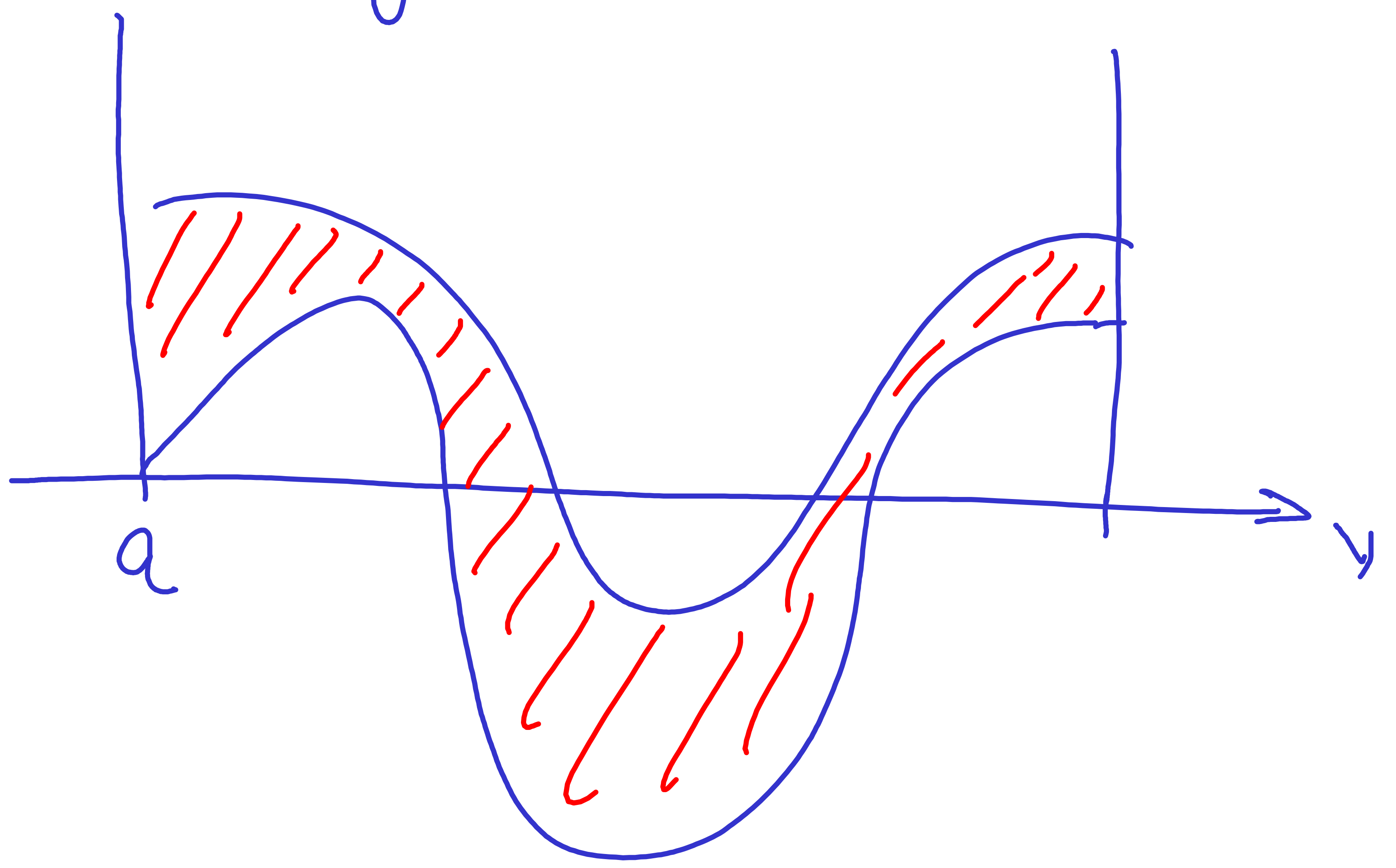
Geometrické aplikace integrálu

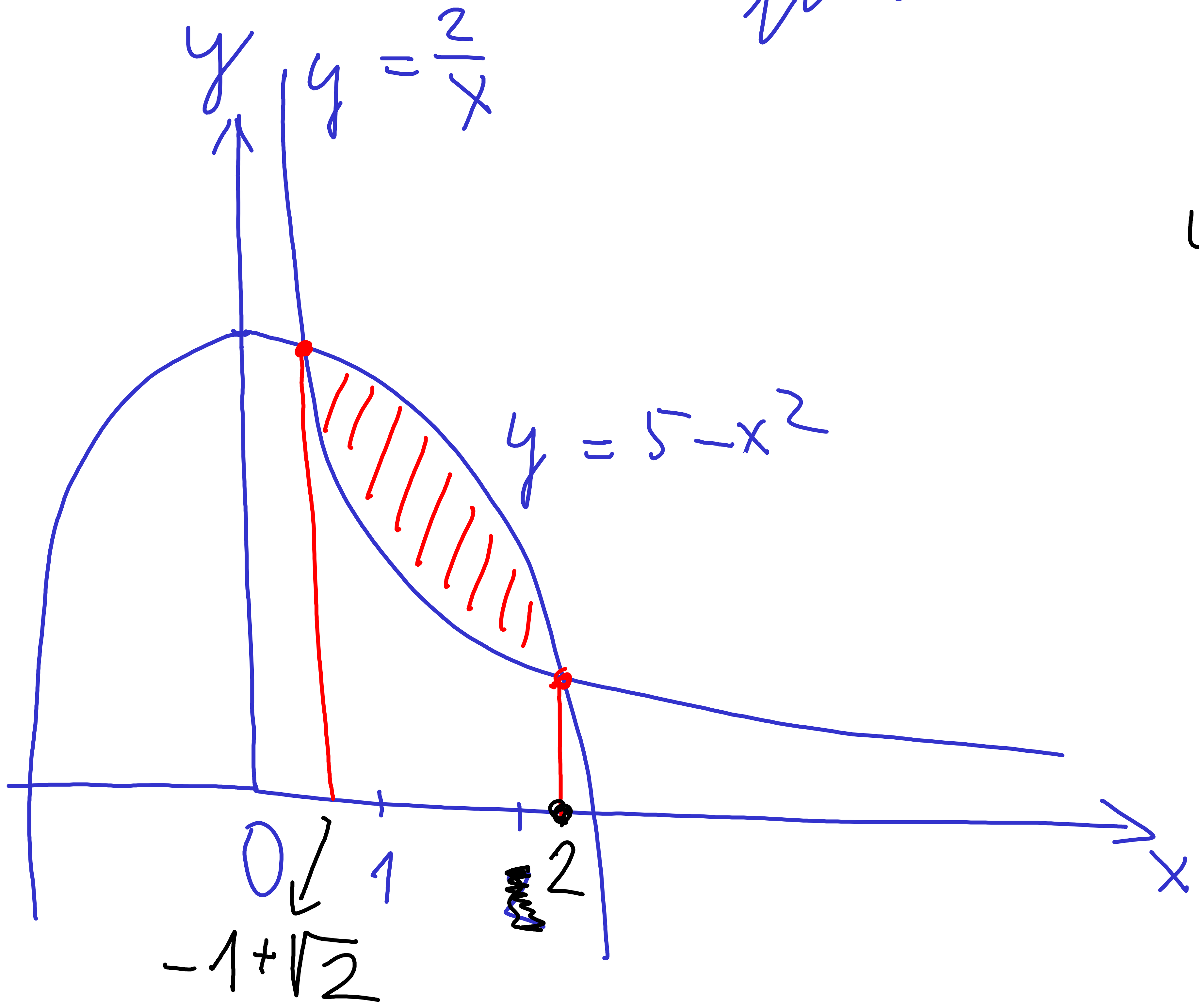
① **Výpočet obsahu** Nechť $f(x) \geq g(x)$ na intervalu $[a,b]$.

Potom obsah plochy mezi grafy funkcí f a g na intervalem $[a,b]$

$$S = \int_a^b \left(f(x) - g(x)\right) dx$$



Příklad: Spočítejte obsah omezené oblasti mezi grafy funkcí $y = 5 - x^2$ a $y = \frac{2}{x}$ pro $x \geq 0, y \geq 0$.



$$y = 5 - x^2 = \frac{2}{x}$$

$$5x - x^3 = 2$$

$$x^3 - 5x + 2 = 0$$

$$(x-2)(x^2+2x-1) = 0$$

$$x_{2,3} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$$

± 1

± 2

2 je kořen

Pouze $-1+\sqrt{2} > 0$.

