Hydrodynamical equations

Derivation and simple solutions

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Derivation of hydrodynamical equations

Boltzmann equation

Particle distribution function $F(t, \boldsymbol{x}, \boldsymbol{\xi})$ gives the number of particles in the element of the phase space $\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\xi} = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$ with coordinates $\boldsymbol{x}$ and momenta $\boldsymbol{\xi}$ as

$$F(t, \boldsymbol{x}, \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\xi}.$$

The time evolution of the particle distribution function under the influence of external force $\boldsymbol{f}$ acting on partice with mass m and taking into account particle collisions is

$$\frac{\partial F}{\partial t} + \frac{\xi_h}{m}\frac{\partial F}{\partial x_h} + f_h\frac{\partial F}{\partial \xi_h} = \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{\mathrm{coll}},$$

which is the Boltzmann equation. Here used the Einstein summation convention for index h.

Boltzmann equation

Using the Poisson bracket

$$\{H, F\} = \frac{\partial H}{\partial x_h}\frac{\partial F}{\partial \xi_h} - \frac{\partial H}{\partial \xi_h}\frac{\partial F}{\partial x_h},$$

the Boltzmann equation for the system that obeys the Hamilton equation can be rewritten as

$$\frac{\partial F}{\partial t} - \{H, F\} = \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{\mathrm{coll}}.$$

For stationary collisionless system the distribution function depends on the particle energy only,

$$\{H, F\} = 0.$$

Momentum equations

The Boltzmann equation can be solved numerically to derive the particle distribution function. However, for most of practical applications, the distribution function is very close to the Maxwelian distribution expressed at given location in the frame comoving with the fluid. In such a case, just mean quantities are of real importance for the description of the flow. These are moments of the Boltzmann equation

$$m \int F \, \mathrm{d}\boldsymbol{\xi} = \rho, \qquad \text{(0th moment, flow density)},$$

$$\frac{1}{m} \int \boldsymbol{\xi} F \, \mathrm{d}\boldsymbol{\xi} = \boldsymbol{v}, \qquad \text{(1st moment, flow velocity)}.$$

These can be derived by multiplying the Boltzmann equation by m and $\boldsymbol{\xi}/m$ and by integrating. However, the equation for n-th moment contains $n+1$-th moment. Consequently, we shall close the equations somehow to avoid obtainig infinite set of equations. This is done for the equation for the 2nd moment using thermodynamical relations for pressure.

The continuity equation

Multiplicating the Boltzmann equation by particle mass m and integrating over the velocity space

$$\underbrace{\int m\,\frac{\partial F}{\partial t}\,\mathrm{d}\boldsymbol{\xi}}_{1}+\underbrace{\int m\,\frac{\xi_h}{m}\frac{\partial F}{\partial x_h}\,\mathrm{d}\boldsymbol{\xi}}_{2}+\underbrace{\int m\,f_h\frac{\partial F}{\partial \xi_h}\,\mathrm{d}\boldsymbol{\xi}}_{3}=\underbrace{\int m\,\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{\mathrm{coll}}\mathrm{d}\boldsymbol{\xi}}_{4}$$

$$\begin{aligned}
1 =&\ m\frac{\partial}{\partial t}\int F\,\mathrm{d}\boldsymbol{\xi} = m\frac{\partial n}{\partial t} = \frac{\partial \rho}{\partial t},\\
2 =&\ \frac{\partial}{\partial x_h}\int \xi_h F\,\mathrm{d}\boldsymbol{\xi} = m\frac{\partial}{\partial x_h}\left(n v_h\right) = \frac{\partial\left(\rho v_h\right)}{\partial x_h},\\
3 =&\ \sum\int f_h[F]_{-\infty}^{\infty}\,\mathrm{d}\boldsymbol{\xi}' = 0 \text{ is } f_h \text{ does not depend on } \boldsymbol{\xi},\\
4 =&\ 0 \text{ for conserved quantity } (m),
\end{aligned}$$

where

- $n = \int F\,\mathrm{d}\boldsymbol{\xi}$ is number density of particles,
- $\rho = mn$ is the density,
- $v_h = \frac{1}{N}\int \xi_h F\,\mathrm{d}\boldsymbol{\xi}$ is the mean speed.

