5. Integration on manifolds

Recall that the transformation formula for multiple integrals (or coordinate change formula for integrals):

Suppose $U \subseteq \mathbb{R}^n$ open subset and $\Phi : U \to \Phi(U)$ a diffeom. between open subsets of $\mathbb{R}^n$.

Let $f : \Phi(U) \longrightarrow \mathbb{R}$ be a smooth function with compact support:

$$\int_{\phi(U)} f = \int_U \underline{(f \circ \Phi) \, | \det D\phi} \, | \qquad (*)$$

.

Looks like the transformation of n-forms on mfds. of dim. n:

Suppose M is a smooth mfd. of dim. n, $\omega \in \Omega^n(M)$ and (U, u) is a chart of M:

Then $\exists!$ C^∞-fct. $\omega^U_{1\ldots n} : U \to \mathbb{R}$ s.t.

$$\omega|_U = \omega^U_{1\ldots n} \, du^1 \wedge \ldots \wedge du^n$$

$$\left(\omega^U_{1\ldots n} = \omega\left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\right) : U \to \mathbb{R} \right).$$

Suppose $v : U \to v(U) \subseteq \mathbb{R}^n$ is another chart for M with domain U. Then $\omega|_U = \omega^V_{1\ldots n} \, dv^1 \wedge \ldots \wedge dv^n$.

Local coordinate expressions of the two functions w.r. to (U,u) resp. (V,v) $\left.\vphantom{\int}\right\}$

$$\omega^U_{1\ldots n} \circ u^{-1} : u(U) \longrightarrow \mathbb{R}$$

$$\omega^V_{1\ldots n} \circ v^{-1} : v(U) \longrightarrow \mathbb{R}$$

$$\omega^U_{1\ldots n}(u^{-1}(y)) = \omega(u^{-1}(y))(T_y u^{-1} e_1, \ldots, T_y u^{-1} e_n)$$

$$\omega^V_{1\ldots n}(v^{-1}(z)) = \omega(v^{-1}(z))(T_z v^{-1} e_1, \ldots, T_z v^{-1} e_n).$$

Now let $\Phi : u(U) \longrightarrow v(U)$, $\Phi := v \circ u^{-1}$

$\Rightarrow$ $v^{-1} \circ \Phi = u^{-1}$ and $\underline{T_y u^{-1}} = \underline{T_{\phi(y)} v^{-1} \circ T_y \phi}$.

$\Longrightarrow$

$$\underbrace{\omega^U_{1\dots n}(u^{-1}(y))} = \omega(u^{-1}(y))\left(T_y u^{-1} e_1, \dots, T_y u^{-1} e_n\right) =$$

$$= \omega(u^{-1}(y))\left(T_{\phi(y)} v^{-1} \circ T_y \phi e_1, \dots, T_{\phi(y)} v^{-1} \circ T_y \phi e_n\right) \to (v^{-1})^* \omega(\phi(y))(T_y\phi e_1, \dots, T_y \phi e_n)$$

$$= \det(D_y\phi)\, \omega(u^{-1}(y))\left(T_{\phi(y)} v^{-1} e_1, \dots, T_{\phi(y)} v^{-1} e_n\right)$$

$$= \underbrace{\det(D_y\phi)}_{\neq 0}\ \underline{\omega^V_{1\dots n}(v^{-1}(\phi(y)))}.$$

$\neq 0 \quad \forall y \in \underline{u(U)}$, since ϕ is a diffeom.

If we assume that U is connected, hence also $u(U)$, then $\det(D_y\phi)$ is either always positive or always negative on $u(U)$.

(*) says that integral over local coordinate expression of ω is well-defined up to a sign. (i.e. independent of choice of chart up to a sign).

5.1 Orientation

Suppose V is a n-dim. vector space.

If $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are two ordered basis of V, then $\exists!$ linear map $A: V \to V$ s.t. $A(a_i) = b_i \ \forall i$.

The two basis $\{a_i\}_{1 \le i \le n}$ and $\{b_i\}_{1 \le i \le n}$ have the same <u>orientation</u>, if $\det(A) > 0$; if $\det(A) < 0$, they are said to have opposite orientations.

- "Having the same orientation" defines an equivalence relation on the set of ordered basis of V and $\exists$ exactly two equivalence classes.

- An orientation on V is the choice of one of these two classes and a vector space with a choice of orientation is called an oriented vector space.

Having chosen an orientation on V, any ordered basis in the chosen class is called positively oriented and any that is not negatively oriented.

- Standard orientation on $\mathbb{R}^n$ is the orientation determined by the standard basis $\{e_1, \dots, e_n\}$.

  A basis $\{a_1, \dots, a_n\}$ is positively oriented w.r.t. to the standard orientation, if $\det((a_1, \dots, a_n)) > 0$.

- Given two n-dim. oriented vector spaces V and W, then a linear isomorphism $A : V \to W$ is called orientation preserving, if A maps a (hence any) positively oriented basis to a positively oriented basis.
  Otherwise it is called orientation-reversing.

