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Pythagoras as a Mathematician

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In this article, two questions are posed: Just how reliable is the evidence concerning Pythagoras's mathematical studies, and can we reconstruct his contribution to mathematics? All known fragments of evidence by fourth-century B.C. authors on Pythagoras's mathematical investigations are examined, and it is shown that all the discoveries they mentioned belong to the sixth century B.C. The opinion that the Pythagoreans ascribed their own discoveries to Pythagoras is refuted, and it is shown that we are able to establish logically his contribution to mathematics. © 1989 Academic Press, Inc.

Der Aufsatz behandelt die Frage, ob es sichere Zeugnisse über Pythagoras' mathematische Beschäftigungen gibt und ob wir auf dieser Grundlage seinen Beitrag zür Mathematik rekonstruieren können. Im Aufsatz werden Zeugnisse der Autoren aus dem 4 Jh. v.u.Z. über Pythagoras' mathematische Forschungen gesammelt, und es wird gezeigt, daß alle seine Entdeckungen wirklich dem Ende des 6 Jh. v.u.Z. angehören. Im Aufsatz wird die ältere Meinung abgelehnt, daß die Pythagoreer ihre Entdeckungen dem Pythagoras zugeschrieben haben, und es wird gezeigt, daß wir in der Lage sind, seinen Beitrag zur Mathematik abzugrenzen. © 1989 Academic Press, Inc.

V state rassmatrivaetsya vopros, sushchestvuyut li nadezhnye svidetelstva o matematicheskikh zanyatiyakh Pifagora i mozhem li my, osnovyvayas na nikh, rekonstruirovat ego vklad v matematiky. Na osnovanii ryada svidetelsv avtorov IV v. do n. e. pokazyvaetsya, chto vse matematicheskie otkrytiya, pripisyvaemye Pifagoru, deistvitelno otnosyatsya k kontsu VI v. do n. e. i vpolne mogli byt sdelany im samim. V state oprovergaetsya mnenie, chto pifgoreitsy pripisyvali svoi nauchnye otkrytiya Pifagoru, i utverzhdaetsya, chto my v sostoyanii opredelit ego vklad v matematiku. © 1989 Academic Press, Inc.

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1

The present paper is devoted to the consideration of three interconnected questions: (1) How reliable is the customary image of Pythagoras as a mathematician? (2) Which particular achievements in mathematics may be attributed to him on the basis of authentic evidence? (3) What role did Pythagoras play in the development of mathematics?

The first question is posed mainly because as early as the beginning of the 20th century, doubt was cast on the reality of Pythagoras's mathematical activity. The tendency to negate his scientific pursuits is shown in the works of E. Sachs [1917], E. Frank [1923], W. Rathmann [1933], A. Rey [1933, 105–106], and W. A. Heidel [1940]. From the 1940s to the 1970s, this point of view was reflected in the books

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of O. Neugebauer [1957, 148–149], W. Burkert [1972, 208–217, 401–426], B. L. van der Waerden [1956, 151–158], J. Philip [1966, 172], Á. Szabó [1978], W. Knorr [1975, 5], and others.

In addition to comments concerning the reliability of particular pieces of evidence, the main arguments advanced against the historicity of the tradition of Pythagoras's mathematical inquiries are as follows.

(1) Pythagoras was first a religious figure and is presented as such in the early sources. Information on his philosophical and scientific activities appeared much later and does not deserve credence.

(2) Even when Pythagoras is associated with mathematics, it is not as an original thinker but as a transmitter of the Egyptian (or Babylonian) mathematical tradition, with which he became acquainted during his travels in the East.

(3) Strictly deductive proof and the resulting mathematical theory became possible only after the inquiries of the Eleatic school (ca. 480–440 B.C.). Moreover, because Parmenides and Zeno developed their theories after the death of Pythagoras (ca. 495 B.C.), the tradition concerning the deductive character of his mathematics does not merit credence.

The first two arguments have been considered in detail in our earlier publications [Zhmud 1985; 1986a]. Here we review only their main conclusions.

Analysis of the very earliest evidence (fifth century B.C.) indicates that at that time Pythagoras was already known not only as the advocate of metempsychosis, but also mainly as a rational thinker, a scientist, and a person of vast knowledge. Evidence of his concrete scientific achievements first appeared in the fourth century B.C., and although many pieces of that evidence have reached us through the works of later authors, there is no doubt that this tradition stems from Pythagoras's lifetime [Zhmud 1985].

The hypothesis of the Eastern roots of Pythagoras's mathematics is based on the legend of his travels in the East, which has not been confirmed by reliable sources. In addition, Greek deductive mathematics developed in a plane completely different from that of the calculating mathematics of the Egyptians and Babylonians. No trace of Eastern influence can be found in the works of the Greek mathematicians who had actually visited Egypt, such as Thales, Democritus, and Eudoxus. Even after the conquests of Alexander the Great, when Greeks found themselves living in close contact with these people, they showed no marked tendency to adopt Eastern mathematical methods. Although Euclid lived in Alexandria for most of his life, we cannot find any evidence of an Egyptian influence in the 13 books of his *Elements*. This is also true with respect to other mathematicians of the third century B.C., such as Archimedes and Apollonius of Perge, both of whom might in principle be acquainted with Eastern mathematics. Only in Hypsicles (mid-second century B.C.) can definite traits of Babylonian mathematics be found. In the sixth to fifth centuries (and possibly even earlier), the extremely few borrowings apply only to practical methods of counting (and these were Egyptian and not Babylonian) and do not relate to the problems that interested Greek mathematicians [Zhmud 1986a,b].

Let us now consider the third argument. The Hungarian historian of mathematics Á. Szabó [1978], who is supported by some other scholars [Burkert 1972, 425; Philip 1966, 200], considers that the mathematics of the sixth to early fifth centuries evolved empirically and that the deductive method was borrowed from the logic of the Eleatic school.

At first glance philosophy seems to be in a more favorable position than mathematics. The first examples of deductive reasoning that we note are the fragments of a philosophical poem of Parmenides and a composition of his disciple Zeno. Parmenides presents his basic tenet: "being is, not-being is not" (28 B 2–4) [1], from which he logically deduces the fundamental signs of being (immutability, unity, unchangeability, etc.) and refutes alternative variants (the origin of being, its qualitative diversity, etc.). Zeno, in denying the possibility of motion and plurality (29 A 25, B 1–2), often resorted to *reductio ad absurdum*. Parmenides was perhaps the first *philosopher* to advance his ideas on the fulcrum of logical argumentation (28 A 28), but this is not to say that he invented this method. It is possible to show that he borrowed the deductive method from mathematics and, therefore, it was already in use in the sixth century B.C.