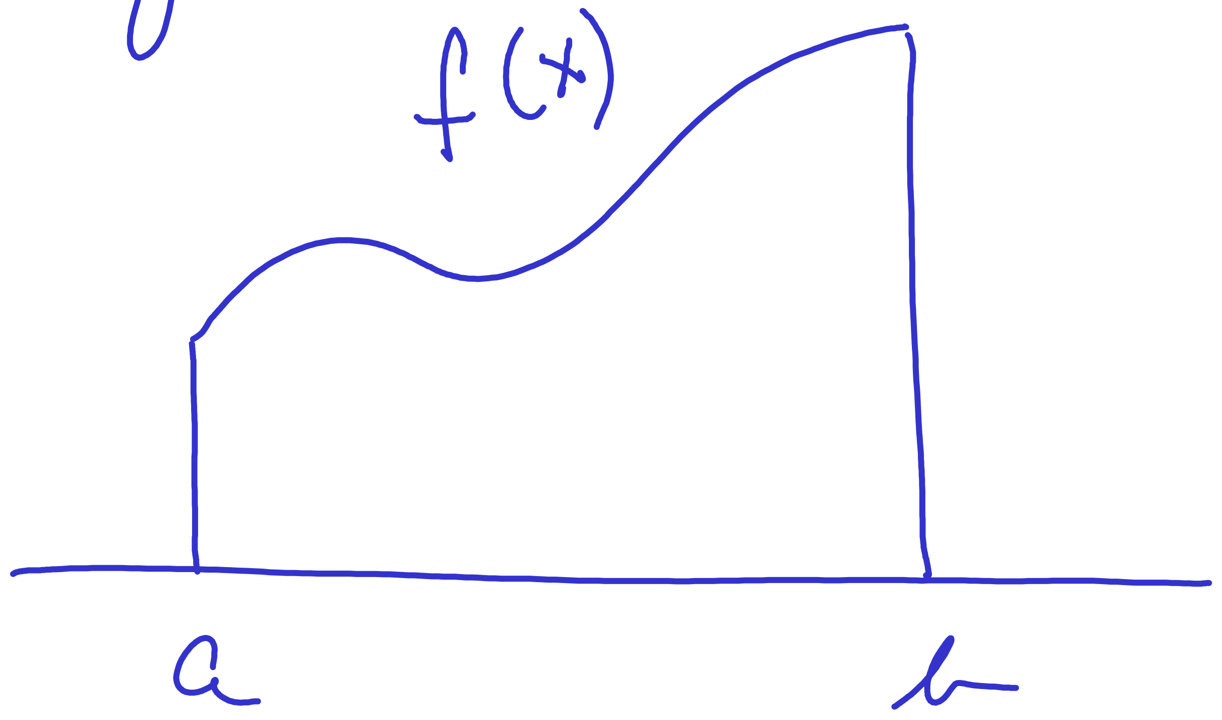
Proto počítáme

$$\int_{\sqrt{2}-1}^{2} \left((5-x^2) - \frac{2}{x} \right) dx = \left[5x - \frac{x^3}{3} - 2 \ln|x| \right]_{\sqrt{2}-1}^{2}$$

$$= \left(5 \cdot 2 - \frac{8}{3} - 2 \ln 2 \right) - \left(5(\sqrt{2}-1) - \frac{(\sqrt{2}-1)^3}{3} - 2 \ln(\sqrt{2}-1) \right)$$

$$= 10 - \frac{8}{3} - 2 \ln 2 - 5\sqrt{2} + 5 + \frac{(\sqrt{2}-1)^3}{3} + 2 \ln(\sqrt{2}-1) = \ldots\ldots$$