On mfds. we can talk about orientations on the tangent spaces, but we need some notion of smoothness:

<u>Def. 5.1</u> M mfd.

(1) M is called <u>orientable</u>, if we can choose an orientation on T_xM $\forall x \in M$ s.t. the following holds:

For any local frame $\{\xi_1, \dots, \xi_n\}$ of TM on an connected open subset $U \subseteq M$, the basis $\{\xi_1(y), \dots, \xi_n(y)\}$ of $T_yU = T_yM$ $\triangleright$ either positively oriented $\forall y \in U$ or negatively oriented $\forall y \in U$.

(2) If M orientable a choice of orientation on $T_x M$ $\forall x \in M$ as in (1) is called an orientation on M.

A orientable mfd. with a chosen orientation is called an oriented mfd.

- If M is connected and orientable, it is easy to see that an orientation is already determined by the orientation on one tangent space. Hence, on a connected orientable mfd. $\exists$ two orientations.

- An open subset of an oriented mfd. is itself in a natural way an oriented mfd.

<u>Def. 5.2</u> Suppose M and N are oriented mfds. and $f: M \to N$ is a local diffeomorphism. Then f is called orientation preserving, if the linear isomorphism $T_x f : T_x M \to T_{f(x)} N$ is orientation preserving $\forall x \in M$.

Orientation via special atlases:

Def. 5.3 M mfd.

① An oriented atlas on M is an atlas $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ for M s.t. for any $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$ the transition map $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \longrightarrow u_\beta(U_\alpha \cap U_\beta)$ has the property that $\det(D(u_\beta \circ u_\alpha^{-1})) > 0$ on $u_\alpha(U_\alpha \cap U_\beta)$.

② Two oriented atlases on M are called orientation equivalent, if their union is again an oriented atlas.

Prop. 5.4 M mfd. of dim. n. Then the following are equivalent:

① M is orientable

② M admits an oriented atlas

③ $\exists$ an n-form $\omega \in \Omega^n(M)$ s.t. $\omega(x) \neq 0$ $\forall x \in M$.

Proof

① $\Rightarrow$ ② Suppose M is orientable and fix an orientation.

Choose an atlas on M, $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ s.t. U_α is connected $\forall \alpha \in I$. $\Rightarrow$ $T_x u_\alpha : \underline{T_x U_\alpha} \longrightarrow T_{u_\alpha(x)} u_\alpha(U_\alpha) = T_{u_\alpha(x)} \mathbb{R}^n \simeq \underline{\underline{\mathbb{R}^n}}$ is either orientation preserving $\forall x \in U_\alpha$ or orientation reversing $\forall x \in U_\alpha$.

In the first case we keep the chart as it is and in the second case we compose it with an orientation reversing linear map $\mathbb{R}^n \to \mathbb{R}^n$ (for instance exchange the first two variables).

In this we we obtain an oriented atlas.

(2) $\Longrightarrow$ (3) Let $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ ~~be~~ an oriented atlas on M and $\{f_i\}_{i \in \mathbb{N}}$ is a partition of unity that is subordinate to the cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M.

For $i \in \mathbb{N}$ choose $\alpha_i \in I$ s.t. $\operatorname{supp}(f_i) \subseteq U_{\alpha_i}$

and define $\omega^i \in \Omega^n(M)$ by $\omega^i := f_i \, du^1_{\alpha_i} \wedge \dots \wedge du^n_{\alpha_i}$

(extended by zero to all of M).

Set $\omega := \sum_{i \in \mathbb{N}} \omega^i$. Since $\operatorname{supp}(f_i)$ is locally finite, ω is a smooth n-form on M.

Fix $x \in M$. Then $\sum_{i \in \mathbb{N}} f_i(x) = 1$ implies $\exists\, i$ s.t. $f_i(x) > 0$

By definition, $\omega^i(x)\left(\frac{\partial}{\partial u^1_{\alpha_i}}, \dots, \frac{\partial}{\partial u^n_{\alpha_i}}\right) = f_i(x) > 0$.

Since the atlas is oriented and all f_j's have non-negative values on M, We have $\omega^j(x)\left(\frac{\partial}{\partial u^1_{\alpha_i}}, \dots, \frac{\partial}{\partial u^n_{\alpha_i}}\right) \geq 0 \quad \forall j$.

$\Rightarrow \omega(x) \neq 0$

3 $\Rightarrow$ 1 $\quad \omega \in \Omega^n(M)$ nowhere vanishing.

For $x \in M$, we call a basis $\{\xi_1, \dots, \xi_n\}$ of T_xM positively oriented, if $\omega(x)(\xi_1, \dots, \xi_n) > 0$. This gives an orientation on M.

$\square$

Remark Prop. shows

- Every oriented atlas determines an orientation and two oriented atlas on M determine the same orientation $\iff$ they are orientation equivalent.

- Similarly, any nowhere vanishing n-form determines an orientation on M and any two such n-form ω and τ determine the same orientation $\iff$ $\exists$ a positive smooth fct. $f : M \to \mathbb{R}$ on M s.t. $\tau = f\omega$.