2

Although not a single fragment from the work of Thales, the first Greek mathematician, remains, information on his five theorems has survived in the ancient tradition. Two theorems are mentioned by the Peripatetic Eudemus of Rhodos (ca. 330 B.C.), author of the invaluable work *The History of Geometry* (fr. 134, 135 Wehrli). The Neoplatonist Proclus reports two other theorems (*In primun Euclidis Elementorum librum commentarii*, G. Friedlein, Ed., p. 157. 10, 250.20), taking his information from the work of Eudemus [Becker 1954, 24–28; Heath 1926 I, 36–37]. One theorem is also mentioned by Diogenes Laertius, who quotes the first-century authoress Pamphyla (Diogenes Laertius I, 24). Quite probably Eudemus learned of Thales's theorems from the Sophist Hippias of Elis (ca. 430 B.C.), known for his interest in mathematics [Snell 1966; Classen 1965]. It is worth noting that Thales's renown as an eminent geometrician was already reflected in Aristophanes's comedies (*Nubes* 180; *Aves* 1009).

What Eudemus writes about Thales's mathematics and, furthermore, *how* he treats it clearly show a good grasp of the subject [van der Waerden 1956, 146]. In one case Eudemus speaks about the proof of a theorem, in another he states that it was "found" by Thales, and in a third case he comments that scientific proof was not given. Eudemus's observations on the deductive character of at least some of Thales's conclusions should not be ignored, especially because the same implication is found in another fragment of Eudemus: "Thales taught some things more abstractly and others, more empirically" (fr. 133 Wehrli).

It is often stated that Thales's argumentation rested on the method of superposition and was not strictly deductive. Still, his proof of the theorem on the equality

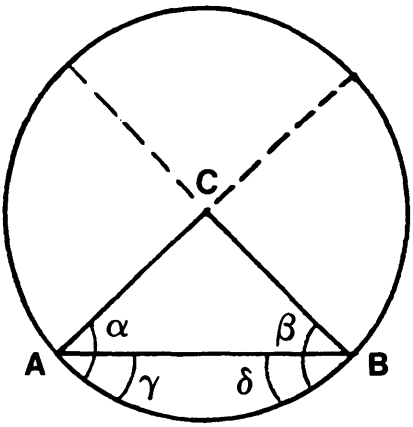

FIGURE 1

of angles at the base of an isosceles triangle, preserved in Aristotle (*Analytica priora* 41 b 13–22), shows that Thales certainly was not content with a visual demonstration [Becker 1966, 38–39; Neuenschwander 1973, 353–354]. His proof proceeded as follows (see Fig. 1): ABC is an isosceles triangle with the apex in the center of a circle. Prove that the angles at the base are equal. $\angle\alpha = \angle\beta$ since both of them are angles of the half circle. $\angle\gamma = \angle\delta$ since two angles of any segment are equal to each other. Subtracting the equal angles γ and δ from the equal angles α and β, we obtain that CAB and CBA are equal to each other.

Let us now consider the level of mathematical investigation of the younger contemporaries of the Eleatics. We know that Democritus (b. ca. 470 B.C.) wrote a book on incommensurable lines (Diogenes Laertius IX, 47); therefore, by this time the irrationality of $\sqrt{2}$ had been demonstrated. Hippocrates of Chios (ca. 440 B.C.) was working on the famous problem of that time, the duplication of the cube. This necessarily had to precede the corresponding problem in planimetry, the duplication of the square, which was closely associated with the discovery of incommensurable lines. From the fragment Hippocrates's work on the quadrature of lunes (Eudemus, fr. 140 Wehrli), it is clear that he was acquainted with a significant part of the contents of Euclid's Books I–IV. It is also clear that these propositions had been proven some time before him, as the rigorous argumentation of Hippocrates himself was warranted only if the propositions on which he relied had the same logical form and were as accomplished as his own. Eudemus ascribes to Hippocrates the first *Elements* (fr. 133 Wehrli), in which the theorems and problems known at that time were gathered and placed in logical order. All this demonstrates a maturity of the mathematics of this time that cannot be explained if we assume that the deductive method was borrowed only in the mid-fifth century.

Here we should mention van der Waerden's [1978] convincing reconstruction of the early Pythagorean textbook of mathematics, which preceded Hippocrates's

Elements and contained the basis of Euclid's Books I–IV [2]. Supported by this reconstruction, which is confirmed by historical evidence on the closeness of Hippocrates and the Pythagoreans (42 A 5; Iamblichus, *De communi mathematica scientia,* N. Festa, Ed., p. 78), we arrive at early fifth-century mathematics, that is, at the same source from which Parmenides and Zeno might have borrowed the idea of deductive proof. According to tradition (28 A 1), Parmenides was close to the Pythagorean milieu, and, therefore, we have every reason to accept the conclusion drawn by Th. Gomperz [1922 I, 139]: the form of Parmenides's system is taken from the mathematics of Pythagoras [3].

In the history of science there are many examples of one branch of science borrowing methods that have proven fruitful in another field of knowledge. However, no one will adopt a method if its first application did not give appreciable results in the field in which it arose. It is clear that deductive argumentation in Eleatic philosophy, and in philosophy as such, certainly does not carry the logical conviction and irrefutability of mathematical proof [4]. Neither Parmenides nor Zeno was able to prove anything; they only tried to do so. Their young contemporaries, the atomists, had already rejected the idea that there is no nonbeing (that is, void). Their cosmos consisted exactly of a void filled with atoms. Zeno's attempt to refute motion and plurality was not successful, nor could it be, although the problems he raised greatly stimulated the development of philosophy. The influence of the Eleatics on later philosophers is due to the depth of the Eleatics' ideas and not to deductive argumentation. Were not some of Heraclitus's ideas accepted, even though his reasoning was very far from proof? In other words, when we compare the very minor success of the deductive method in philosophy with what it gave to mathematics, the question of "whom was it borrowed from?" seems rhetorical [5].

