Magnetohydrodynamics

Coupling hydrodynamics with Maxwell equations

Jiří Krtička

Masaryk University

Basic equations

Basic assumption

We shall consider *plasma* as a gas, in which the ionization can not be neglected. In such a gas the hydrodynamical equations are affected by the electromagnetic field. On the other hand, Maxwell equations shall account for the presence of plasma. This leads to a concept of coupled *magnetohydrodynamical (MHD) equations*.

We shall assume that plasma is electrically neutral, that is, the charge of selected macroscopic volume is zero.

Equation of continuity

We shall assume that plasma consists of: electrons (e), ions (i), and neutrals (n). The equation of continuity holds for all three components,

$$\frac{\partial n_{\mathrm{e}}}{\partial t} + \nabla \cdot (n_{\mathrm{e}} \boldsymbol{v}_{\mathrm{e}}) = 0,$$
$$\frac{\partial n_{\mathrm{i}}}{\partial t} + \nabla \cdot (n_{\mathrm{i}} \boldsymbol{v}_{\mathrm{i}}) = 0,$$
$$\frac{\partial n_{\mathrm{n}}}{\partial t} + \nabla \cdot (n_{\mathrm{n}} \boldsymbol{v}_{\mathrm{n}}) = 0.$$

Multiplying these equations by appropriate masses of particles and summing them up, we derive the mean equation of continuity

$$\frac{\partial}{\partial t} (n_{\mathrm{e}} m_{\mathrm{e}} + n_{\mathrm{i}} m_{\mathrm{i}} + n_{\mathrm{n}} n_{\mathrm{n}}) + \nabla \cdot (n_{\mathrm{e}} m_{\mathrm{e}} \boldsymbol{v}_{\mathrm{e}} + n_{\mathrm{i}} m_{\mathrm{i}} \boldsymbol{v}_{\mathrm{i}} + n_{\mathrm{n}} m_{\mathrm{n}} \boldsymbol{v}_{\mathrm{n}}) = 0.$$

Introducing the total density $\rho = \sum n_\alpha m_\alpha$ and mean velocity $\boldsymbol{v} = \sum n_\alpha m_\alpha \boldsymbol{v}_\alpha / \left(\sum n_\alpha m_\alpha\right)$, the continuity equation takes the same form as for one-component fluid,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

Equation of continuity for electric charge

Multiplying the three continuity equations by charge of each component, we derive equation of continuity for electric charge in the form of

$$\frac{\partial}{\partial t}\left(n_\mathrm{e} q_\mathrm{e} + n_\mathrm{i} q_\mathrm{i}\right) + \nabla \cdot \left(n_\mathrm{e} q_\mathrm{e} \boldsymbol{v}_\mathrm{e} + n_\mathrm{i} q_\mathrm{i} \boldsymbol{v}_\mathrm{i}\right) = 0.$$

Introducing the total electric charge density $\rho_\mathrm{e} = \sum n_\alpha q_\alpha$ and the current density $\boldsymbol{j} = \sum n_\alpha q_\alpha \boldsymbol{v}_\alpha$, the equation of continuity for electric charge has the form of

$$\frac{\partial \rho_\mathrm{e}}{\partial t} + \nabla \cdot \boldsymbol{j} = 0.$$

However, we assume quasineutrality of plasma ($\rho_\mathrm{e} = 0$), therefore the equation of continuity for electric charge simplifies to

$$\operatorname{div} \boldsymbol{j} = 0.$$

Equation of motion

The equation of motion holds for each component of the flow separately,

$$\underbrace{\frac{\partial}{\partial t}\left(m_\alpha n_\alpha \boldsymbol{v}_\alpha\right)}_{1} + \underbrace{\nabla\left(n_\alpha m_\alpha \boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha\right)}_{2} =$$

$$\underbrace{\boldsymbol{f}_\alpha}_{3} - \underbrace{\nabla p_\alpha}_{4} + \underbrace{q_\alpha n_\alpha \boldsymbol{E}}_{5} + \underbrace{\frac{1}{c} q_\alpha n_\alpha \boldsymbol{v}_\alpha \times \boldsymbol{B}}_{6} + \underbrace{\sum_\beta}_{7} R_{\alpha\beta},$$

where we accounted for the Lorentz force (5+6) and for friction between components (7). Summing the equation over individual components:

