ ... t^*

By the above $F_{t^*} \neq G_{t^*}$



But: $\varepsilon = \min\{R(t^*), r(t^*)\}$, take resp. δ,
take $t < t^*$ and $|t - t^*| < \delta$,



Recall $F_t = G_t$

F_{t^*} is dir ext of $F_t = G_t$

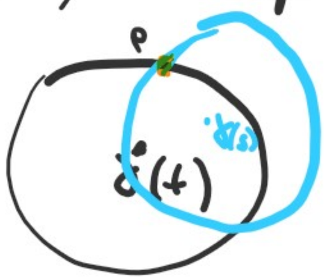
G_{t^*} — // — $G_t = F_t$ $\Rightarrow F_{t^*} = G_{t^*}$ — contrad.

∎

In fact, extension along path is extension along appropr. chain:

<u>Lemma</u>: Let $\{(B_{R(t)}(\gamma(t)), F_t)\}$ gives the extension of the $(B_{R(0)}(\gamma(0)), F_0)$ along a path γ. Then $R(t)$ is contin on $[0,1]$.

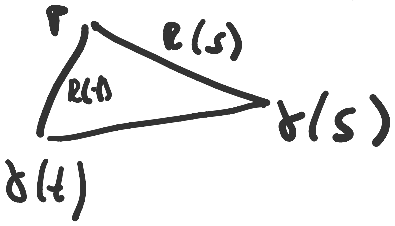
<u>Proof</u>: take $t \in [0,1]$; $\varepsilon = R(t)$; take resp. δ,
take $s: |s-t| < \delta$

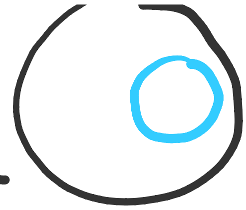


Then $\partial B_{\gamma(t)} \cap \partial B_{\gamma(s)} \neq \emptyset$,

because $\overline{B_{\gamma(s)}} \not\subset B_{\gamma(t)}$ ←

because $B_{\gamma(s)} \not\subset B_{\gamma(t)}$ ⟵ impossible because then we could extend F_s to a larger disc!





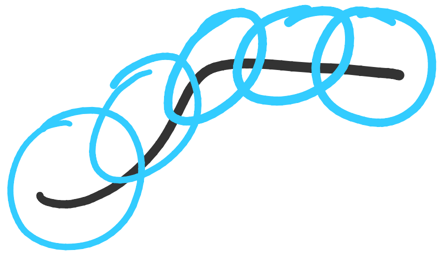
$\Longrightarrow$ by triangle ineq: $|\gamma(t)-\gamma(s)| \geq |R(t)-R(s)|$

since this $<\varepsilon$ $\Longrightarrow$ $|R(t)-R(s)| < \varepsilon$ $\blacksquare$

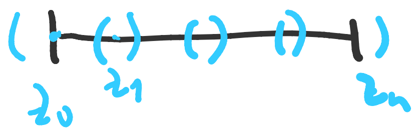
Corol: $r := \min_{t\in[0,1]} R(t) > 0.$

Now, $\forall t$, take $\varepsilon = \min R(t)$, take resp. δ; gives covering of $[0,1]$ by $\{B_\delta(t)\}^{<R}$; choose finite subcovering $\Longrightarrow$ $a = z_0, z_1, \ldots, z_n = b$:

$\bigcup B_{R_j}(z_j) \supset [\gamma]$



By def, $\forall F_j$ is the dir anal exten of F_{j-1}.



And vice versa:

extension along a chain provides extension along a path γ: $[\gamma] \subset \{\text{union of discs}\}$

along a path γ: $[\gamma] \subset \{$union of discs$\}$

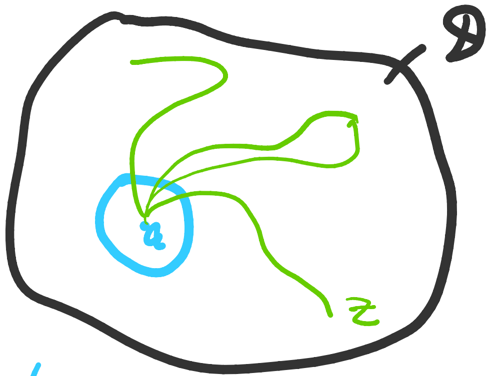
(exercise)

$\rightsquigarrow$ the two things coincide

(contin exten / chain exten)

Def: a (complete) anal function (according to Weierstrass) is the result of the analytic contin of a given element along all possible paths in some Domain $\mathcal{D}$, starting at the center of the original element;

(assuming extension along $\forall$ path exists!)



(B_R, F_a)

„Result" = the collection of all the elements obtained at all the points in $\mathcal{D}$.

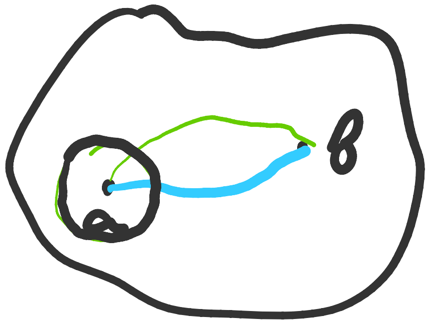
Question: how many elem...