3

Before we turn to the mathematical inquiries of Pythagoras, it is necessary to mention still another problem. Often even those scholars who admit that Pythagoras was engaged in mathematics leave open the question concerning his concrete contribution to this science [Vogt 1909]. The difficulty in reconstructing this contribution is usually seen as due to the custom in the Pythagorean school of ascribing its scientific achievements to its founder [Allman 1889, 21; Heath 1926 I, 411; Guthrie 1962 I, 149], and as a result, we cannot distinguish Pythagoras's own contribution.

It must be noted that this opinion is not confirmed by fifth- to fourth-century tradition nor by later evidence. We do not know of a *single* Pythagorean who ascribed his own discovery to Pythagoras. There is no reliable evidence that this tendency even existed in the Pythagorean school. The first (and only) allusion of this sort is found in the work of Iamblichus (*De vita Pythagorica* 158, 198), a Neopythagorean of the third to fourth centuries A.D. known for his lively fantasy. The discoveries in mathematics traditionally attributed to Pythagoras are *never* connected with any of the Pythagoreans (except for one passage in Proclus which we consider later). Of course, certain astronomical discoveries are connected, in addition to Pythagoras, with Parmenides and Oenopides [Burkert 1972, 303–308], but since it is clear that *they themselves* did not attribute these discoveries to Pythagoras, this confusion, so common in the late tradition, has nothing to do with our question.

Iamblichus (*De vita Pythagorica* 88, 246–247; *De communi mathematica scientia,* N. Festa, Ed., p. 77) repeats several legends about Hippasus, who, according to one version, "divulged" to the uninitiated the secrets of Pythagorean mathematics and was punished for this by the gods. Another version states that he ascribed the discovery of the dodecahedron *to himself,* while actually "it all belongs to that man" (namely, to Pythagoras). There is no reason to see in this statement a motto of fifth-century Pythagorean mathematicians, as Burkert [1972, 197] attempts to prove.

Beginning in the third century B.C. many religious and philosophical treatises were ascribed to Pythagoras and his followers, particularly to Archytas, but they have nothing in common with the tendency discussed here. Pseudo-Pythagorean writings, which appeared after the Pythagorean school had already dissolved, did not follow any sort of school tradition but simply showed a widespread fashion of that time. Even before these writings, Platonists and Peripatetics as well as physicians of the Hippocratic school ascribed their works to their teachers. Even the poet Epicharmus, who was interested in philosophy but had not established a school, had some forged writings ascribed to him at the end of the fifth century B.C. (23 A 10). However, the main feature of pseudo-Pythagorean writings, showing a break with tradition, is that there are no references to scientific discoveries of Pythagoras, nor any interest at all in scientific problems. The authors had no discoveries of their own to attribute to Pythagoras, nor any desire to ascribe to him any fraudulent ones [Thesleff, 1965].

Hence, there is sufficient evidence to acknowledge that Iamblichus's assertions are unauthentic. The two chapters of his book in which these statements were mentioned have already been considered by E. Rohde [1901, 155–156, 160–161] to have no support in the previous tradition. Indeed, in both places Iamblichus speaks of pseudo-Pythagorean writings widespread in his time, most of which were attributed to Pythagoras. It was this fact that gave him the idea that Pythagoreans ascribed "with the rare exception" their discoveries to their teacher: "Because very few of their compositions are claimed to be their own" (*De vita Pythagorica* 198)! Iamblichus's reasoning is so transparent that it is simply incredible that his conclusions were able to bewitch several generations of scholars, especially when we recall that, besides him, none of the classical writers mentions the tendency of Pythagoreans to attribute their scientific discoveries to the founder of the school.

This conclusion allows us to remove these contrived difficulties from the reconstruction of Pythagoras's mathematics. If the Pythagoreans did not ascribe their own achievements to their teacher, then, following tradition, we can certainly decide which theories belong to Pythagoras and which belong to his followers.

4

In order to verify this conclusion, let us turn to the most reliable sources, to fourth-century-B.C. authors who mention the mathematical inquiries of Pythagoras.

(1) In the epideictic speech *Busiris* the Athenian rhetor Isocrates claims that Pythagoras borrowed his philosophy from the Egyptians, more precisely, from the Egyptian priests (*Busiris* 28). According to Isocrates, this "Egyptian philosophy" consisted of geometry, arithmetic, and astronomy (*Busiris* 23). These disciplines, of course, had nothing to do with priestly concerns, but this is in good agreement with other sources concerning the teachings of these disciplines in the Pythagorean school [Morrison 1958, 201–203].

(2) Plato's disciple Xenocrates (*Darstellung der Lehre und Sammlung der Fragmente,* R. Heinze, Ed., fr. 9) testifies to Pythagoras's discovery of the numerical expression of harmonic intervals. This discovery is closely associated with the theory of proportionals, which probably preceded it. The mathematical theory of music provided the finishing touches to the circle of related disciplines taught at the Pythagorean school. This circle included geometry, arithmetic, astronomy, and harmonics—the future quadrivium of the Middle Ages. The credit for uniting them belongs not to Theodorus of Cyrene or to Hippias of Elis, who taught the quadrivium in the second half of the fifth century B.C., but to Pythagoras [Marrou 1965, 99, 267, 272; Loria 1914, 29], who linked music not only with mathematics but also with astronomy—in the famous doctrine of the harmony of the spheres.

(3) The fragment of Aristotle's monograph on the Pythagoreans reads as follows: "Pythagoras, son of Mnesarchus, first devoted himself to the study of mathematics, in particular of numbers (*τὰ μαϑήματα καὶ τοὺς ἀριϑμούς*), but later he could not refrain from the miracle-making of Pherecydes" (*Qui ferebantur librorum fragmenta,* V. Rose, Ed., fr. 191). The belief that these words were actually written by Aristotle has often been contested [Heidel 1940, 8; Burkert 1972, 412; Philip 1966, 23], but no convincing arguments have been advanced.

(4) In another passage Aristotle again touches on Pythagorean education, based on mathematics: "Simultaneously with these philosophers [Leucippus and Democritus—L. Zh.] and earlier than them, the so-called Pythagoreans were the first to engage in the study of the mathematical sciences, greatly advancing them; being educated in them (*ἐντραφέντες*), the Pythagoreans began to consider their principles as the principles of all things" (*Metaphysica* 985 b 23). But whom could Aristotle have had in mind when he spoke of mathematicians living before Democritus and Leucippus? In the first third of the fifth century there was the Pythagorean mathematician and philosopher Hippasus, but he considered fire, not numbers, as the principle of all things—Aristotle himself wrote of this (*Metaphysica* 984 a 7). Therefore, Pythagoras must be the mathematician he had in mind [Jäger 1946 I, 162, 456], although quite possibly not only him.