1 : use the definitions of ρ and $\boldsymbol{v}$ and the continuity equation,

2 : assume $\boldsymbol{v}_\alpha \cdot \nabla \boldsymbol{v}_\alpha \approx \boldsymbol{v} \cdot \nabla \boldsymbol{v}$,

3 : introduce the total volume force $\boldsymbol{f} = \sum_\alpha \boldsymbol{f}_\alpha$,

4 : introduce the total pressure $p = \sum_\alpha p_\alpha$,

5 : assume quasineutrality $\sum_\alpha q_\alpha n_\alpha = 0$ (implying no influence of $\boldsymbol{E}$),

6 : introduce the current density $\boldsymbol{j} = \sum n_\alpha q_\alpha \boldsymbol{v}_\alpha$,

7 : and assuming Newton's third law $\sum_{\alpha\beta} R_{\alpha\beta} = 0$, we arrive at

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{f} + \frac{1}{c} \boldsymbol{j} \times \boldsymbol{B}.$$

Maxwell's equations

We shall write the the Maxwell's equations in cgs units in vacuum (i.e., $\boldsymbol{H} = \boldsymbol{B}$ and $\boldsymbol{E} = \boldsymbol{D}$) and neglecting the displacement current $1/c\,\partial\boldsymbol{D}/\partial t$. From the Ampère's law

$$\operatorname{rot}\boldsymbol{H} = \frac{1}{c}\frac{\partial\boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}$$

then follows

$$\boldsymbol{j} = \frac{c}{4\pi}\operatorname{rot}\boldsymbol{B}.$$

From the equation of motion with frictional term follows the Ohm's law

$$\boldsymbol{j} = \sigma\boldsymbol{E}'$$

written in the frame where $\boldsymbol{v}' = 0$. Transforming into a general frame

$$\boldsymbol{j} = \sigma\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right).$$

From this follows for the electric intensity (using the Ampère's law)

$$\boldsymbol{E} = \frac{\boldsymbol{j}}{\sigma} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B} = \frac{c}{4\pi\sigma}\operatorname{rot}\boldsymbol{B} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}.$$

Maxwell's equations

From the induction equation

$$-\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{rot}\,\boldsymbol{E}$$

follows using $\boldsymbol{E} = \frac{c}{4\pi\sigma}\mathrm{rot}\,\boldsymbol{B} - \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}$ the equation for the magnetic field

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{c^2}{4\pi\sigma}\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{B} + \mathrm{rot}\,(\boldsymbol{v} \times \boldsymbol{B})\,.$$

For an ideal plasma (ideal MHD) the conductivity $\sigma \to \infty$ and

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{rot}\,(\boldsymbol{v} \times \boldsymbol{B})\,.$$

In this case the Ohm's law simplifies to

$$\boldsymbol{E} = -\frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}.$$

A side note: magnetic diffusion

The term inversely proportional to σ in the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{c^2}{4\pi\sigma} \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{B} + \mathrm{rot}\,(\boldsymbol{v} \times \boldsymbol{B})$$

describes magnetic field diffusion, because

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{c^2}{4\pi\sigma} \Delta \boldsymbol{B}$$

is a diffusion equation. From this equation follows the characteristic diffusion time

$$\tau = \frac{4\pi\sigma}{c^2} L^2,$$

which is typically very large for astrophysical plasmas due to large L. Therefore, in most applications the diffusion term can be safely neglected.

