Analytic extension

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$\dim \operatorname{Aut}(B_1) = n^2+2n$, $\dim \operatorname{Aut}(P_1) = 3n$

$\Rightarrow B_1 \not\simeq P_1$ (H. Poincaré)

<u>Another proof</u>: if $\exists F: B_1 \to P_1$, $\|F(z)\|_\infty = \|z\|_2$ $\forall z$

$\Rightarrow$ $\underbrace{\{\|z\|_\infty = \tfrac{1}{2}\}}_{\text{nonsmooth}} \overset{}{\underset{\text{diff}}{\longleftrightarrow}} \underbrace{\{\|z\|_2 = \tfrac{1}{2}\}}_{\text{smooth}}$ – contrad.

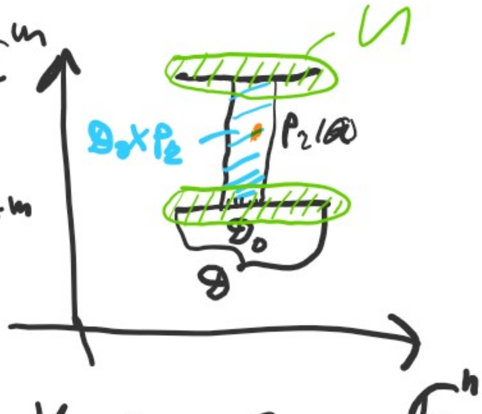
Forced analyitic continuation

<u>Example</u>: $\mathcal{D} = \{|z_1|<1\} \cup \{|z_2|<1\} \subset \mathbb{C}^2 \Rightarrow$

$\forall f \in \mathcal{O}(\mathcal{D})$ extends hol-ly to $\mathbb{C}^2$!

<u>Def</u>: let $\mathcal{D} \subset \mathbb{C}^n$, $P_1(a) \subset \mathbb{C}^m$, $\mathcal{D}_0 \subset \mathcal{D}$ subdomain

Consider $\Omega_0 := \mathcal{D}_0 \times P_1(a) \cup U$, U-open nbd of the set: $\mathcal{D} \times \partial^m P_1(a) \subset \mathbb{C}^n \times \mathbb{C}^m \sim \mathbb{C}^{n+m}$



$\Omega_0 \subset \mathbb{C}^{n+m}$ dom. Ω_0 is called a <u>Hartogs figure</u>

<u>Def</u>: the <u>envelope</u> of a Hartogs figure is $\mathcal{D} \times P_1 \cup U$

Theorem: Any function holomorphic in a

Theorem: Any function holomorphic in a Hartogs figure, extends hol-ly to its envelope.

Proof: Let f be our function, and the domain described as above; define the extension for f as follows: $F(z,w) := \frac{1}{(2\pi i)^m}\int\limits_{\partial^m P_2} \frac{f(z,\xi)\,d\xi}{\xi - w}$

where $z \in \mathcal{D} \subset \mathbb{C}^n$; $w \in P_2 \subset \mathbb{C}^m$

$\xi \in \partial^m P_2$

expres. under the integ is contin; it is hol in w (actually infinite); it is hol in z $\forall$ fixed ξ, w – since f is hol in U! $\Rightarrow F(z,w) \in \mathcal{O}(\mathcal{D} \times P_2)$ an an int., hol-ly depending on params.

It remains to show that $F = f$ for $(z,w) \in \mathcal{D}_0 \times P_2$ – – but this follows from the Cauchy f-la, applied to $f(z,\xi)$ (fixed ξ) in $P_2(a) \Rightarrow F$ extends f to $\mathcal{D} \times P_2$ from $\mathcal{D}_0 \times P_2 \Rightarrow$ gives the exten to the envelope. ∎

Remark: the exten phenom in the Thm applies immediately to linear images of Hartogs figures (actually to bihol images as well).

Corollary: $\forall f \in \mathcal{O}(P_2(a) \setminus K)$ $K \subset P_2(a)$ – set

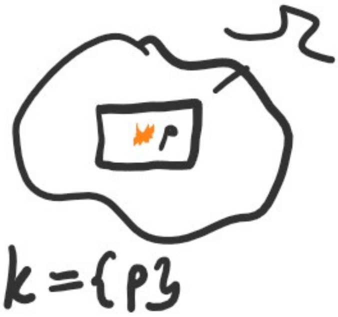
Corollary: $\forall f \in (P_2(a)\setminus K)$, $K \subset P_2(a)$ – cpt subset, extends hol to $P_2(a)$. $P_2\setminus K$ – conn

In partic., $\forall f \in \mathcal{O}(\mathbb{C}^n\setminus K)$ extends to $f \in \mathcal{O}(\mathbb{C}^n)$. $\mathbb{C}^n\setminus K$ – conn

In partic., **all** isolated singularities of hol func-s are **removable!**

Proof: follows immed-ly after shrinking $P_2(a)$: $P_2\setminus K \supset$ Hartogs figure, for which P_2 is the envelope!