(5) Aristoxenus, a disciple of the last Pythagoreans and later of Aristotle, con-

sidered that "Pythagoras honored the study of numbers (ἡ περὶ τοὺς ἀριθμοὺς πραγματεία) more than anyone. He made great advances in it, withdrawing it from the practical calculations of merchants and likening all things to numbers" (fr. 23 Wehrli). Since in the last part of the fragment he speaks of even and odd numbers, giving them a typically Pythagorean formulation (cf. Iamblichus, *In Nicomachi arithmeticam introductionem,* H. Pistelli, Ed., p. 12), it is probable that by the "study of numbers" or "theory of numbers" Aristoxenus understood the theory of even and odd numbers, preserved in Book IX of Euclid. Becker has brilliantly shown that this theory relates to the very earliest stage of Pythagorean mathematics [Becker 1934; van der Waerden 1979, 397]. Taking into account a fragment of Aristoxenus that Becker did not mention, we may associate this theory directly with Pythagoras. Apparently the adjoining theory of figured numbers also belongs to him.

(6) Proclus in his commentary to Book I of Euclid presents the famous *Catalogue of Geometers,* the material of which in its basic features refers to Eudemus. The following is said of Pythagoras: "After them (Thales and Mamercus) Pythagoras transformed the philosophy of geometry, making it a form of liberal education, considering its principles abstractly, and examining the theorems immaterially and intellectually. He discovered the theory of proportionals and the construction of cosmic bodies" (Eudemus, fr. 133 Wehrli). Despite the numerous objections put forth against the authenticity of this passage [6], most specialists attribute at least the first sentence to Eudemus. Actually, it would be very strange if Eudemus in naming well-known mathematicians had left out Pythagoras. As he was well acquainted with Thales, he would know as much as Aristotle and Aristoxenus did about Pythagoras.

Vogt [1909, 31] noted the coincidence of the exact wording of "Pythagoras transformed the philosophy of geometry, making it a form of liberal education" with the passage in Iamblichus (*De communi mathematica scientia,* N. Festa, Ed., p. 70). But this cannot serve as proof that Proclus, not finding anything in Eudemus about Pythagoras, inserted the words of Iamblichus (Burkert especially insisted on this) in the catalogue. Since the second part of the same sentence appears in an abbreviated form in the catalogue of Hero of Alexandria (*Definitiones* 136, J. L. Heiberg, Ed., Vol. IV, p. 108), who lived 200 years before Iamblichus [Heath 1922 I, 154], it is obvious that Proclus and Iamblichus had a common source [7].

In the second sentence of the passage, Pythagoras is credited with the discovery of the theory of proportionals and the construction of cosmic bodies. Although the reading of the "theory of proportionals" (τῶν ἀνὰ λόγων πραγματεία) is widely accepted, it is based on only one of Proclus's manuscripts [8], while in others the "theory of irrationals" is mentioned (τῶν ἀλόγων πραγματεία). Nonetheless, the first reading seems preferable in many respects. As applied to the time of Pythagoras, we cannot speak of a "theory" of irrationals, only of the discovery of the irrationality of $\sqrt{2}$. Eudemus, and also Proclus, must have known this. The theory of proportionals is closely related with the acoustical investigations of Pythagoras and with his mathematical discoveries: evidently on the basis of this theory, he proved his famous theorem. Moreover, other authors also have mentioned Pythagoras in connection with the theory of proportionals [Nicomachus, *Introductio arithmetica* II, 22; Iamblichus, *In Nicomachi arithmeticam introductionem*, H. Pistelli, Ed., p. 118].

If Pythagoras really had discovered the irrationality of $\sqrt{2}$, the association of such a famous discovery with a no-less-famous name would have found some sort of reverberation in Greek literature. Yet no author of antiquity has mentioned this. On this point all the information we have associates the discovery with the name of Hippasus [9].

The case of the construction of the cosmic bodies, that is, the five regular polyhedra, is more complicated. Eudemus could not have attributed the construction of all five bodies to Pythagoras, because in the scholia to Euclid (XIII, I, p. 291 Stamatis) it is stated that the first three bodies (the dodecahedron, cube, and tetrahedron) were discovered by the Pythagoreans, and the octahedron and icosahedron by Theaetetus. This information, as currently accepted, must be attributed to Eudemus. Traditionally, the construction of the dodecahedron is associated with Hippasus (18 A 4); moreover, it assumes the discovery of irrationality, which was not likely to have been accomplished by Pythagoras. All this results in the conclusion that Pythagoras was responsible for the construction of the first two polyhedra, the cube and the pyramid [10].

The version which states that Pythagoras was the author of all five bodies is encountered even before Proclus, in the doxographic tradition (Aëtius II, 6, 5). This is important for us, because it is well known that there are many arbitrary interpretations and misunderstandings in the works of the doxographers. In any case, it is clear that only later authors associated Pythagoras with *someone else's* discoveries and not early Pythagoreans with their own.

(7) Diogenes Laertius relates that a certain Apollodorus the Calculator credits Pythagoras with the proof of the theorem that the squares of the sides of a right-angled triangle are equal to the square of the hypotenuse (Diogenes Laertius VIII, 12). Here Diogenes quotes an epigram in honor of this discovery:

> When Pythagoras that famous figure found
> A noble offering he laid down!

Cicero was the first to quote these two lines—and after him Vitruvius, Plutarch, Athenaeus, Diogenes Laertius, Porphyry, and Proclus [11]. The unanimity with which Pythagoras is proclaimed the author of this theorem, the absence of other contenders, and its closeness to his other discoveries speak for the authenticity of Apollodorus's words. Although when he lived is not known exactly (clearly he lived before the first century B.C.), in accordance with Burkert's convincing suggestion [1972, 428] we may identify him with the philosopher Apollodorus of Cyzicus (latter half of the fourth century B.C.)

