RECALL:

M mfd., $E \subseteq TM$ smooth distribution of rank k,

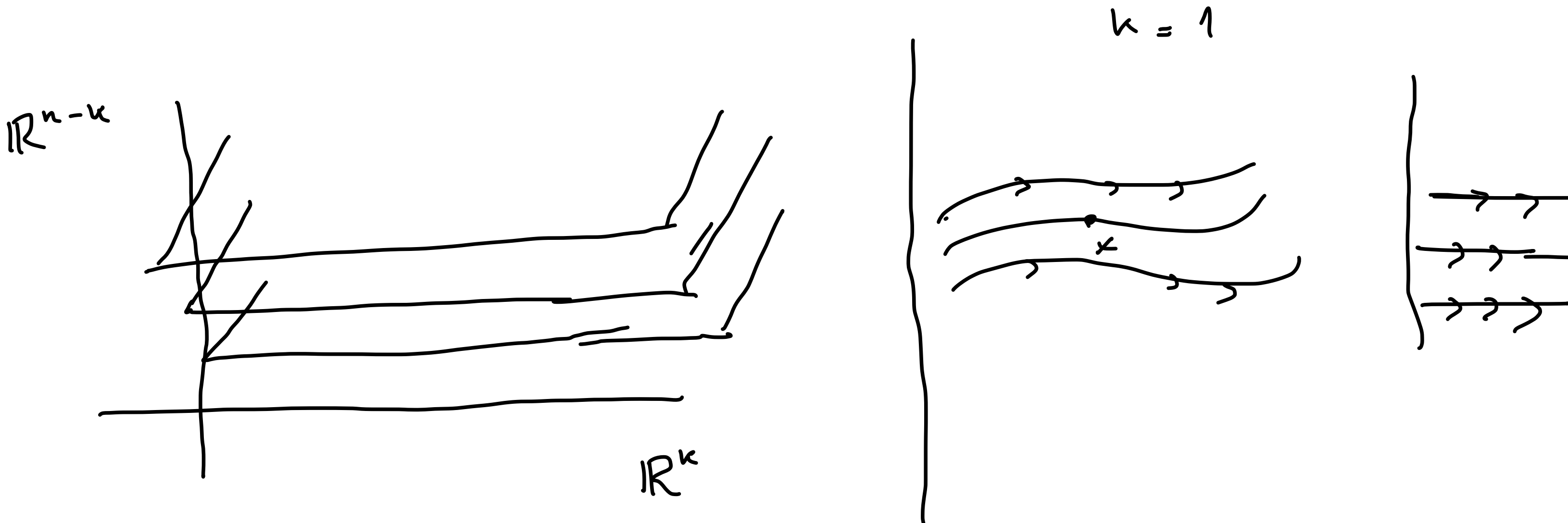
E is integrable $\Longleftrightarrow$ E is involutive

If E is involutive, then for each $x \in M$ $\exists$ a chart (U, u) with $x \in U$ s.t.

$\dim(M) = n$

- $u(U) = W \times \tilde{W} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ $\qquad W \subseteq \mathbb{R}^k$, $\tilde{W} \subseteq \mathbb{R}^{n-k}$
- for each $a \in \tilde{W}$ the subset $u^{-1}(W \times \{a\}) \subseteq M$ is integral submfd. for E.

This says, given an involutive distribution, locally around each point $\exists$ a chart (U,u) where U is filled by integral submanifolds; in the corresponding coordinates they are given by horizontal subspaces $\mathbb{R}^k \times \{a\}$ of $\mathbb{R}^n$.



Charts as in Thm. 3.38 are called <u>distinguished charts</u> for (M, E) and the integral submflds. $u^{-1}(\underline{W \times \{a\}}) \subseteq M$ are called <u>plaques</u>.

Note that, if (U_α, u_α) and (U_β, u_β) are distinguished chart for (M, E) with $U_\alpha \cap U_\beta \neq \emptyset$, then the transition map is of the form:

$$u_\beta \circ u_\alpha^{-1} : \underbrace{u_\alpha(U_\alpha \cap U_\beta)}_{\subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}} \longrightarrow \underbrace{u_\beta(U_\alpha \cap U_\beta)}_{\subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}} \qquad (*).$$

$$(x, y) \longmapsto (f(x,y), g(y)) \qquad f, g \text{ smooth}.$$

differential of $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 0 & \partial g / \partial y \end{pmatrix}$

i.e. transition maps map subsets $W_\alpha \times \{a\}$ to $W_\beta \times \{b\}$.