The continuity equation

This gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho v_h\right)}{\partial x_h} = 0,$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \boldsymbol{v}\right) = 0,$$

which is the continuity equation.

The continuity equation: interpretation

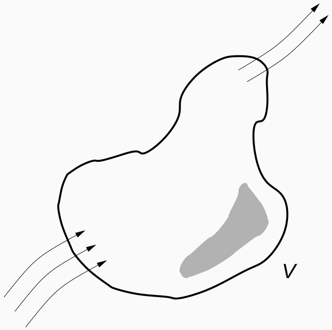
Integration over volume fixed in space gives

$$-\int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V = \int_V \nabla\cdot(\rho\boldsymbol{v})\,\mathrm{d}V,$$

or, using the Stokes theorem

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \oint_{\partial V} \rho\boldsymbol{v}\,\mathrm{d}\boldsymbol{S},$$

which is the expression of the law of conservation of mass.



The continuity equation: Lagrangian picture

Introducing the *Lagrangian derivative*, describing the time change of any quantity $q(t, \boldsymbol{x})$ following a moving fluid particle,

$$\frac{\mathrm{D}q(t,\boldsymbol{x})}{\mathrm{D}t} = \frac{\partial q(t,\boldsymbol{x})}{\partial t} + \frac{\partial q(t,\boldsymbol{x})}{\partial x_h}\frac{\partial x_h}{\partial t},$$

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla$$

the continuity equation can be rewritten as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0,$$

which for incompressible fluid ($\rho = \text{const.}$) is

$$\nabla\cdot\boldsymbol{v} = 0.$$

Equation of motion

Multiplying the Boltzmann equation by ξ_i and integrating

$$\underbrace{\int \xi_i \frac{\partial F}{\partial t}\,\mathrm{d}\boldsymbol{\xi}}_{1} + \underbrace{\int \xi_i \frac{\xi_h}{m}\frac{\partial F}{\partial x_h}\,\mathrm{d}\boldsymbol{\xi}}_{2} + \underbrace{\int \xi_i\, f_h \frac{\partial F}{\partial \xi_h}\,\mathrm{d}\boldsymbol{\xi}}_{3} = \underbrace{\int \xi_i \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{\mathrm{coll}}\mathrm{d}\boldsymbol{\xi}}_{4}$$

$$\begin{aligned}
1 =\ & \frac{\partial}{\partial t}\int \xi_i F\,\mathrm{d}\boldsymbol{\xi} = m\frac{\partial}{\partial t}(nv_i) = \frac{\partial(\rho v_i)}{\partial t},\\
2 =\ & \frac{1}{m}\frac{\partial}{\partial x_h}\int \xi_i\xi_h F\,\mathrm{d}\boldsymbol{\xi} = m\frac{\partial}{\partial x_h}\int (c_i+v_i)(c_h+v_h)F\,\mathrm{d}\boldsymbol{\xi} =\\
& m\frac{\partial}{\partial x_h}\left[v_i v_h\int F\,\mathrm{d}\boldsymbol{\xi} + v_h\int c_i F\,\mathrm{d}\boldsymbol{\xi} + v_i\int c_h F\,\mathrm{d}\boldsymbol{\xi} + \int c_i c_h F\,\mathrm{d}\boldsymbol{\xi}\right] =\\
& \frac{\partial}{\partial x_h}(mnv_i v_h + 0 + 0 + p_{hi}) = \frac{\partial}{\partial x_h}(\rho v_i v_h + p_{hi}),\\
3 =\ & \textstyle\sum_h \int f_h[\xi_i F]_{-\infty}^{\infty}\,\mathrm{d}\boldsymbol{\xi}' - \int\sum_h \delta_{ih} f_h F\,\mathrm{d}\boldsymbol{\xi} = -nf_i = -\rho g_i,\\
4 =\ & 0 \text{ for conserved quantity } (\xi), \text{ where}
\end{aligned}$$

- $c_h = \xi_h/m - v_h$ is the thermal speed,
- $p_{hi} = m\int c_i c_h F\,\mathrm{d}\boldsymbol{\xi}$ is the pressure tensor, $p_{hi} = p\delta_{hi}$,
- $g_i = f_i/m$ is force per unit of mass (acceleration).