Burkert correctly observes that the motif of sacrificing the oxen, which contradicts the later opinion that Pythagoras was a vegetarian, may be considered as evidence of the antiquity of the epigram and nothing else. We know that precisely in the fourth century Aristoxenus insisted that Pythagoras did not refuse meat (fr. 25 Wehrli), while Aristotle said he refused only certain parts of animals (*Qui fereban-tur librorum fragmenta,* V. Rose, Ed., fr. 194). It is interesting that Proclus—the only one who doubted the authorship of Pythagoras—proceeded apparently from the assumption that Pythagoras was not capable of sacrificing animals (*In primum Euclidis Elementorum librum commentarii,* G. Friedlein, Ed., p. 426).

(8) The following is the last piece of evidence worth considering: Hero of Alexandria (*Geometrica* 8, J. L. Heiberg, Ed., vol. IV, p. 218), and after him Proclus (*In primum Euclidis Elementorum librum commentarii,* G. Friedlein, Ed., p. 428) attribute to Pythagoras the method for determining the sides of a right-angled triangle (the Pythagorean triplets). It is known that they both used the writings of Eudemus—and this information most probably comes from him [Heath 1926 I, 36; von Fritz 1945, 252]. Another source is difficult to suppose here.

Thus, we may preliminarily group together those concrete mathematical problems that Pythagoras was apparently involved with: the theory of proportionals, the theory of even and odd numbers, the Pythagorean theorem, the method of determining the Pythagorean triplets, and the construction of the first two regular polyhedra. Understandably, we cannot assume that these are the only discoveries of Pythagoras in mathematics. It is scarcely possible to obtain a complete picture of his activities from the fragmentary evidence of fourth-century authors. These achievements must be considered the foundation on which we must necessarily rely in further reconstructions of the mathematics of Pythagoras, bringing to bear both later evidence and the inner logic of the development of mathematics itself.

Before proceeding further, however, we must note, first, the lack of a contradiction in the foregoing evidence and the close interconnection of those mathematical problems that they reported, and, second, that all the discoveries of Pythagoras fully correspond to the level of Greek mathematics at the end of the sixth century. Pythagorean mathematics of the first half of the fifth century (the discovery of irrationality, the method of application of areas, etc.) naturally continued the endeavors of the founder of the school, yet all this was attributed not to Pythagoras but to the Pythagoreans in general, or concretely to Hippasus. Therefore, neither within the Pythagorean school nor beyond it was there any attempt to ascribe to Pythagoras the scientific achievements of another, at least in the field of mathematics.

But is it possible that this tendency appeared at a later period, so that as time passed Pythagoras was made the author of new discoveries? No, this also is not confirmed by the material available to us.

Two historians of the late fourth century B.C., Anticlides and Hecataeus of Abdera, in speaking of Pythagoras's mathematical inquiries (*Die Fragmente der griechischen Historiker,* F. Jacoby, Ed., 140 F 1; 264 F 25), do not present any concrete facts. The poet Callimachus (third century B.C.) mentions the study of triangles and Pythagoras's discovery of a sort of "figure" (*Fragmenta nuper reperta,* R. Pfeiffer, Ed., fr. 191, 58–62). It is customary to see in this a hint of the well-known theorem, which indirectly confirms an early dating of Apollodorus's quoted epigram. Plutarch, in citing this epigram, was doubtful as to which it referred: to Pythagoras's theorem or to the method of application of areas, which he deemed the more important discovery (*Non posse suaviter vivi secundum Epicuri praecepta* II. 1094b; *Quaestiones convivales* 720 a). It is quite clear that Plutarch had no source directly naming Pythagoras as the author of the method of application of areas.

The Neo-Pythagorean, Nicomachus of Gerasa (ca. 100 A.D.), writes that arithmetic, geometric, and harmonic proportions (*Introductio arithmetica* II, 22), as well as the three correspondent means (*Introductio arithmetica* II, 28.6), were known to Pythagoras. Iamblichus added that in Pythagoras's time the harmonic mean was called the "subcontrary," while the contemporary term was introduced by Hippasus (*In Nicomachi arithmeticam introductionem*, H. Pistelli, Ed., p. 100). In another instance Iamblichus claims that Pythagoras knew another proportion, a "musical" one (*In Nicomachi arithmeticam introductionem*, p. 118). And finally, he credits Pythagoras with the discovery of friendly numbers, where the sum of the factors of one is equal to the sum of the factors of the other, for instance, 220 and 284 (*In Nicomachi arithmeticam introductionem*, p. 35).

The above is virtually all that may be found on the mathematical discoveries of Pythagoras; other evidence was also discussed above. We note that none of the above-mentioned authors associates Pythagoras's name with anything that could not in principle be credited to Pythagoras. Actually, only Iamblichus's information on friendly numbers goes beyond the limits of the information of the fourth-century authors. This consensus is quite surprising. Proclus's report about the five regular polyhedra scarcely flaws this, especially if we take into account that he lived more than 1000 years after Pythagoras.

Omitting detailed analysis, we stress that analogous situations are found in acoustics and astronomy. In the latter case, it is somewhat more complicated; however, here too we can show that the disagreement of the sources is due to common distortions that are encountered in thousands of other cases and is not due to any singularity of the Pythagorean school.

With Pythagorean philosophy the picture is very different: its founder was credited with ideas that in no circumstances could have belonged to him. The interpretation of his philosophy in the spirit of Platonism dates to the fourth century B.C., but Speusippus and Xenocrates, Plato's disciples, developed it, not the Pythagoreans [Burkert 1972, 57–71]. For unexplained reasons Aristotle's disciple Theophrastus preferred this particular Platonic version, and because of him it is referred to in most of the later sources, greatly complicating the reconstruction of early Pythagorean philosophy. Fortunately, the situation in the history of science is quite different.

5

Let us now return to what we mentioned earlier: the close interconnection of all Pythagoras's mathematical discoveries. Of course, this cannot be the sole basis for reconstruction; it is well known that the solution of two logically close problems may occur many decades apart. Nevertheless, it may be considered additional confirmation of the authenticity of the collected evidence.

One of the important links between arithmetic, geometry, and harmonics was the theory of proportionals [Allman 1889, 48–49]. Pythagoras was undoubtedly aware of the three means—arithmetic, $c = (a + b)/2$; geometric, $c = \sqrt{ab}$; and harmonic, $c = 2ab/(a + b)$—and also of the "musical" proportion $a : (a + b)/2 = 2ab/(a + b) : b$, directly related to his acoustic experiments. Pythagoras discovered the numerical expression of harmonic intervals by the division of the string of the monochord into ratios 12 : 6, 12 : 8, 12 : 9 (Gaudentius, *Introductio harmonica* 11). The same relations (6 : 9 = 8 : 12) also occur in the musical proportion, where the inner terms are the arithmetic and harmonic means between the extremes of the proportion. Hippasus used this proportion in his acoustic experiment with copper discs (Aristoxenus, fr. 90 Wehrli).

