Recall from yesterday:

M mfd. of dim. n, orientable and oriented.

$\omega \in \Omega^n_c(M)$ n-form with compact support.

$\mathrm{supp}(\omega)$ compact $\Longrightarrow$ $\exists$ finitely many charts from the maximal oriented atlas $(U_1, u_1), \dots, (U_\ell, u_\ell)$

s.t. $\mathrm{supp}(\omega) \subseteq U_1 \cup \dots \cup U_\ell$

Further choose $f_i : M \to [0,1]$ smooth $(i = 1, \dots, \ell)$ s.t. $\mathrm{supp}(f_i) \subseteq U_i$ and $\left.\sum_{i=1}^{\ell} f_i \right|_{\mathrm{supp}(\omega)} \equiv 1$.

$$\int_M \omega := \sum_{i=1}^{\ell} \int_{u_i(U_i)} \underbrace{f_i\omega(u_i^{-1}(y))\left(\frac{\partial}{\partial u_i^1}(u_i^{-1}(y)), \dots, \frac{\partial}{\partial u_i^n}(u_i^{-1}(y))\right)}_{\substack{\shortparallel \\ \text{local coordinate expression} \\ \text{of } f_i\omega}}$$

<u>Prop. 5.5</u> M mfd. of dim. n

Then $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ is a surjective linear map.

<u>Proof</u> Linearity follows from the definition and the linearity of the integral of fcts. in $\mathbb{R}^n$.

Surjectivity: We need to show $\exists\, \omega \in \Omega^n_c(M)$ s.t. $\int_M \omega \neq 0$.

Choose a chart (U, u) and smooth <u>nonzero</u> function $f : M \to \mathbb{R}_{\geq 0}$ with compact support contained in U.

$\omega := f\, du^1 \wedge \dots \wedge du^n$ can be extended by zero to an element in $\Omega^n_c(M)$.

$$\Longrightarrow \int_M \omega = \int_{u(U)} f \circ u^{-1} > 0 .$$

$\square$.

Special cases :

① $M = \mathbb{R}$ equipped with its standard orientation, for $a < b$ and any $\omega = f\,dt \in \Omega^1(\mathbb{R})$ (t is coordinate in $\mathbb{R}$), we have

$$\int_{[a,b]} \omega = \int_a^b f(t)\, \underline{dt}$$

② Line integrals : $V \subseteq \mathbb{R}^n$ open subset, $\omega = \sum_{i=1}^n \omega_i\, dx^i \in \Omega^1(V)$ and $\gamma : I \longrightarrow V$ C^∞-curve, $I \subseteq \mathbb{R}$ open interval :

$\Rightarrow$ $\gamma^*\omega \in \Omega^1(I)$ and for $a, b \in I$

$$\int_{[a,b]} \gamma^*\omega = \sum_{i=1}^n \int_{[a,b]} (\omega_i \circ \gamma)(\gamma_i'(t))\, dt .$$

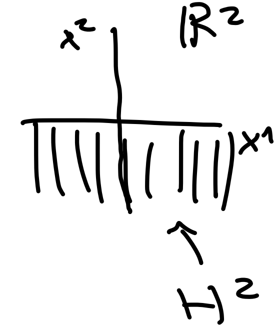
Line integral of ω along $\gamma|_{[a,b]} =: \alpha$

also written as $\int_\alpha \omega$

5.3 Manifolds with boundary

Def. 5.6

- A n-dimens. (smooth) manifold with boundary is a Hausdorff second countable topolog. space M equipped with a maximal C^∞-atlas of chart with values in the half space $H^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^1 \leq 0\}$

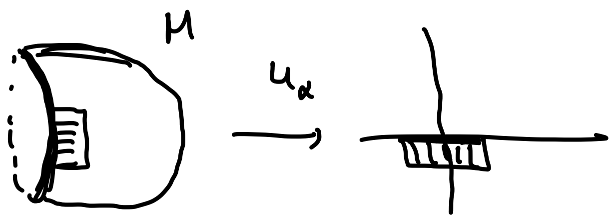


- A C^∞-atlas $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ of M with values in H^n is a collection ~~of~~ $u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq H^n \subseteq \mathbb{R}^n$ of homeomorphisms, where $U_\alpha \subseteq M$ and $u_\alpha(U_\alpha) \subseteq H^n$ are open subsets s.t.