Výpočet objemu rotačního tělesa

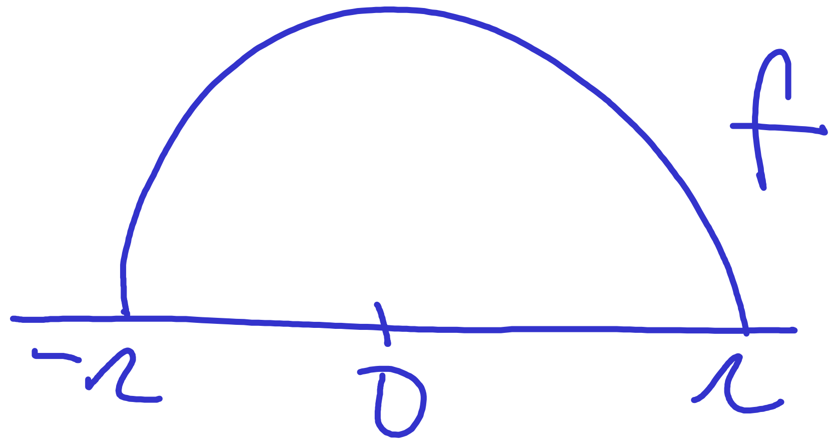


$f(x) \geq 0$ na $[a,b]$

Rotací kolem osy x vznikne těleso, jehož objem je

$$V = \pi \int_a^b f^2(x)\,dx$$

Objem koule o poloměru r



$f(x) = \sqrt{r^2 - x^2}$

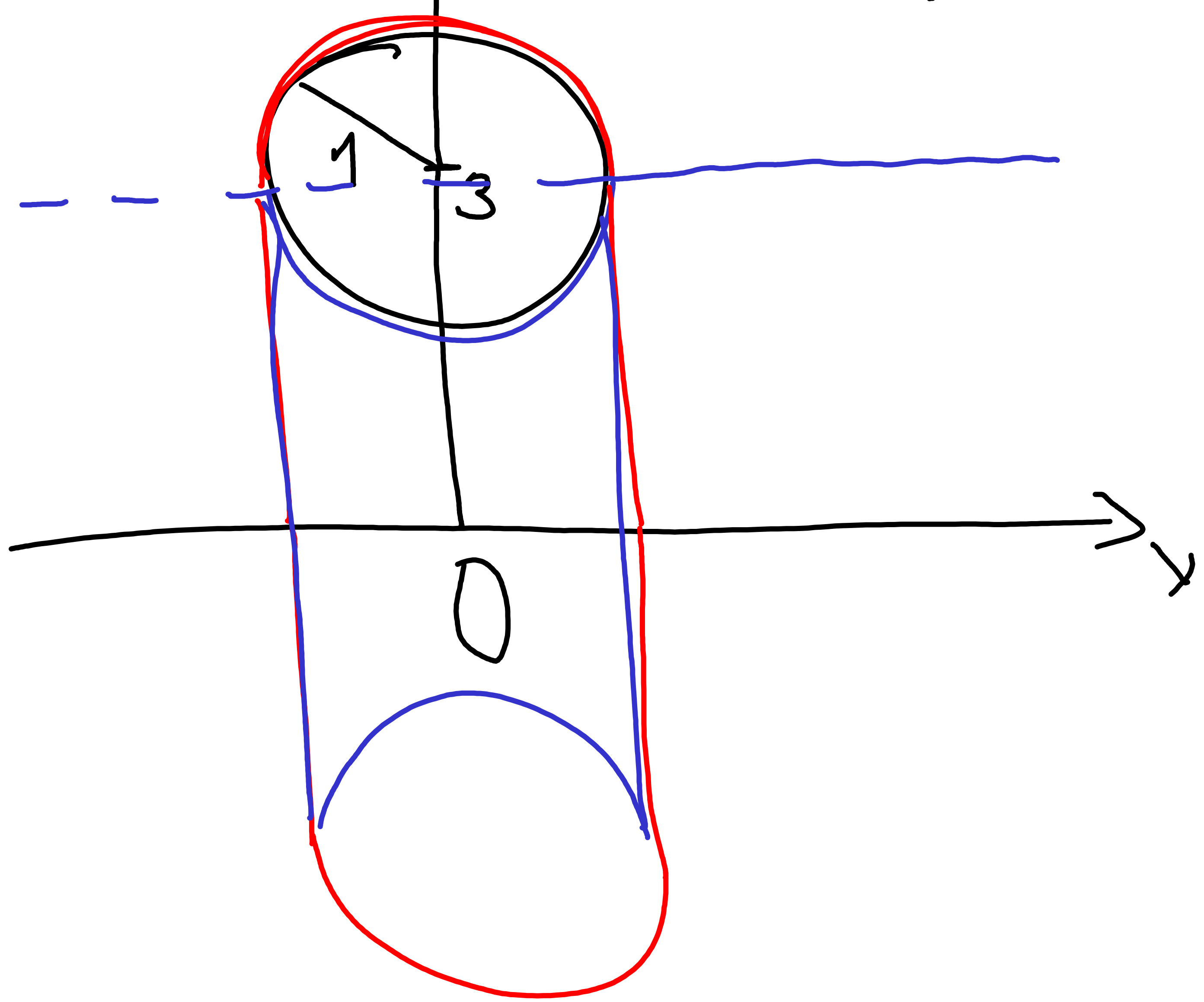
Objem koule bude

$$V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} =$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \pi \left\{ \left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right\} = \pi \left(\frac{2}{3} r^3 + \frac{2}{3} r^3 \right)$$