Fraenkel [1938] found an interesting confirmation of Pythagoras's claim to the theory of proportionals. He showed that some of Heraclitus's ideas were expressed as geometric proportions. For example, god/man = man/child (22 B 79), god/man = man/monkey (22 B 82–83), and drunken man/child = child/sober man (22 B 117). Fränkel justly assumed that Heraclitus did not find the geometric proportion by himself, but took it from the Pythagoreans.

The arithmetical theory of proportion applicable to commensurable magnitudes was probably used by Pythagoras in proving his famous theorem [Heath 1922 I, 147–148; 1926 I, 353–354; van der Waerden 1979, 359; Neuenschwander 1973, 369].

The next section of Pythagorean arithmetic is the theory of even and odd numbers, the first example in the theory of numbers. It was considered by Becker [1934], whose work has been followed by most historians of Greek mathematics [Reidemeister 1949, 31–32; van der Waerden 1979, 396–397], to be preserved almost unchanged in the work of Euclid (IX, 21–34). The first five propositions of this theory (in abbreviated form) will serve as an example:

21. The sum of even numbers is even.
22. The sum of an even number of odd numbers is even.
23. The sum of an odd number of odd numbers is odd.
24. An even number minus an even number is even.
25. An even number minus an odd number is odd.

The proof of these propositions is based on the definitions of Book VII, which follow one another in rigorous logical order. Although Euclid sometimes presented numbers as segments (this is rather the exception than the rule) and the Pythagoreans used counting stones (*psephoi*), the basic idea does not change. Becker's article [1934, 538] and Knorr's book [1975, 141–143] in even more detail show that the proofs preserved by Euclid are easily illustrated with *psephoi*. It is absolutely improbable that Pythagoras presented his propositions without proofs and that these were added later; most of the propositions of this theory are obvious to anyone acquainted with elementary calculations. Therefore, Aristoxenus and Aristotle could not credit Pythagoras with the discovery or "illustration" of the fact that the sum of even numbers is even, but only with the proof of this and other similar propositions. Just as Thales did in geometry, Pythagoras began in arithmetic with the simplest facts, which no one earlier had felt required proof. How quickly he advanced in the elaboration of the deductive method is shown by the fact that four propositions of this theory (IX, 30–31, 33–34) are demonstrated by indirect proof. Á. Szabó was the first to note this; however, he refused to admit that these proofs are as old as the propositions [Szabó 1978, 247]. The only argument that he advances—the absence of historical evidence—does not withstand criticism. Sources in early Greek mathematics are so few that to expect evidence for each proof would be utterly utopian.

Considering the mathematical side of the question, we must recognize the reasonableness of Becker's conclusions; he suggested that the theory of even and odd numbers should be considered *en bloc*. (Certain minor changes noted by him did not concern Propositions 30–31 and 33–34.) The propositions demonstrated by indirect proof follow very naturally from the ones demonstrated directly, not differing from them in complexity. Thus, for instance, the demonstration of Propositions 33 and 34 demands nothing except Definitions 8 and 9 of Book VII. Would it not be extremely odd to suppose that the initial direct proof had been exchanged for an indirect one? Greek mathematics avoided such operations. Everything suggests that the theory has come down to us in its original form. Two important conclusions follow from this: (1) the visualizability of mathematical facts and their deductive proof certainly do not stand in irreconcilable contradiction, as Szabó tries to prove; (2) indirect proof is an integral part of mathematics, starting at the very early stage of its development [12], and only later were attempts made to use it in Eleatic philosophy.

Another example of a very early use of indirect proof is the theorem on the equality of the sides subtended by the equal angles of a triangle (Euclid 1, 6). This is the converse of the theorem proven by Thales on the equality of the angles of an isosceles triangle. It relates to the early Pythagorean mathematical textbook reconstructed by van der Waerden and evidently was proven either in Pythagoras's generation or in the next generation after his [Zaitsev 1985, 186–187].

The second link between geometry and arithmetic was the theory of figured numbers (triangular, square, oblong, etc.), establishing an interconnection between numbers and geometric figures. Although there is no direct evidence ascribing this theory to Pythagoras, a whole series of arguments speaks in favor of his authorship.

The construction of figured numbers with the aid of a gnomon amounts to the summation of a simple arithmetic series, for instance, of the odd and even numbers (see Fig. 2):

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{square number}$$

$$2 + 4 + 6 + \cdots + 2n = n(n + 1) \quad \text{oblong number.}$$

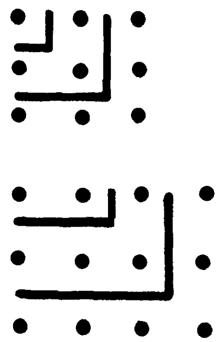

FIGURE 2

In type it obviously belongs to the same early Pythagorean psephic arithmetic which includes the theory of even and odd numbers. Aristotle wrote of those who "reduce numbers to the forms of triangles and squares" (*Metaphysica* 1092 b 12), apparently referring to the early Pythagoreans. His contemporary Speusippus, in the writing *On Pythagorean Numbers,* called some of them polygonal (*Scripta, accedunt fragmenta,* P. Lang, Ed., fr. 4).

At the same time, it is evident that the theory of figured numbers predates the method of application of areas (Book II of Euclid), which appeared in the first half of the fifth century B.C. and which also uses a gnomon. Finally, it is accepted that the method of determining the Pythagorean triplets, attributed to Pythagoras by Hero and Proclus, was determined with the aid of square numbers (see below). Hence, we have sufficient reason to align ourselves with the scholars who consider Pythagoras the author of this theory [Allman 1889, 31–33; Heath 1922 I, 76; van der Waerden 1956, 158–164].