Equation of motion

This gives

$$\frac{\partial\left(\rho v_i\right)}{\partial t}+\frac{\partial}{\partial x_h}\underbrace{\left(\rho v_i v_h+p\,\delta_{hi}\right)}_{\Pi_{ik}}=\rho g_i,$$

which is, after differencing and using the continuity equation,

$$\rho\frac{\partial v_i}{\partial t}+\rho v_h\frac{\partial v_i}{\partial x_h}=-\frac{\partial p}{\partial x_i}+\rho g_i,$$

where Π_{ik} is the momentum flux density tensor, or

$$\rho\frac{\partial \boldsymbol{v}}{\partial t}+\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}=-\nabla p+\rho\boldsymbol{g},$$

the momentum equation. Introducing the Lagrangian derivative the momentum equation has a form of Newton's second law

$$\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=-\nabla p+\rho\boldsymbol{g}.$$

Energy equation

Multiplicating the Boltzmann equation by $\xi_i\xi_j/m$ and integrating

$$\underbrace{\int \frac{1}{m}\xi_i\xi_j \frac{\partial F}{\partial t}\,\mathrm{d}\boldsymbol{\xi}}_{1} + \underbrace{\int \frac{1}{m^2}\xi_i\xi_j\xi_h \frac{\partial F}{\partial x_h}\,\mathrm{d}\boldsymbol{\xi}}_{2} + \underbrace{\int \xi_i\xi_j \frac{f_h}{m}\frac{\partial F}{\partial \xi_h}\,\mathrm{d}\boldsymbol{\xi}}_{3} = \underbrace{\int \frac{1}{m}\xi_i\xi_j \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{\text{coll}}\mathrm{d}\boldsymbol{\xi}}_{4}$$

$$\begin{aligned}
1 =&\ \frac{1}{m}\frac{\partial}{\partial t}\int \xi_i\xi_j F\,\mathrm{d}\boldsymbol{\xi} = m\frac{\partial}{\partial t}\int (c_i+v_i)(c_j+v_j)F\,\mathrm{d}\boldsymbol{\xi} = \frac{\partial}{\partial t}\left(\rho v_i v_j + p_{ij}\right),\\
2 =&\ \frac{1}{m^2}\frac{\partial}{\partial x_h}\int \xi_i\xi_j\xi_h F\,\mathrm{d}\boldsymbol{\xi} = m\frac{\partial}{\partial x_h}\int (c_i+v_i)(c_j+v_j)(c_h+v_h)F\,\mathrm{d}\boldsymbol{\xi} = \\
&\ \frac{\partial}{\partial x_h}\left(\rho v_i v_j v_h + v_h p_{ij} + v_i p_{hj} + v_j p_{hi}\right),\\
3 =&\ \begin{cases} 0, \text{ terms with } h \neq i \text{ and } h \neq j \text{ (direct integration)},\\ -f_i n v_j - f_j n v_i, \text{ terms with } h = i \text{ or } h = j \text{ (per-partes)}, \end{cases}\\
4 =&\ 0 \text{ when contraction is performed, where}
\end{aligned}$$

- $p_{hij} = \int c_h c_i c_j F\,\mathrm{d}\boldsymbol{\xi}/m$ is $p_{hij} = 0$ when neglecting viscosity.

Energy equation

After the contraction and multiplication by $\frac{1}{2}$ we derive

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{3}{2}p\right) + \frac{\partial}{\partial x_h}\left(\frac{1}{2}\rho v_h v^2 + \frac{5}{2}p v_h\right) - \rho v_i g_i = 0,$$

or, introducing the specific energy $\rho\epsilon = \frac{3}{2}p$,

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{\rho v^2}{2}\right) + \nabla\cdot\left[\rho\boldsymbol{v}\left(\epsilon + \frac{v^2}{2}\right) + p\boldsymbol{v}\right] - \rho\boldsymbol{v}\boldsymbol{g} = 0,$$

which is the energy equation.