Basic equations of ideal MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{f} + \frac{1}{4\pi} \left(\operatorname{rot} \boldsymbol{B}\right) \times \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{rot}\left(\boldsymbol{v} \times \boldsymbol{B}\right)$$

$$\operatorname{div} \boldsymbol{B} = 0$$

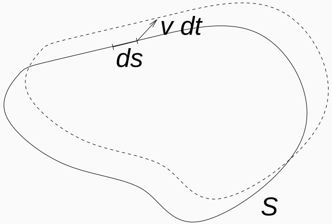
equation for energy

Some applications

Flux freezing

Let us study a closed curve $\mathscr{C} = \partial S$ that is co-moving with the plasma and calculate the change of magnetic flux through $\mathscr{C}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{S} =$$
$$\int_S \frac{\partial \boldsymbol{B}}{\partial t}\,\mathrm{d}\boldsymbol{S} + \oint_{\partial S} \boldsymbol{B}\cdot(\boldsymbol{v}\times\mathrm{d}\boldsymbol{s}) =$$
$$\int_S \frac{\partial \boldsymbol{B}}{\partial t}\,\mathrm{d}\boldsymbol{S} + \oint_{\partial S} \boldsymbol{B}\times\boldsymbol{v}\,\mathrm{d}\boldsymbol{s} =$$
$$\int_S \left(\frac{\partial \boldsymbol{B}}{\partial t} - \mathrm{rot}\,(\boldsymbol{v}\times\boldsymbol{B})\right)\mathrm{d}\boldsymbol{S} = 0.$$



This means that the flux remains constant in time for any arbitrary closed curve. In turn, this implies that magnetic field-lines must move with the plasma. Or, in scientific jargon, the field-lines are frozen into the plasma.

Flow along $\boldsymbol{B}$ is not affected by the field, while the flow perpendicular to $\boldsymbol{B}$ tears down the field. In a tube the product SB is constant, for which the continuity equation $\rho S v = \text{const.}$ implies $\rho \sim B/v$.

Two limiting cases

From the momentum equation

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{f} + \frac{1}{4\pi} \left(\mathrm{rot}\, \boldsymbol{B}\right) \times \boldsymbol{B}$$

two limiting cases follow. For negligible magnetic field $(\mathrm{rot}\, \boldsymbol{B}) \times \boldsymbol{B}/(4\pi) \approx 0$ the flow is not affected by the magnetic field and moves freely. But we still have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{rot}\left(\boldsymbol{v} \times \boldsymbol{B}\right),$$

and the magnetic field is from the flux freezing condition carried with the flow.

In the oposite case the magnetic field dominates and the flow follows the magnetic field.

Introducing the magnetic pressure

The magnetic term in the equation of motion can be rewritten as

$$\frac{1}{4\pi}\left(\operatorname{rot}\boldsymbol{B}\right)\times\boldsymbol{B}=\frac{1}{4\pi}\left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{B}-\nabla\left(\frac{B^2}{4\pi}\right).$$

Consequently, the equation of motion takes the form of

$$\rho\frac{\partial\boldsymbol{v}}{\partial t}+\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}=-\nabla\left(p+\frac{B^2}{4\pi}\right)+\boldsymbol{f}+\frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}.$$

From this equation, the term $B^2/(4\pi)$ can be regarded as *magnetic pressure*.

Magnetostatics

The magnetic pressure plays a crucial role in magnetostatics, which is an application of MHD on stationary ($\partial/\partial t = 0$) and static ($\boldsymbol{v} = 0$) systems. For these systems the momentum equation gives

$$\nabla \left(p + \frac{B^2}{4\pi} \right) = \boldsymbol{f} + \frac{1}{4\pi} \boldsymbol{B} \cdot \nabla \boldsymbol{B}$$