The basic propositions of this theory were not included in Euclid's collection. They appeared in popular form in the books of later authors, such as Nicomachus (*Introductio arithmetica* I, 7–11, 13–16, 17) and Theon of Smyrna (*Eorum quae in mathematicis ad Platonis lectionem utilia sunt expositio,* E. Hiller, Ed., p. 26–42), and also in Iamblichus's commentaries on Nicomachus. Nicomachus did not adduce the proofs; nonetheless, they evidently were included in the material that he used and to which he added practically nothing. This follows at least from the propositions coinciding with Euclid's: he gives demonstrations while Nicomachus omits them because he wrote for a public that was not interested in them. If Pythagoras rigorously proved all the elementary propositions of even and odd numbers, then he must have also constructed the theory of figured numbers on deductive grounds. Knorr [1975, 142–145] gives a very plausible reconstruction of this theory, although he doubts whether the Pythagoreans developed it as strictly axiomatically as he himself does.

From the study of triangular and square numbers we can proceed to stereometric problems and attempt to construct a body limited by equilateral triangles and squares—in this case we obtain the tetrahedron and cube. Study of the

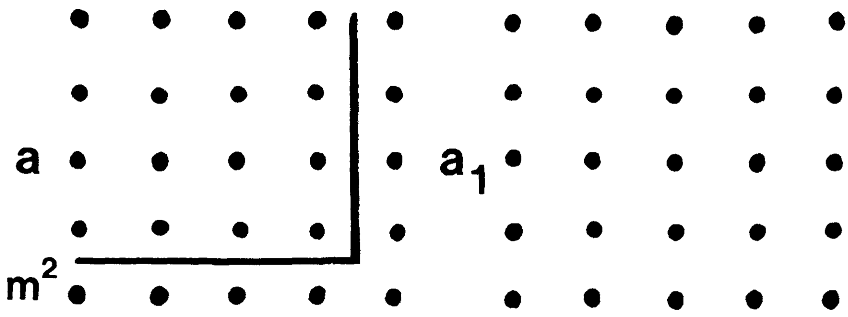

FIGURE 3

properties of square numbers quite probably led to the finding of the method of determining the Pythagorean triplets [Allman 1889, 31–33; Heath 1926 I, 356–360; von Fritz 1945, 252; van der Waerden 1956, 163–164) which may be represented as follows. Adding a gnomon to the square, we obtain the next square; therefore, we must find such a gnomon as would be a square number (see Fig. 3). a is the side of the square and gnomon $m^2 = 2a + 1$ from

(1) $a = (m^2 - 1)/2$;

(2) $a_1 = a + 1 = (m^2 + 1)/2$.

In order for m^2 to satisfy (1) and (2), m must be odd. From this we obtain

$$m^2 + \left(\frac{m^2 - 1}{2}\right)^2 = \left(\frac{m^2 + 1}{2}\right)^2,$$

where m is any odd number.

Above we cited Iamblichus, who attributed to Pythagoras the discovery of friendly numbers, each of which is equal to the sum of the factors of the other. Although on the whole Iamblichus is an unreliable source, in this case we seem to have no reason to doubt him. It is quite another matter if we turn to a related problem—perfect numbers, that is, those equal to the sum of their factors, for instance,

$$1 + 2 + 3 = 6 \quad \text{or} \quad 1 + 2 + 4 + 7 + 14 = 28.$$

Perfect numbers were treated by Nicomachus and also by Theon of Smyrna and Iamblichus. Nicomachus gives the basic rule for finding them: if the sum of the parts of a geometric series is a prime number, then multiplying it by the last member of the series will give a perfect number (*Introductio arithmetica* I, 16, 1–4). In Nicomachus the proof of this rule is omitted as usual, but it can be found in Euclid (IX, 36).

Heath [1922 I, 74–76], Becker [1934, 134–136], and van der Waerden [1956, 161] have attributed perfect numbers either to Pythagoras directly or to the early Pythagoreans. However, Burkert [1972, 431–433] refutes this, suggesting that perfect numbers were not discovered earlier than the second half of the fourth century. Actually, we encounter the first perfect numbers only in Euclid. Aristotle testifies that the Pythagoreans called "10," and not "6" or "28," a perfect number (*Metaphysica* 1084 a 32). Nothing was said of these numbers in the fragment of Speusippus, although prime numbers are mentioned here.

In the absence of direct evidence we can scarcely insist on the early Pythagorean origin of perfect numbers or ascribe them to Pythagoras himself. Nonetheless, we must admit that the method of finding them is in itself so simple that it certainly might have been found during Pythagoras's lifetime. The rule for finding perfect numbers (IX, 36) directly results from the theory of even and odd numbers (IX, 21–34), and its proof with minor changes can be given with the support of only Propositions 21–34 [Becker 1934, 134–136; van der Waerden 1979, 399–400]. If this proof is actually the original one, it may belong to the very early stage of Pythagorean arithmetic.

6

In examining the mathematical studies of Pythagoras, we cannot but note the predominance of the arithmetic part over the geometric. This fact has already been noted [Michel 1958, 5–6; Knorr 1975, 132–134], and it can scarcely be explained only by the state of the sources since there are several important testimonies to this effect. Archytas (47 B 4) had already relegated arithmetic to first place, considering it more rigorous than geometry, which should indicate the maturity of Pythagorean arithmetic already in the first half of the fifth century. Diogenes Laertius, citing the historian Anticlides, wrote that Pythagoras paid more attention to the "arithmetical aspect of geometry" (VIII, 12). The testimonies of Aristotle (*Qui ferebantur librorum fragmenta,* V. Rose, Ed., fr. 191) and Aristoxenus (fr. 23 Wehrli) also support this statement.

Nevertheless, it is quite probable that several other geometric theorems of the first four books of Euclid belong to Pythagoras, although evidence of this has not survived. Naturally, the list of his achievements presented here should not be considered exhaustive.

On the other hand, we should not wonder at the comparatively small number of the mathematical discoveries of Pythagoras. The Greeks often wrote of the mathematical coloring of Pythagoras's philosophy, but they never considered him a mathematician *par excellence,* mainly because he was not. His talent manifested itself in the broadest possible areas—politics, religion, philosophy, science—so that mathematics could not occupy a leading position with him. As for the first "professional" mathematicians—Hippocrates, Theaetetus, and Eudoxus—we can suppose that they gave wholehearted spiritual energy to their systematic study of mathematics, but this could scarcely be characteristic of Pythagoras, for whom politics and religion were equally important.

Still, in order to give a balanced evaluation of his role in the development of mathematics, it is necessary to consider Pythagoras in a real historical perspective. Pythagoras belonged to only the second generation of Greek mathematicians, and we must not compare him with Archytas or Eudoxus, but with his predecessor Thales, for whom mathematics was also not the main sphere of his intellectual activity. In this comparison there are sufficient facts to speak of a new stage of development of Greek mathematics beginning with Pythagoras.