Energy equation: some manipulations

Multiplication of momentum equation by v_i and summation gives

$$\rho v_i \frac{\partial v_i}{\partial t} + \rho v_i v_h \frac{\partial v_i}{\partial x_h} = -v_i \frac{\partial p}{\partial x_i} + v_i \rho g_i,$$

or

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) - \frac{1}{2}v^2 \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_h}\left(\frac{1}{2}\rho v^2 v_h\right) - \frac{1}{2}v^2}_{=0 \text{ (continuity equation)}} \frac{\partial(\rho v_h)}{\partial x_h} = -v_i \frac{\partial p}{\partial x_i} + v_i \rho g_i.$$

Substracting this from the energy equation yields equation for the internal energy

$$\frac{\partial(\rho\epsilon)}{\partial t} + \nabla \cdot (\rho\epsilon\boldsymbol{v}) = -p\nabla\cdot\boldsymbol{v},$$

which can be rewritten using the continuity equation as

$$\rho\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = -p\nabla\cdot\boldsymbol{v}.$$

Energy equation: second law of thermodynamics

The conservation of entropy for isentropic flow requires that

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0,$$

which for the specific entropy of ideal gas $s = c_V \ln(pv^{\varkappa}) + \text{const.}$
$= c_V \ln(p\rho^{-\varkappa}) + \text{const.}$ is (using $p = \frac{2}{3}\rho\epsilon$)

$$\frac{\partial\left(\rho\epsilon\rho^{-\varkappa}\right)}{\partial t} + \boldsymbol{v}\cdot\nabla\left(\rho\epsilon\rho^{-\varkappa}\right) = 0.$$

Derivating and multiplying by $\rho^{\varkappa}$ we arrive at

$$\frac{\partial\left(\rho\epsilon\right)}{\partial t} + \nabla\cdot\left(\rho\epsilon\boldsymbol{v}\right) - \rho\epsilon\nabla\cdot\boldsymbol{v} - \varkappa\epsilon\frac{\partial\rho}{\partial t} - \varkappa\epsilon\boldsymbol{v}\cdot\nabla\rho = 0.$$

Eliminating the last two terms using the equation of continuity and noting that $\varkappa - 1 = \frac{2}{3}$ for ideal gas we derive the equation for the internal energy once again

$$\frac{\partial\left(\rho\epsilon\right)}{\partial t} + \nabla\cdot\left(\rho\epsilon\boldsymbol{v}\right) = -p\nabla\cdot\boldsymbol{v}.$$

Many faces of the beast

Collecting the nuggets: the hydrodynamical equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \rho \boldsymbol{g},$$

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{\rho v^2}{2}\right) + \nabla \cdot \left[\rho \boldsymbol{v}\left(\epsilon + \frac{v^2}{2}\right) + p\boldsymbol{v}\right] = \rho \boldsymbol{v} \boldsymbol{g}$$

- system of nonlinear first-order partial differential equations
- unknowns ρ, $\boldsymbol{v}$, p, and ϵ (+equation of state)
- initial and boundary conditions crucial
- inviscid flow, no magnetic field
- some special analytic solutions, general solution only numerically
- stationary solutions are important ($\partial/\partial t = 0$, but $\boldsymbol{v} \neq 0$)

The hydrodynamical equation in planar symmetry

In a planar symmetry the hydrodynamic quantities do not depend on x and y coordinates, there is no flow in $\boldsymbol{x}$ and $\boldsymbol{y}$ directions ($\boldsymbol{v} = v(z)\boldsymbol{z}$) and the hydrodynamical equations are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g.$$

The hydrodynamical equations in spherical coordinates

In spherical coordinate system, the components of the velocity vector are $\boldsymbol{v} = (v_r, v_\theta, v_\phi)$ and the components of force are $\boldsymbol{g} = (g_r, g_\theta, g_\phi)$. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \rho v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

and the components of equation of motion take the form of

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r,$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r} = -\frac{1}{r\rho}\frac{\partial p}{\partial \theta} + g_\theta,$$

$$\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot\theta}{r} = -\frac{1}{r\rho\sin\theta}\frac{\partial p}{\partial \phi} + g_\phi.$$

The hydrodynamical equations in spherical symmetry

In a spherical symmetry the hydrodynamic quantities do not depend on θ and ϕ coordinates, there is no flow in $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ directions ($\boldsymbol{v} = v(r)\boldsymbol{r}$) and the hydrodynamical equations are ($v \equiv v_r$)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g.$$