<u>Def. 3.39</u> A <u>foliated atlas of dimension k</u> on a mfd. $(M, \mathcal{A})$ of dim. n is a subatlas $\mathcal{A}'$ of $\mathcal{A}$ consisting of charts $(U, u) \in \mathcal{A}$ s.t.

- $u(U) = W \times \tilde{W} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ for open subsets $W \subseteq \mathbb{R}^k$, $\tilde{W} \subseteq \mathbb{R}^{n-k}$
- transitions maps are of the form $(*)$

<u>Def. 3.40</u> A k-dimensional foliation $\mathcal{F}$ on a mfd. M of dim. n is a maximal foliated atlas of dim. k.

Frobenius Thm. shows that any involutive smooth distribution E on a mfd. M of rank k defines a k-dimensional foliation $\mathcal{F}^E$. Conversely, any foliation $\tilde{\mathcal{F}}$ of dim. k determines a smooth distribution of rank k on M given by

$$E_x := T_{u(x)} u^{-1} \left(\underline{T_w \mathbb{R}^k \times \{0\}} \right) \subseteq T_x M, \quad x \in M,$$

where $u(x) \in \underline{\mathbb{R}^k \times \mathbb{R}^{n-k}}$ equals $u(x) = w + \tilde{w}$ with $w \in W \subseteq \mathbb{R}^k$, $\tilde{w} \in \tilde{W} \subseteq \mathbb{R}^{n-k}$ and $(U,u) \in \underline{\mathcal{F}}$.

By $(*)$, E_x is well-defined, meaning independent of the chart $(U,u) \in \mathcal{F}$ with $x \in U$.

Some application of Thm. 3.38 to the study of PDEs :

<u>Ex.</u> Consider the following system of PDEs for a function

$f : \mathbb{R}^3 \to \mathbb{R}$:

$$(*) \quad \begin{aligned} -2z^2 \frac{\partial f}{\partial x} + 2x \frac{\partial f}{\partial z} &= 0 \\ -3z^3 \frac{\partial f}{\partial y} + 2y \frac{\partial f}{\partial z} &= 0 \end{aligned}$$

coordinates on $\mathbb{R}^3$ (x, y, z).

(It is a linear system of first order PDEs) .

When does $(*)$ have any non-constant solution f ?

$X = -2z^2 \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial z}$, $Y = -3z^3 \frac{\partial f}{\partial y} + 2y \frac{\partial}{\partial z}$

X, Y span a rank 2 distribution $\underline{E}$ on the open subset

$V = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\} \subseteq \mathbb{R}^3$

Moreover, $[X,Y] = -12xz\frac{\partial}{\partial y} + 8yz\frac{\partial}{\partial x} = \underline{\frac{4x}{z}Y - \frac{4y}{z}X}$

$\Longrightarrow$ E is involutive distribution on V.

By Frobenius Thm. (Thm. 3.38) $\exists$ locally around any $(x_0, y_0, z_0) \in V$ a chart (U, u) s.t. E is spanned by $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$.

(*) $X.f = 0$, $Y.f = 0$ is equivalent to $\frac{\partial f}{\partial u^1} = \frac{\partial f}{\partial u^2} = 0$ in the coordinates $\begin{pmatrix} u^1(x,y,z) \\ u^2(x,y,z) \\ u^3(x,y,z) \end{pmatrix}$

Hence, $f = u^3$ is a solution and any solution in a sufficiently small neighbhd. of (x_0, y_0, z_0) is of the form $f(x,y,z) = g(u^3(x,y,z))$, where g is a smooth function in one variable.