$\mathbb{C}^n = \mathbb{C}^{n-1}\times\mathbb{C}$, $n \geq 2$



Corollary: $f \in \mathcal{O}(B_2(a)\setminus K) \Rightarrow f$ extends hol-ly to $B_2(a)$ ($B_2\setminus K$ – conn)

Proof: „eat" K step-by-step by rotated Hartogs figures

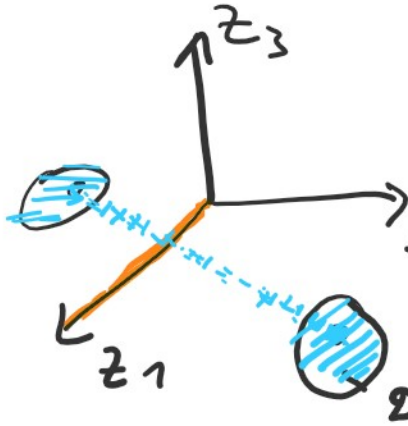
Thm (Removal of cpt singularities)

$\Omega \subset \mathbb{C}^n$, K – cpt in $\Omega \Rightarrow \forall f \in \Omega\setminus K$ extends hol-ly to Ω (provided $\Omega\setminus K$ is extends hol-ly to Ω (provided $\Omega \setminus K$ is connected)

can be proved by Hartogs figures

Sometimes, non-compact sing can be removed...

Ex: $L \subset \mathbb{C}^3$; then $\forall f \in \mathbb{C}^3 \setminus L$ extends to $\tilde{f} \in \mathbb{C}^3$.
cx line



Proof: by lin. trans., we may assume: $L = \{z_2 = z_3 = 0\}$

$\mathcal{D} \overset{\mathbb{C}^2}{=}$ disc in the (z_1, z_3)-space

$\mathcal{D} \cap \{z_3 = 0\} \neq \emptyset$; $\mathcal{D}_0$ - subdisc with $\mathcal{D}_0 \cap \{z_3 = 0\} = \emptyset$

multiply by disc in the z_2-space $\Rightarrow$ Hartogs Figure $\Rightarrow$ remove $\mathcal{D} \cap \{z_3 = 0\}$; $\Rightarrow$ remove entire $\{z_2 = z_3 = 0\}$

Def: Let $\mathcal{D} \subset \mathbb{C}^n$; then $\tilde{\mathcal{D}} \supset \mathcal{D}$ is called a hol exten fn $\mathcal{D}$, if $\forall f \in \mathcal{O}(\mathcal{D})$ extends to $\tilde{f} \in \mathcal{O}(\tilde{\mathcal{D}})$

Ex: $\mathcal{D}$ - Hart fig, $\tilde{\mathcal{D}}$ - envelope

$\mathcal{D} = \Omega \setminus K$, $\tilde{\mathcal{D}} = \Omega$ ($\Omega \setminus K$ - conn)

Propos: If $\tilde{\mathcal{D}}$ is a hol exten of $\mathcal{D}$ and $f \in \mathcal{O}(\mathcal{D})$, then the extended func. $\tilde{f}$ in $\tilde{\mathcal{D}}$ takes in $\tilde{\mathcal{D}}$ only the values that f takes in $\mathcal{D}$.

only the values that f takes in D.

$f(p)=A$

Proof: assume, otherwise, that $\hat{f}$ takes a value A, and $f(D)\not\ni A$; then consider $g := \frac{1}{f(z)-A}$

$g\in\mathcal{O}(D)$ (since $A\notin f(D)$), but by uniq, to points where $\hat{f}\neq A$, g can be extended only as $\frac{1}{\hat{f}(z)-A} = \hat{g} \Rightarrow \lim_{z\to p}\hat{g}=\infty \Rightarrow g$ actually does not extend to $\hat{D}$ hol-ly! - contrad ■

Corollary: if D - bounded, then $\hat{D}$ - also bounded

Proof: the function $f_j(z)=z_j \in\mathcal{O}(D)$ and are bounded $\Rightarrow$ by the propos: $f_{j=z_j}$ stay bounded in $\hat{D} \Rightarrow \hat{D}$ - bounded ■

Remark: $\exists$ domains which admit no hol exten

$z=re^{2\pi i\cdot q},\ q\in\mathbb{Q}$

$\sum_{n=1}^{\infty} z^{n!}\in\mathcal{O}(B_1)$, $B_1\subset\mathbb{C}$, and doesn't extend anywhere across the bdry

$$f(z_1,z_2) := \sum_{k=1}^{\infty} z_1^{k!} + \sum_{\ell=1}^{\infty} z_2^{\ell!} \in\mathcal{O}(P_1(0)),\quad P_1(0)\subset\mathbb{C}^2$$

and doesn't extend anywhere!

and doesn't extend anywhere!

