Tutorial 4—Global Analysis

19.10.2021

1. We have seen in the first tutorial that $\text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of dimension $r(n+m-r)$ in. For $X \in \text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ compute the tangent space
$$T_X\text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \subset T_X\text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

2. We have seen in the first tutorial that the Grassmannian manifold $\text{Gr}(r,n)$ can be realized as a submanifold of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of dimension $r(n-r)$. For $E \in \text{Gr}(r,n)$ compute the tangent space
$$T_E\text{Gr}(r,n) \subset T_E\text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^n).$$

3. Consider the general linear group $\text{GL}(n,\mathbb{R})$ and the special linear group $\text{SL}(n,\mathbb{R})$. We have seen that they are submanifolds of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (even so called Lie groups) and that $T_{\text{Id}}\text{GL}(n,\mathbb{R}) \cong M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.
   (a) Compute the tangent space $T_{\text{Id}}\text{SL}(n,\mathbb{R})$ of $\text{SL}(n,\mathbb{R})$ at the identity Id.
   (b) Fix $A \in \text{SL}(n,\mathbb{R})$ and consider the conjugation $\text{conj}_A : \text{SL}(n,\mathbb{R}) \to \text{SL}(n,\mathbb{R})$ by A given by $\text{conj}_A(B) = ABA^{-1}$. Show that conj_A is smooth and compute the derivative $T_{\text{Id}}\text{conj}_A : T_{\text{Id}}\text{SL}(n,\mathbb{R}) \to T_{\text{Id}}\text{SL}(n,\mathbb{R})$.
   (c) Consider the map $\text{Ad} : \text{SL}(n,\mathbb{R}) \to \text{Hom}(T_{\text{Id}}\text{SL}(n,\mathbb{R}), T_{\text{Id}}\text{SL}(n,\mathbb{R}))$ given by $\text{Ad}(A) := T_{\text{Id}}\text{conj}_A$. Show that Ad is smooth and compute $T_{\text{Id}}\text{Ad}$.

4. Consider $\mathbb{R}^n$ equipped with the standard inner product of signature (p,q) (where $p+q=n$) given by
$$\langle x, y\rangle := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i$$
and the group of linear orthogonal transformation of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ given by
$$\text{O}(p,q) := \{A \in \text{GL}(n,\mathbb{R}) : \langle Ax, Ay\rangle = \langle x, y\rangle \quad \forall x, y \in \mathbb{R}^n\}.$$
   (a) Show that
$$\text{O}(p,q) = \{A \in \text{GL}(n,\mathbb{R}) : A^{-1} = I_{p,q}A^tI_{p,q}\},$$
where $I_{p,q} = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}$, and that $\text{O}(p,q)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

(b) Show that $\mathrm{O}(p,q)$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$ with respect to matrix multiplication μ and that $\mu : \mathrm{O}(p,q) \times \mathrm{O}(p,q) \to \mathrm{O}(p,q)$ is smooth (i.e. that $\mathrm{O}(p,q)$ is a Lie group.)

(c) Compute the tangent space $T_{\mathrm{Id}}\mathrm{O}(p,q)$ of $\mathrm{O}(p,q)$ at the identity Id.