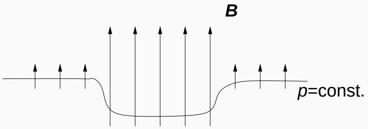
or

$$\nabla p = \frac{1}{4\pi} \left(\mathrm{rot}\, \boldsymbol{B} \right) \times \boldsymbol{B} + \boldsymbol{f}.$$

Magnetostatics: two examples

In absence of external fields $\boldsymbol{f} = 0$, for vertically constant magnetic field $\partial \boldsymbol{B}/\partial z = 0$ in the z direction the momentum equation simplifies to

$$\nabla \left(p + \frac{B^2}{4\pi} \right) = 0 \Rightarrow p + \frac{B^2}{4\pi} = \text{const.}$$



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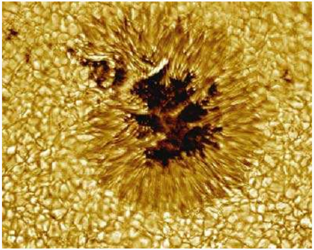

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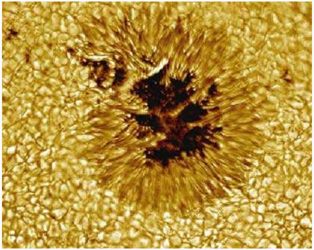
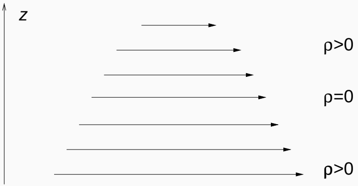


For atmosphere with horizontal magnetic field the hydrostatic equilibrium equation reads

$$\frac{\mathrm{d}}{\mathrm{d}z} \left(p + \frac{B^2}{4\pi} \right) = -\rho g.$$

For $\mathrm{d}B^2/\mathrm{d}z < 0$ this enables enables to support the matter ($\rho > 0$) above the regions of zero density ($\rho = 0$). This corresponds to solar prominences.



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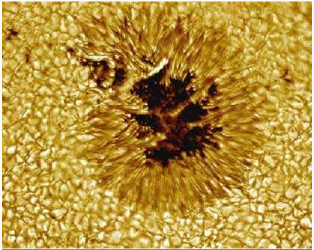

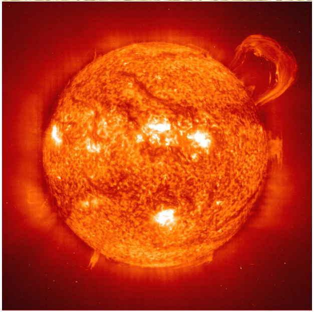

Polarization electric field in the stellar atmosphere

Let us go back to the momentum equations of individual components and assume an isothermal atmosphere in hydrostatic equilibrium without magnetic field ($\boldsymbol{B} = 0$) composed of ionized hydrogen only. In such a case the momentum equations simplify to

$$kT\nabla n_\mathrm{i} - m_\mathrm{i} n_\mathrm{i}\, \boldsymbol{g} - e n_\mathrm{i}\, \boldsymbol{E} = 0,$$

$$kT\nabla n_\mathrm{e} - m_\mathrm{e} n_\mathrm{e}\, \boldsymbol{g} + e n_\mathrm{e}\, \boldsymbol{E} = 0.$$

Summing these two equation (taking into an account that $m_\mathrm{e} \ll m_\mathrm{i}$ and $n_\mathrm{i} \approx n_\mathrm{e}$) we derive

$$\frac{1}{n_\mathrm{i}}\nabla n_\mathrm{i} = \frac{m_\mathrm{i}}{2kT}\, \boldsymbol{g},$$

which is ordinary hydrostatic equilibrium equation. Substracting these equations we derive equation for *polarization electic field* in the form of

$$\boldsymbol{E} = -\frac{m_\mathrm{i}}{2e}\, \boldsymbol{g}.$$

Suggested reading

E. Battaner: Astrohysical fluid dynamics

L. Mestel: Stellar magnetism