- $M = \bigcup_{\alpha \in I} U_\alpha$

- $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \longrightarrow u_\beta(U_\alpha \cap U_\beta)$ are smooth,

which means that they can be extended to smooth maps defined on open subsets of $\mathbb{R}^n$ containing $u_\alpha(U_\alpha \cap U_\beta)$.



- A point $x \in M$ is called a boundary point of M, if
  <u>$\exists$ a chart</u> (U_α, u_α) s.t. $u_\alpha(x) \in u_\alpha(U_\alpha) \cap \{0\} \times \mathbb{R}^{n-1}$
  $$= u_\alpha(U_\alpha) \cap \partial H^n$$

  where $\partial H^n = \{(x^1, \dots, x^n) \in H^n : x^1 = 0\}$.

  We write $\partial M = \{x \in M : x \text{ is a boundary points}\}$

  Note that $x \in \partial M \iff \forall$ chart $(U_\alpha, u_\alpha) \in \mathcal{A}$ with $x \in U_\alpha$
  $$u_\alpha(x) \in \partial H^n.$$

- Points $x \in M \setminus \partial M$ are called interior points.
  $$x \in M \setminus \partial M \iff u_\alpha(x) \in H^n \setminus \partial H^n \quad \forall \text{ charts } (U_\alpha, u_\alpha) \in \mathcal{A}.$$

Prop. 5.7 M n-dim. mfd. with boundary (with $\partial M \neq \emptyset$)

Then ∂M is an $(n-1)$-dim. manifold without boundary.

Proof. An atlas for ∂M is given by

$$\left\{ \left(\underline{U_\alpha \cap \partial M}, \, u_\alpha \big|_{U_\alpha \cap \partial M} \right) : (U_\alpha, u_\alpha) \in \mathcal{A} \right\},$$

where $\mathcal{A}$ is a atlas for the manifold M with boundary.

($u_\alpha(U_\alpha) \cap \{0\} \times \mathbb{R}^{n-1}$ is open in $\underline{\mathbb{R}^{n-1}}$).

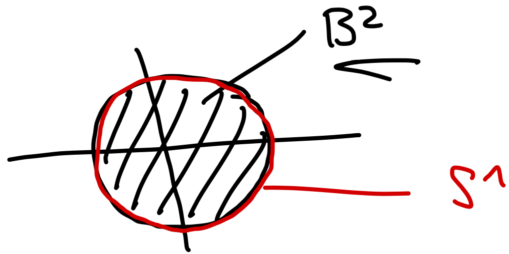
$\square$

Ex.

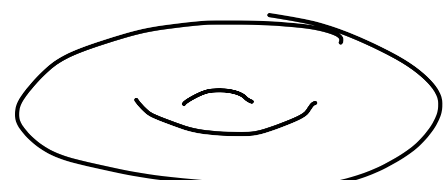
$M = B^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$

$\partial M = S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$

$n = 1$



Ex. Rotation of B^2 around circle leads to a manifold M with boundary $\partial M = T^2$



All concepts such as smooth functions, vector fields, different. forms, tensors etc. make sense on manifolds with boundary.

- If $i : \underline{\partial M} \hookrightarrow M$ is the natural inclusion, then it is smooth and for any k-form ω on M, $i^*\omega$ is a k-form on ∂M.

  ($x \in M$: $\underline{T_x \partial M} \subseteq T_x M$).

An orientation on a manifold with boundary (defined as for manifolds without boundary) induces an orientation on ∂M.

Suppose $\mathcal{A} = \{(U_\alpha, u_\alpha)\}_{\alpha \in I}$ is an oriented atlas for M and

consider $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_{\alpha\beta})$ for two charts of $\mathcal{A}$.

As observed $u_\alpha(U_\alpha \cap U_\beta) \cap \underline{\{0\} \times \mathbb{R}^{n-1}}$ is mapped to $u_\beta(U_\alpha \cap U_\beta) \cap \{0\} \times \mathbb{R}^{n-1}$

At a point $x = (0, x^2, \ldots, x^n) \in u_\alpha(U_\alpha \cap U_\beta)$:

$$D_x(u_\beta \circ u_\alpha^{-1}) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ \underline{} & & & \\ v & & \underline{A} & \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^{n-1}$, $A \in M_{n-1 \times n-1}(\mathbb{R})$.