<u>Ex.</u> Consider the following system of PDEs for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

$$(**) \quad \begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \underline{\alpha}(x,y,f(x,y)) \\ \frac{\partial f}{\partial y}(x,y) &= \underline{\beta}(x,y,f(x,y)) \end{aligned}$$

α, β are smooth functions defined on an open subset $V \subseteq \mathbb{R}^3$

(Overdetermined system of PDEs of possibly non-linear first order equation).

Q When does (**) have a solution?

Necessary conditions for α and β:

$$\boxed{\frac{\partial}{\partial y}\left(\underline{\alpha(x,y,f(x,y))}\right) = \frac{\partial}{\partial x}\left(\beta(x,y,f(x,y))\right)}$$

(since $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

Chain rule $\Longrightarrow$ $\boxed{\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}}$ (***)

which must hold at any $(x,y,z) \in V$ where there is a solution with $f(x,y)=z$.

By Frobenius Thm, (***) is not only necessary it is also sufficient: it implies that for any $(x_0, y_0, z_0) \in V$ $\exists$ an open neighbhd. U of $(x_0, y_0) \in \mathbb{R}^2$ and a unique solution of (**) $f: U \to \mathbb{R}$ with $f(x_0, y_0) = z_0$.

Why? (**) prescribes the tangent plane to the graph of f in terms of coordinates of the graph. Collection of tangent planes defines a rank 2 distribution on V and (***) is equivalent to involutivity.

Suppose $f : U \to \mathbb{R}$ were a solution (on an open subset $U \subseteq \mathbb{R}^2$) of $(**)$.

Then $\psi : U \to \mathbb{R}^3$ (ψ is a parametrization of the submfd. $gr(f) \subseteq \mathbb{R}^3$)

$$\psi(x,y) = (x, y, f(x,y))$$

is a diffeom. onto $gr(f)$

$$\left(T_{(x,y)}\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \right)$$

$T_{\psi(x,y)} gr(f)$ is spanned by

$$T_{(x,y)}\psi \left(\frac{\partial}{\partial x}(x,y) \right) = \frac{\partial}{\partial x}(x,y) + \frac{\partial f}{\partial x}(x,y) \frac{\partial}{\partial z}(x,y)$$

$$= \alpha(x, y, f(x,y))$$

$$T_{(x,y)}\psi \left(\frac{\partial}{\partial y}(x,y) \right) = \frac{\partial}{\partial y}(x,y) + \frac{\partial f}{\partial y}(x,y) \frac{\partial}{\partial z}(x,y)$$

$$= \beta(x,y, f(x,y)).$$

$\Leftarrow$ f is a solution.

$$X := \frac{\partial}{\partial x} + \alpha(x,y,z)\frac{\partial}{\partial z}$$

$$Y := \frac{\partial}{\partial y} + \beta(x,y,z)\frac{\partial}{\partial z}$$

vector fields on V.

span a rank 2 distribution E on V.

E is involutive $\Longleftrightarrow$ (***) holds

(f is is a solution of (**) $\Longleftrightarrow$ gr(f) is an integral submfd.)

If this is the case, then through any point $(x_0, y_0, z_0) \in V$ $\exists$ an integral submfd. $N \subseteq V \subseteq \mathbb{R}^3$ of E, which locally has the form gr(f) for a function $f: U \to \mathbb{R}$, U open neighb. of (x_0, y_0) with $f(x_0, y_0) = z_0$.

On the opposite ending of integrable distributions (among all distributions) are the so-called bracket-generating distributions:

<u>Def: 3.41</u> A smooth distribution $E \subseteq TM$ on a mfd. M is called bracket-generating, if any local frame $\{\xi_1, \dots, \xi_k\}$ of E together with its iterated Lie brackets $[\xi_i, \xi_j]$, $[\xi_e, [\xi_i, \xi_j]]$ -etc. form a local frame for TM.

<u>Remark</u> If a local frame is bracket-generating around some point, then so is any other frame around that point.