The hydrodynamical equations in cylindrical coordinates

In cylindrical coordinate system, the components of the velocity vector are $\boldsymbol{v} = (v_R, v_\phi, v_z)$ and the components of force are $\boldsymbol{g} = (g_R, g_\phi, g_z)$. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\rho v_\phi) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

and the components of equation of motion take the form of

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} + v_z\frac{\partial v_R}{\partial z} - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + g_R,$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + v_z\frac{\partial v_\phi}{\partial z} + \frac{v_R v_\phi}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \phi} + g_\phi,$$

$$\frac{\partial v_z}{\partial t} + v_R\frac{\partial v_z}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_z}{\partial \phi} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z.$$

Hydrostatic equilibrium

Static case: $\partial/\partial t = 0$, $\boldsymbol{v} = 0$

In a static case the equation of continuity is fulfilled identically and the momentum equation leads to

$$\nabla p = \rho \boldsymbol{g}$$

the equation of hydrostatic equilibrium. The energy equation $Q_{\mathrm{rad}} = 0$ gives the radiative equilibrium equation.

Atmosphere in hydrostatic equilibrium

The equation of hydrostatic equilibrium in homogeneous gravitational field directed along the z-axis is

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g,$$

which, using the ideal gas equation of state $p = \rho kT/(\mu m_{\mathrm{H}})$, leads to

$$\frac{\mathrm{d}\,(\rho T)}{\mathrm{d}z} = -\frac{\mu g m_{\mathrm{H}}}{k}\rho.$$

In isothermal atmosphere $T = \text{const.}$ this has the solution

$$\rho = \rho_0 e^{-z/H}, \qquad H = \frac{kT}{\mu m_{\mathrm{H}} g},$$

where H is the atmospheric *scale-height*. For $z \to \infty$ is $\rho \to 0$, as it should be.

Density scale height: sweeping the whole Universe

Because many astrophysical object are close to the hydrostatic equilibrium, the density scale height is one of the most important charasteristics length scales in astrophysics. Besides others, it determines the density scale of atmosphere.

	T [K]	$g/g_\odot$	H
neutron star	10^5	10^{10}	0.3 mm
white dwarf	10^4	10^3	100 m
Earth	300	4×10^{-2}	10 km
Sun	6000	1	200 km
Supergiant	10^4	10^{-3}	$1\ R_\odot$
galaxy cluster	10^7	10^{-13}	10^5 pc

Atmosphere in hydrostatic equilibrium: spherical symmetry

The equation of hydrostatic equilibrium in spherically symmetric isothermal case is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g,$$

which, with $g = GM/r^2$, has the solution

$$\rho = \rho_0 \exp\left(\frac{\mu m_\mathrm{H} GM}{kT}\frac{1}{r}\right).$$

There are two problems with this solution applied for gas spheres. For $r \to 0$ the equation is not applicable, because one should insert $M = M(r)$: *Lane-Emden equation*. Moreover, for $r \to \infty$ is $\rho \to \rho_0 \neq \rho_\mathrm{ISM}$. Solution: *Bonnor-Ebert spheres* with external pressure. Matter may escape from the regions, where the thermal speed is higher than the escape speed: *atmospheric escape*: loss of planetary atmospheres, solar-type (coronal) winds.

Lane-Emden equation

Consider a spherical mass in equilibrium. The hydrostaic equilibrium equation is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g = -\frac{\rho G M(r)}{r^2}.$$

The polytropic relation $p = C\rho^{1+1/n}$ with the definition of mass inside radius r, which is $M(r) = 4\pi \int_0^r \rho r'^2 \,\mathrm{d}r'$, gives after differentiation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^{1+1/n}\right)\right] = -\frac{4\pi G}{C}\rho.$$

Introducing new variables θ and ξ via $\rho = \lambda\theta^n$ and $\xi = r/\alpha$, where λ is arbitrary dimensional constant and

$$\alpha = \sqrt{\frac{C(1+n)}{4\pi G \lambda^{1+1/n}}}$$

we arrive at *Lane-Emden equation*

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n.$$

Hydrostatic atmospheres with radiative force

The equation of hydrostatic equilibrium in spherically symmetric atmosphere in radiative equlibrium is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g + \rho g_{\mathrm{rad}},$$

with the radiative force $g_{\mathrm{rad}} = \frac{1}{c}\int \kappa\rho F_\nu \,\mathrm{d}\nu = \frac{\kappa\rho L}{4\pi c r^2}$. The temperature is given by the energy transport equation

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4aT^3}\frac{\kappa\rho L}{4\pi c r^2}.$$