Since, $u_\beta \circ u_\alpha^{-1}$ maps interior to interior points, i.e. points with negative x^1 coordinate to ones with negative x^1 coordinate, we have $\lambda > 0$

Hence, $\det(D_x u_\beta \circ u_\alpha^{-1}) > 0$ (because charts are of oriented atlas)

implies that $\det(A) > 0$. Since A is the derivative of the transition map of the charts for ∂M induced by (U_α, u_α) and (U_β, u_β).

Hence, the atlas on ∂M induced by $\mathcal{U}$ is also oriented.

Suppose M is a manifold with boundary of dim. n.

If $\omega \in \Omega_c^{n-1}(M)$, then ω vanishes on the open subset $M \setminus \mathrm{supp}(\omega)$ and hence so does $d\omega$ (Thm. 4.18) which implies $\operatorname{supp}(d\omega) \subseteq \operatorname{supp}(\omega)$. In particular, $d\omega \in \Omega_c^n(M)$.

Thm. 5.8 (Stokes)

Suppose M is an oriented n-dim. manifold with boundary ∂M.
For any $\omega \in \Omega_c^{n-1}(M)$ we have:

$$\int_M d\omega = \int_{\partial M} \omega \quad \left(= \int_{\partial M} i^* \omega \right)$$

In particular, if M is a manifold without boundary ($\partial M = \emptyset$), then $\int_M d\omega = 0$.

<u>Proof</u>. Assume $\omega \in \Omega_c^{n-1}(M)$

- Let (U_j, u_j) $j = 1, \dots, \ell$ be charts of an oriented atlas of M s.t. $\operatorname{supp}(\omega) \subseteq U_1 \cup \dots \cup U_\ell$ and $f_j : M \to [0,1]$ smooth functions s.t. $\operatorname{supp}(f_j) \subseteq U_j$ and $\sum_{j=1}^{\ell} f_j \big|_{\operatorname{supp}(\omega)} \equiv 1$.

- Then $\left(U_j \cap \partial M, u_j\big|_{U_j \cap \partial M}\right)$ and $f_j\big|_{\partial M}$ for $j = 1, \dots, \ell$ can be used to compute $\int_{\partial M} \omega = \underline{\underline{\sum_{j=1}^{\ell} \int_{U_j \cap \partial M} f_j \omega}}$

- Also, note that $\omega = \sum_{j=1}^{\ell} f_j \omega$ implies $d\omega = \sum_{j=1}^{\ell} d(f_j\omega)$

and we have $\operatorname{supp}(d(f_j\omega)) \subseteq \operatorname{supp}(f_j \omega) \subseteq U_j$.

Hence, $\int_M d\omega = \sum_{i=1}^{\ell} \int_{U_j} d(f_j\omega)$.

It suffices to show that $\int_{U_j} d(f_j\omega) = \int_{\partial U_j} f_j \omega \quad \forall j$.

Without loss of generality we may assume $\operatorname{supp}(\omega)$ is contained in the domain of a single chart (U, u).

Then

$$\omega = \sum_{j=1}^{n} \omega_j \, du^1 \wedge \ldots \wedge \widehat{du^j} \wedge \ldots \wedge du^n$$

for smooth fcts. $\omega_j : M \to \mathbb{R}$ with compact support in U.

- The tangent space $T_x \partial M$ for $x \in \partial M$ is spanned by $\frac{\partial}{\partial u^i}(x)$, $i \geq 2$

Hence, $du^1 \big|_{\partial M} \equiv 0$ and so $\omega\big|_{\partial M} = \omega_1 \, du^2 \wedge \ldots \wedge du^n$

This implies $\int_{\partial M} \omega = \int_{\partial u(U)} \omega_1 \circ u^{-1} = \int_{\{0\} \times \mathbb{R}^{n-1}} \omega_1 \circ u^{-1}$

because ω_1 has compact support contained in U.