Dom. of hol: $\approx$ a domain with no hol. extensions.

Precisely:

Def: a domain $D \subset \mathbb{C}^n$ is called a dom. of hol.,
if $\exists f \in \mathcal{O}(D)$, s.t. $\forall a \in D$, if $r := \mathrm{dist}_\infty(a, \partial D)$,
then $f|_{\underset{\hat{D}}{P_r(a)}}$ doesn't extend to any bigger polydisc $P_R(a)$, $R > r$.

(that is, f can't be extended across the bdry at $\forall$ bdry pt $z \in \partial D$).

In partic, f can't be extended $\forall \tilde{D} \supsetneq D$.

Q: Why such an involved def?
$\exists$ examples (see Shabat) where $\nexists$ hol exten of D, But $\forall f \in \mathcal{O}(D)$ extends through some of the bdry pts ($\exists$ an „extension" of f, but it is multi-valued)

We avoid such phenomonon!

Goal: describe domains of hol (geom-ly).

Answer: dom of hol are the ones

Answer: dom of hol are the ones which are „convex" – in an appr sense!

Def: $\mathcal{D} \subset \mathbb{C}^n$; $S \subset \mathcal{O}(\mathcal{D})$; let cpt $K \subset \mathcal{D}$; then the S-convex hull $\hat{K}_S$ of K is the set:

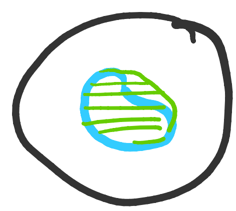
$$\hat{K}_S = \{ p \in \mathcal{D} : |f(p)| \leq \max_K |f| \quad \forall f \in S \}$$

In partic, one can take $S = \mathcal{O}(\mathcal{D})$; then $\hat{K}$ is called the hol-ly convex hull.

Def: a domain $\mathcal{D} \subset \mathbb{C}^n$ is called S-convex ($S \subset \mathcal{O}(\mathcal{D})$), if $\forall$ cpt $K \subset \mathcal{D}$, it holds that $\hat{K}_S \Subset \mathcal{D}$ ($\overline{\hat{K}_S}$ – cpt in $\mathcal{D}$).
($\Leftrightarrow$ $\hat{K}_S$ is bounded and is away from $\partial\mathcal{D}$: $\mathrm{dist}(\hat{K}_S, \partial\mathcal{D}) > 0$).

$\hat{K}_S \supset K$.

Remark: if $S \supset \{z_1, .., z_n\} \Rightarrow$ $\hat{K}_S$ is bounded



Ex: $S = \{\mathbb{C}\text{-lin. func-s}\} \subset \mathcal{O}(\mathcal{D}) \Rightarrow$

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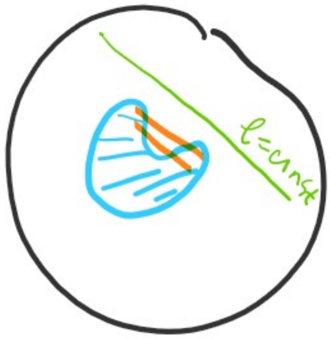
$\hat{K}_S$ – the usual convex hull

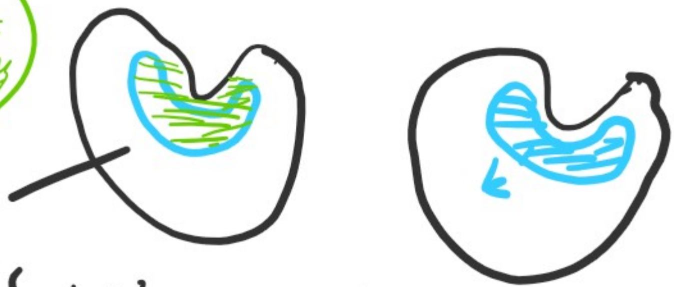


$\forall \ell$ we add such segm-s

$\Rightarrow \hat{K}_S = \text{convex hull} \cap D$

(transferring to $\mathbb{C}$-lin – exercise)

Conclusion: S-convexity of D is the usual geom. convexity



Here $\hat{K}_S$ is not a cpt

$\Rightarrow D$ is not S-convex

Homework: $S = \{z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}\} \subset \mathcal{O}(D)$

D - Reinhard dom $\Rightarrow$ S-convexity $=$ log-convexity