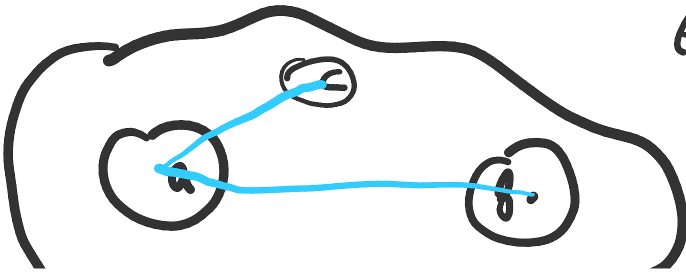
<u>Question</u>: how many elements at a point can we get in this way?

<u>Theorem</u> (Homotopy Theorem).

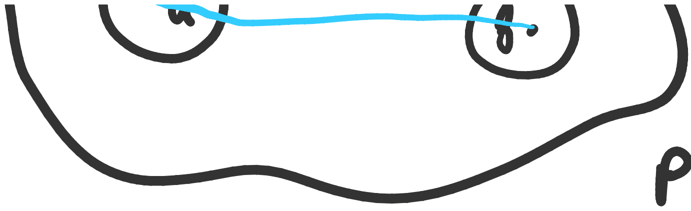
Let $\mathcal{F}$ be a complete anal func in a dom $\mathcal{D}$. Fix $a, b \in \mathcal{D}$.
Consider <u>homotopic</u> paths γ^0, γ^1 in $\mathcal{D}$ connecting a, b. Then the extensions of any element at the pt a along γ^0 and γ^1 resp. gives <u>the same</u> result!

<u>Remark</u>: in the def of compl anal function, it makes no difference at which pt to start, and with which element to start!





Because $\forall$ other



because $\forall$ other element at further pt is obtained by anal contin

<u>Reminder - a homotopy</u>: γ^0, γ^1 are <u>homotopic</u> in $\mathcal{D}$, (both connect a, b)

if $\exists$ a contin on $[0,1]\times[0,1]$ map $\gamma(s,t)$ valued in $\mathcal{D}$, such that:

(i) $\gamma(s,0)=a$, $\gamma(s,1)=b$ (all curves $\gamma(s,t)$ with fixed s connect a and b)

(ii) $\gamma(0,t)=\gamma^0(t)$, $\gamma(1,t)=\gamma^1(t)$

$s=0 \leftrightarrow \gamma^0$; $s=1 \leftrightarrow \gamma^1$

<u>Proof</u>: Original element F_0 extends along each $\gamma^s = \{\gamma(s,t), 0\le t\le 1\}$

We will prove, in fact, that all extension along γ^s coincide.

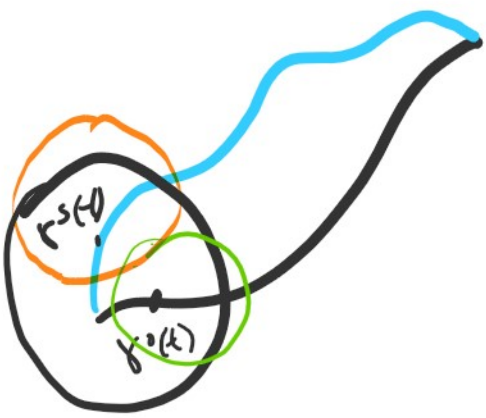
Consider γ^0; take its min $R(t) = r$;

Now use uniform contin of $\gamma(s,t)$ on $[0,1]\times[0,1]$, Now use uniform contin of $\gamma(s,t)$ on $[0,s]\times[0,1]$, take $\varepsilon = \frac{\varepsilon}{2}$, take resp δ: if $dist((s_1,t_1),(s_2,t_2)) < \delta \Rightarrow |\gamma(s_1,t_1) - \gamma(s_2,t_2)| < \varepsilon$

Now we will show that for s – δ-close to 0, actually $F_{s,1} = F_{0,1}$.

we have that $F_{s,0} = F_{0,0} = F_0$.

By contrad, let $F_{s,1} \neq F_{0,1}$



If t is δ-close to 0, then we claim that $F^{s,t}$ is the direct contin. of $F^{0,t}$

t is δ-close to 0; s is δ-close to 0 $\Rightarrow \gamma(s,t)$ is $\varepsilon+\varepsilon=\varepsilon$ close to $\gamma^0(t)$; so, the „green" and the „orange" discs intersect; both „green" and „orange" elements are direct contin of $F_{0,0} \Rightarrow$ they are contin of each other.

Take t^* – min t for which $F_{s,t}$ is <u>not</u> the dir contin of $F_{0,t}$ (like in the proof the diz contin of $F_{0,t}$ (like in the proof of uniq); then, repeating the prev arg, we get a contrad (absolutely the same picture). $F_{0,t^*} \leadsto F_{s,t^*}$

So, for s δ-close to 0, $F_{s,1} = F_{0,1}$

So, we can „move a bit" along S from 0.

Now we again repeat the trick!

$s^* = \min s$, for which $F_{s,1} \neq F_{0,1}$; and again same picture with discs, and a contrad! $\blacksquare$