Ex. Standard contact distribution on $\mathbb{R}^3$; $(x,y,z) \in \mathbb{R}^3$ coordinates on $\mathbb{R}^3$

$$E = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right\rangle \subseteq T\mathbb{R}^3$$

$$\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right] = \frac{\partial}{\partial z} \notin E$$

$\Rightarrow$ E is not integrable; in fact it is bracket-generating ($\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \frac{\partial}{\partial z}$ span $T\mathbb{R}^3$)

More generally, one has the notion of a contact mfd:

It is an odd dimension mfd. M, say of dim. $2n+1$, equipped with a bracket-generating distribution of rank $2n$.

such that the Levi-bracket given by

$$\mathcal{L}_x : E_x \times E_x \longrightarrow T_xM/E_x \simeq \mathbb{R} \qquad x \in M,$$
$$(\xi, \eta) \longmapsto q_x([\tilde{\xi}, \tilde{\eta}](x))$$

where $\hat{\xi}, \hat{\eta}$ are extensions of ξ, η to local vector fields around x and $q_x : T_xM \longrightarrow T_xM/E_x$ is the natural projection,

is non-degenerate. For any $x \in M$. ($\xi \in E_x$, then $\mathcal{L}_x(\xi, \eta) = 0 \ \forall \eta \in E_x \Longrightarrow \xi = 0$).

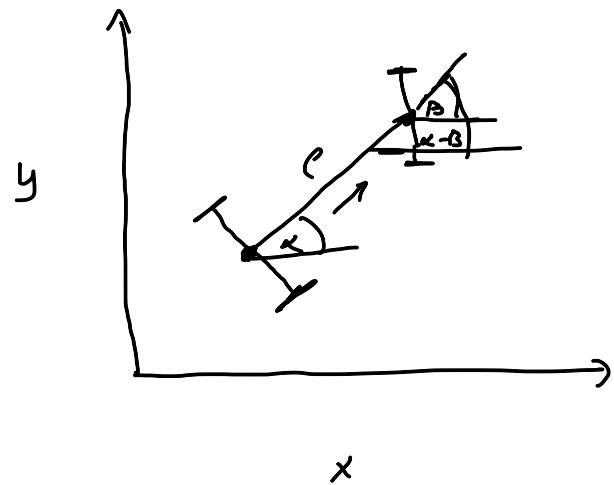
Ex Driving a car.

Configuration / phase space of a car: $M = \mathbb{R}^2 \times S^1 \times S^1$

(x,y) position of midpoint of rear axle

α ... angle of chassis to x-axis

β ... steering angle of front wheels.



Moving the car traverses a curve: $c(t) = (x(t), y(t), \alpha(t), \beta(t))$ in M.

Non-holonomic constraints : constraints on position and velocity that can not be integrated to constraints on position only :

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ is parallel to } \underline{\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) + \ell\cos(\alpha(t)) \\ y(t) + \ell\sin(\alpha(t)) \end{pmatrix} \text{ —"— to } \begin{pmatrix} \cos(\alpha-\beta) \\ \sin(\alpha-\beta) \end{pmatrix}$$

$$x'(t)\sin(\alpha(t)) - y'(t)\cos\alpha(t) = 0$$

$$x'(t) - \ell\sin(\alpha(t))\alpha'(t)\sin(\alpha(t)-\beta(t)) - (y'(t) + \ell\cos(\alpha(t))\alpha'(t)) \cdot \cos(\alpha(t)-\beta(t)) = 0$$

2 linear equations for $\begin{pmatrix} x'(t) \\ y'(t) \\ \alpha'(t) \\ \beta'(t) \end{pmatrix}$

$\rightsquigarrow$ solutions

$$\begin{pmatrix} x'(t) \\ y'(t) \\ \alpha'(t) \\ \beta'(t) \end{pmatrix} = \lambda(t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \mu(t) \begin{pmatrix} \ell \cos\alpha(t) \cos\beta(t) \\ \ell \sin\alpha(t) \cos\beta(t) \\ -\sin\beta(t) \\ 0 \end{pmatrix}$$

$$X := \frac{\partial}{\partial \beta} \quad \text{(steer)}$$

$$Y := \ell \cos\beta \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} \right) - \sin\beta \frac{\partial}{\partial \alpha} \quad \text{(drive)}.$$

The two 'control vector fields' X and Y span a bracket-generating distribution on M (distribution describes the space of possible ~~vecto~~ velocities)

(TM is spanned by X, Y, $[X,Y]$, $[Y,[X,Y]]$).