The basis of mathematics, the deductive method, was discovered by Thales and was applied to mathematical facts whose veracity was visible and almost self-evident, for instance, in the following statement: the diameter divides a circle into two equal parts. But Thales was not satisfied with mere visualizability, and his proofs were not a mere demonstration of this. Those proofs that have come down to us (Aristotle, *Analytica priora* 41 b 13–22; *Metaphysica* 1051 a 26) show the normal procedure of logical reasoning.

Pythagoras's theorem does not possess the visualizability of Thales's theorems and is therefore an important step forward. The oft-mentioned tendency [Reidemeister 1949, 51–52; von Fritz 1955, 93] of early Greek mathematics to move the center of gravity from the visualizability of geometric construction (preserved in particular in such important terms as ϑεώρημα and δείκνυμι) to abstract logical proof should be credited precisely to Pythagoras. Actually, Eudemus wrote about this, stressing the more abstract character of Pythagoras's geometry in comparison with Thales's (fr. 133 Wehrli).

Although in regard to Pythagoras's time we cannot speak of any developed theory in geometry, the need for one was expressed in the explicit formulations of both the first basic axioms of geometry [van der Waerden 1978, 357] and the first geometric definitions (Aristotle, *De anima* 409 a 6; *De sensu* 43 a 31). It was not by chance that Favorinus asserted that Pythagoras was the first to give definitions in mathematics (Diogenes Laertius VIII, 48).

Thales was the first to study "angular" geometry rather than the "linear" geometry of the Egyptians and Babylonians [Gandz 1929], while Pythagoras took the next step and founded stereometry by constructing the regular tetrahedron and the cube.

In addition to geometry, he transferred the deductive method to the still untouched region of arithmetic, creating the first examples of the theory of numbers: the theory of even and odd numbers and the theory of figured numbers. With these theories began the separation, attested to by Aristoxenus, of theoretical arithmetic from the art of practical calculation. It was probably here that the indirect proof was first employed, although it might have just as easily sprung from geometry.

Pythagoras's theory of proportionals became the link between arithmetic and geometry, and in harmonics it paved the way for the first mathematically formulated physical regularity in Greek. We need not stress the immense importance of this first successful attempt at using the quantitative method in the investigation of nature.

The extension of numerical regularities to the movements of the heavenly bodies, although at first arbitrary, had a direct impact on the formation of Greek mathematical astronomy.

Last but not least, Pythagoras was the founder of a famous school of mathematics that determined its development in Greece for many decades to come. In considering Pythagorean mathematics, we should keep in mind not only Pythagoras, Hippasus, Theodorus of Cyrene, and Archytas, who all directly belonged to the Pythagorean community, but also those who were their disciples, who received the principles of this science from the Pythagoreans, such as Democritus, Hippocrates, Hippias of Elis, Theaetetus, and Eudoxus. It is easy to see that there were practically no outstanding mathematicians of the fifth to the first third of the fourth century B.C. outside this group.

Naturally, the reason for this notable success did not lie simply in the devotion of the mathematicians to that line of Pythagorean thought that considered number the key to knowledge. Although such a viewpoint has been frequently expressed, no one has satisfactorily explained how this conviction could help anyone in mathematical research in contrast to, say, the use of mathematics in the investigation of nature. In any case, Hippasus or Theodorus, who showed no trace of number philosophy, achieved in mathematics much greater success than Philolaus, who claimed that "without numbers no cognition is possible" (44 B 4). The flowering of the exact sciences in the Pythagorean school, in addition to the general effect of the Greek cultural revolution revealed by Zaitsev [1985], is also associated with the fact that at the time of Pythagoras four related sciences—arithmetic, geometry, harmonics, and astronomy [van der Waerden 1979, 330–336]—were already unified and this quadrivium occupied a stable place in Pythagorean education. This aided in the constant amassing of new knowledge and its preservation, and at the same time gave access to the study of mathematics to young men at the very age most favorable for its study and for independent research. This tradition, upheld by the Sophists and supported by Plato's authority, outlived both antiquity and the Middle Ages and has retained its value today.

ACKNOWLEDGMENT

I thank my reviewer for having improved my English.

NOTES

1. We cite the pre-Socratic fragments from *Die Fragmente der Vorsokratiker*, H. Diels & W. Kranz, Eds., 8th edition (Dublin/Zürich: Weidmann, 1966). We cite Eudemus's fragments from *Die Schule des Aristoteles: Texte und Kommentar*, F. Wehrli, Ed., 2nd edition (Basel–Stuttgart: Schwabe, 1967–1974).

2. This was mentioned earlier by P. Tannery [1887, 81] and A. Rey [1935, 58–75]; see also Heath [1922 I, 2, 165–169].

3. This point of view was shared by I. Heiberg [1912, 10], J. Burnet [1920, 69], A. Rey [1933, 191, 202–203], K. Reidemeister [1949, 10], H. Cherniss [1951, 336], and L. Tarán [1965, 4].

4. For more details see Zaitsev [1985, 180–190].

5. For criticism of Szabó's ideas see Berka [1981, 125–131] and Knorr [1981, 145–186].

6. Summarized in Burkert [1972, 409–412].

7. According to van der Waerden [1980, 26], Burkert now has abandoned the idea that Proclus inserted in the catalog the wording of Iamblichus.

8. For the history of the question see Vogt [1909], Heath [1922 I, 84–85, 154–155], and Stamatis [1977, 188].

9. The evidence has been collected in the article by K. von Fritz [1945]. See also Knorr [1975, 50–51]. In a recent article, E. Stamatis attributes this discovery to Pythagoras [Stamatis 1977].

10. Heath's idea that the Pythagoreans constructed all five regular polyhedra, although perhaps not in such a rigorous mathematical way as that presented by Euclid [Heath 1922 I, 158–162], is refuted with forceful arguments by Waterhouse [1972].

11. For evidence and detailed analysis see Heath [1922 I, 144–145; 1926 I, 350–356].

12. Van der Waerden, although he does not ascribe the theory of even and odd numbers to Pythagoras, dates it at ca. 500 [van der Waerden 1979, 392]. Becker has expressed himself more cautiously: the first half of the fifth century [1954, 38].

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