- Given orientable mfd., a choice of orientation on M is equivalent to a choice of a maximal (or an oriented) equivalence (las) oriented atlas, and also to a choice of nowhere vanishing n-form ω up to multipl. by a smooth positive function.

<u>Ex</u>. $M = \mathbb{R}^n$ is orientable.

<u>Ex</u> Möbius band is not orientable.

<u>Ex</u> $\mathbb{RP}^n$ is orientable $\Longleftrightarrow$ n is odd. (see Tutorial).

5.2 Integrals

Suppose M is an oriented n-dim. mfd, and let $\mathcal{U} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ be an oriented atlas (giving rise to the fixed orientation).

Then we can define an integral for n-forms on M as follows:

We write $\operatorname{supp}(\omega) := \overline{\{x \in M : \omega(x) \neq 0\}}$ for the support of $\omega \in \Omega^n(M)$ and $\Omega^n_c(M)$ for the space of

n-forms with compact support.

Suppose $\omega \in \Omega^n_c(M)$. Since $\operatorname{supp}(\omega)$ is compact, $\exists$ finitely many charts (U_i, u_i) $i = 1, \dots, \ell$ of the maximal oriented atlas $\mathcal{A}$ (defining the orientation) s.t. $\operatorname{supp}(\omega) \subseteq U_1 \cup \dots \cup U_\ell$. Further, let $f_j : M \to [0,1]$ be smooth functions $j = 1, \dots, \ell$ s.t. $\operatorname{supp}(f_j) \subseteq U_j$ and $\left(\sum_{j=1}^{\ell} f_j\right)\Big|_{\operatorname{supp}(\omega)} \equiv 1$.

((Choose partition of unity subordinate to $\{U_1, \dots, U_\ell, M \setminus \operatorname{supp}(\omega)\}$

and let f_1 be the sum of all functions with support in U_1,

f_2 be the remaining functions whose support is contained in U_2, ...)

$$\int_M \omega := \sum_{i=1}^{\ell} \int_{u_i(U_i)} \overbrace{\underbrace{f_i\omega\left(u_i^{-1}(y)\right)\left(T_y u^{-1} e_1, \dots, T_y u^{-1} e_n\right)}_{f_i\omega(u_i^{-1}(y))\left(\frac{\partial}{\partial u_i^1}(u_i^{-1}(y)), \dots, \frac{\partial}{\partial u_i^n}(u_i^{-1}(y))\right)}}^{(u_i^{-1})^* f_i\omega(e_1,\dots,e_n) = \text{local coordinate expression of } f_i\omega|U_i}$$

$$\omega = \sum f_i \omega \qquad f_i\omega_i \in \Omega^n_c(U_i).$$

Since $\mathrm{supp}(f_i\omega)$ is a compact subset of U_i, the right-hand side equals a finite sum of integrals over functions with compact support. Hence, this integral is finite.

Let us check that $\int_M \omega$ is well-defined, i.e. independent of all the choices: Suppose (V_j, v_j) are finitely many charts of $\mathcal{A}$ and $g_j : M \to [0,1]$ functions we chose constructed w.r.t. these charts.

$$\Longrightarrow \quad \omega = \sum_j g_j \omega \quad \text{and hence} \quad \underline{f_i \omega} = \sum_j \underline{f_i g_j \omega}$$

$$\Longrightarrow \quad \sum_i \int_{u_i(U_i)} f_i \omega\left(u_i^{-1}(y)\right)\left(T_y u_i^{-1} e_1, \dots, T_y u_i^{-1} e_n\right)$$

$$= \sum_{i,j} \int_{u_i(U_i)} f_i g_j \omega\left(u_i^{-1}(y)\right)\left(T_y u_i^{-1} e_1, \dots, T_y u_i^{-1} e_n\right)$$

$$\underset{\substack{\mathrm{supp}(f_i g_j \omega) \\ \subseteq U_i \cap V_j}}{=} \quad \sum_{i,j} \int_{u_i(U_i \cap V_j)} \text{———}//\text{————————}$$

$$\int_{u_i(U_i\cap V_j)} f_i g_j w\,(u_i^{-1}(y))\,(T_y u_i^{-1} e_1, \dots, T_y u_i^{-1} e_n)$$

$$= \int_{u_i(U_i\cap V_j)} \det(D(v_j \circ u_i^{-1}))\, f_i g_j w\,(u_i^{-1}(y))\,(T_{(v_j\circ u_i^{-1})(y)} v_j^{-1} e_1, \dots, T_{(v_j\circ u_i^{-1})(y)} v_j^{-1} e_n)$$

$$= \int_{v_j(U_i\cap V_j)} f_i g_j w\,(v_j^{-1}(y))\,(T v_j^{-1} e_1, \dots, T v_j^{-1} e_n) = \int_{v_j(V_j)} \longrightarrow, \longrightarrow$$

transform-
rule for integrals
and $\det(D(v_j \circ u_i^{-1})) > 0$

since $f_i g_j w$ vanishes
outside of $v_j(U_i \cap V_j)$.

Hence, $\int_M \omega$ is well-defined.