- By Thm. 4.18, $d\omega = \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial u^j} \underline{du^j} \wedge du^1 \wedge \dots \wedge \underline{\widehat{du^j}} \wedge \dots \wedge du^n$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial \omega_j}{\partial u^j} du^1 \wedge \dots \wedge du^n$$

$$\Longrightarrow \int_M d\omega = \sum_{j=1}^{n} (-1)^{j-1} \int_{u(U)} \frac{\partial(\omega_j \circ u^{-1})}{\partial x^j} = \sum_{j=1}^{n} (-1)^{j-1} \int_{\underbrace{[-\infty,0] \times \mathbb{R}^{n-1}}_{= H^n}} \frac{\partial(\omega_j \circ u^{-1})}{\partial x^j}$$

since ω_j's have compact support contained in U

Fubini: Thm. for integrals allows to decompose this into integrals over the individual coordinates, where order of the order of integration does not matters.

$$\Longrightarrow \int_H d\omega = \int_{\mathbb{R}^{n-1}} \Big(\overbrace{\int_{-\infty}^{0} \frac{\partial(\omega_1 \circ u^{-1})}{\partial x^1} \, dx^1}^{(*)\; = \omega_1 \circ u^{-1}(0, x^2, \dots, x^n)} \Big) dx^1 \dots dx^n$$

$$+ \sum_{j=2}^{n} (-1)^{j-1} \int_{(-\infty,0] \times \mathbb{R}^{n-2}} \Big(\underbrace{\int_{-\infty}^{\infty} \frac{\partial(\omega_j \circ u^{-1})}{\partial x^j} \, dx^j}_{} \Big) dx^1 \dots \widehat{dx_j} \dots dx_n$$

$= 0$ by FTC + ω_j have compact support.

(*) by Fundam. Thm. of calculus and ω_1 having compact support.

$$\Longrightarrow \int_M d\omega = \int_{\mathbb{R}^{n-1}} (\omega_1 \circ u^{-1})(0, x^2, \dots, x^n)\, dx^2 \dots dx^n$$

$$= \int_{\partial M} \omega$$

$\square$.

5.4. Excursion: de Rham cohomology

By Prop. 4.14, we know $(\Omega^*(M), \wedge)$ is an (unital, associative) graded-anti-commutative algebra over $C^\infty(M, \mathbb{R})$.

- $$\Omega^*(M) = \bigoplus_{k=0}^{\dim(M)} \Omega^k(M)$$

- $$\Omega^k(M) \wedge \Omega^\ell(M) \subseteq \Omega^{k+\ell}(M)$$

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega \qquad \omega \in \Omega^k(M), \eta \in \Omega^\ell(M).$$

Moreover, we have a linear map $d : \Omega^*(M) \to \Omega^*(M)$, which is a graded derivation of degree 1: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ (*) $\forall \omega \in \Omega^k(M)$.

By Thm. 4.18: $d \circ d = 0$

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \dots \to \Omega^{\dim(M)}(M) \to 0$$

<u>Def. 5.9</u> $\omega \in \Omega^*(M)$

1. ω is called closed, if $d\omega = 0$
2. ω is called exact, if $\exists \eta \in \Omega^*(M)$ s.t. $\omega = d\eta$

Note that $d^2 = 0$ implies that every exact form is closed.

- $\mathrm{Ker}(d) =: Z^*(M) \subseteq \Omega^*(M)$ is a subspace of $\Omega^*(M)$ (subspace of closed differential forms)
  $\mathrm{Ker}(d) = \{\omega \in \Omega^*(M) : d\omega = 0\}$

and also a subalgebra of $\Omega^*(M)$, since (*).

$B^*(M) := \mathrm{Im}(d) \subseteq Z^*(M) \subseteq \Omega^*(M)$ is a subspace.

and a two-sided ideal in $Z^*(M)$: $\left(\begin{array}{l} B^*(M) \wedge Z^*(M) \subset B^*(M) \\ Z^*(M) \wedge B^*(M) \subseteq B^*(M) \end{array} \right)$

$\eta = d\eta'$, $\omega \in Z^*(M)$

$\in \Omega^{k+1}(M)$, $\eta' \in \Omega^k(M)$

$$d(\eta' \wedge \omega) = d\eta' \wedge \omega + (-1)^k \eta' \wedge \underbrace{d\omega}_{=0} = d\eta' \wedge \omega = \eta \wedge \omega$$

$$\Longrightarrow \quad H^*(M) := Z^*(M) / B^*(M) = \bigoplus_{k \geq 0} \underbrace{Z^k(M) / B^k(M)}_{=: H^k(M)}.$$

is a (unital, associative) graded-anti-commutative algebra over $\mathbb{R}$.

It is called the de Rham cohomology algebra of M and $H^k(M)$ the k-th de Rham cohomology space (or group).

For $\omega \in Z^k(M)$ we write $[\omega] \in H^k(M)$ for its cohomology class.