This can be rewritten in terms of $p_{\mathrm{rad}} = (a/3)T^4$ as

$$\frac{\mathrm{d}p_{\mathrm{rad}}}{\mathrm{d}r} = \frac{4aT^3}{3}\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{\kappa\rho L}{4\pi c r^2} = -\rho g_{\mathrm{rad}}.$$

Therefore, the equation of hydrostatic equilibrium is

$$\frac{\mathrm{d}p_{\mathrm{tot}}}{\mathrm{d}r} = \frac{\mathrm{d}(p + p_{\mathrm{rad}})}{\mathrm{d}r} = -\rho g.$$

Atmospheres close to the Eddington limit

Dividing the last two equations

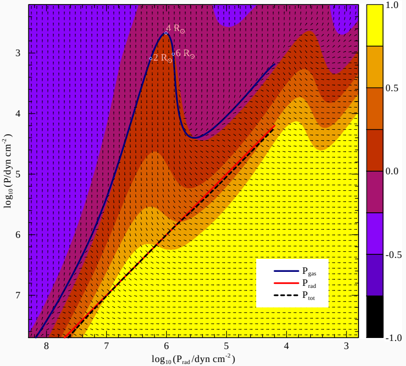
$$\frac{\mathrm{d}p_{\text{tot}}}{\mathrm{d}p_{\text{rad}}} = \frac{\mathrm{d}p}{\mathrm{d}p_{\text{rad}}} + 1 = \frac{g}{g_{\text{rad}}} = \frac{1}{\Gamma}, \qquad \text{or} \quad \frac{\mathrm{d}p}{\mathrm{d}p_{\text{rad}}} = \frac{1}{\Gamma} - 1,$$

where the generalized *Eddington factor* is

$$\Gamma = \frac{\kappa \rho L}{4\pi c G M}.$$

The derivative $\mathrm{d}p/\mathrm{d}p_{\text{rad}}$ is a function of Γ only. Moreover, the the point at which the envelope solution crosses the Eddington limit $\Gamma = 1$ needs to be an extremum in p (Gräfener et al. 2012).

Envelope inflation close to the Eddington limit



Logarithm of the Eddingtor factor Γ (colors) in the $p_{\rm rad} - p$ plane with Iglesias & Rogers (1996) opacities. Black arrows denote slopes $\mathrm{d}p/\mathrm{d}p_{\rm rad}$.

The numerical solution for $23\,M_\odot$ star almost precisely follows a path with $\Gamma = 1$ and crosses the Eddington limit at the lowest gas pressure (corresponding to the Fe-opacity peak). The gas density increases outwards leading to density inversion. This explains why WR and LBV stars have extended envelopes.

(Gräfener et al. 2012)

Eddington limit

The Eddington parameter due to pure electron scattering can be evaluated as

$$\Gamma = \frac{\sigma_\mathrm{T} \frac{n_\mathrm{e}(r)}{\rho(r)} L}{4\pi c G M}, \quad \text{in scaled quantities,} \quad \Gamma \approx 10^{-5} \left(\frac{L}{1\,\mathrm{L}_\odot}\right) \left(\frac{M}{1\,\mathrm{M}_\odot}\right)^{-1}.$$

Therefore, our Sun is very far from the Eddington limit $\Gamma = 1$, while hot O stars with $L \approx 10^6\,\mathrm{L}_\odot$ and $M \approx 10\,\mathrm{M}_\odot$ are very close to the Eddington limit $\Gamma \approx 0.1$.

From the Eddington limit $\Gamma = 1$ one can estimate the minimum mass of radiating object that can be gravitationally bound. From the above equation

$$M_\mathrm{Edd} \approx 10^{-5}\mathrm{M}_\odot \left(\frac{L}{1\,\mathrm{L}_\odot}\right).$$

The observed bolometric flux from quasars of the order of $10^{13}\,\mathrm{L}_\odot$ yields the Eddington mass of $10^8\mathrm{M}_\odot$ providing a hint that the objects are powered by a supermassive black hole.

Suggested reading

G. K. Batchelor: An Introduction to Fluid Dynamics

D. Mihalas & B. W. Mihalas: Foundations of Radiation Hydrodynamics

F. H. Shu: The physics of astrophysics: II. Hydrodynamics

A. Feldmeier: Theoretical Fluid Dynamics