$$= \frac{4}{3} \pi r^3$$

Další příklad – objem anuloidu



Rotací této kružnice kolem osy x vznikne anuloid

Kružnice má rovnici $x^2 + (y-3)^2 = 1$

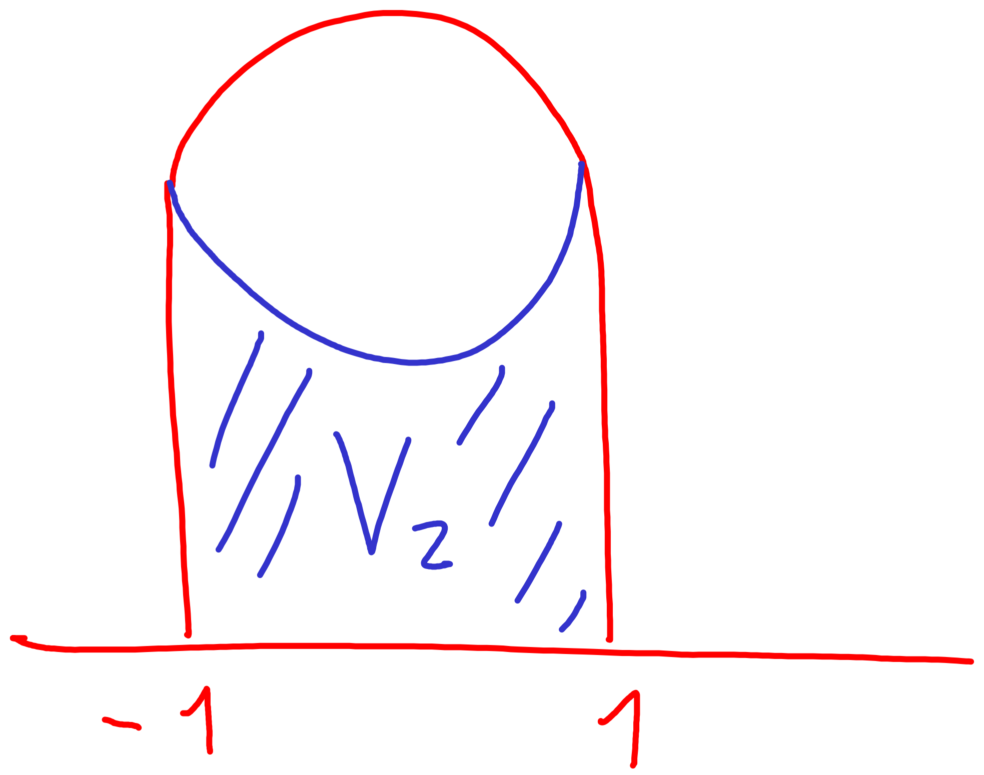
$$(y-3)^2 = 1 - x^2$$

$$|y-3| = \sqrt{1-x^2}$$

$$f(x) = y = 3 + \sqrt{1-x^2}$$

$$g(x) = y = 3 - \sqrt{1-x^2}$$

Objem rotačního tělesa



$$V_1 = \pi \int_{-1}^{1} \left(3+\sqrt{1-x^2}\right)^2 dx = \pi \int_{-1}^{1} \left(9 + 6\sqrt{1-x^2} + 1-x^2\right) dx$$

$$= \pi \int_{-1}^{1} \left(10-x^2 + 6\sqrt{1-x^2}\right) dx =$$

$$= \pi \int_{-1}^{1} (10-x^2)\, dx + 6\pi \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$V = V_1 - V_2$$

$$V_2 = \pi \int_{-1}^{1} \left(3-\sqrt{1-x^2}\right)^2 dx = \pi \int_{-1}^{1} \left(9 - 6\sqrt{1-x^2} + 1-x^2\right) dx$$

$$V = V_1 - V_2 = \pi \int_{-1}^{1} \left(9 + 6\sqrt{1-x^2} + 1-x^2\right) dx - \pi \int_{-1}^{1} \left(9 - 6\sqrt{1-x^2} + 1-x^2\right) dx$$

$$= \pi \int_{-1}^{1} 12\sqrt{1-x^2}\, dx = 12\pi \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

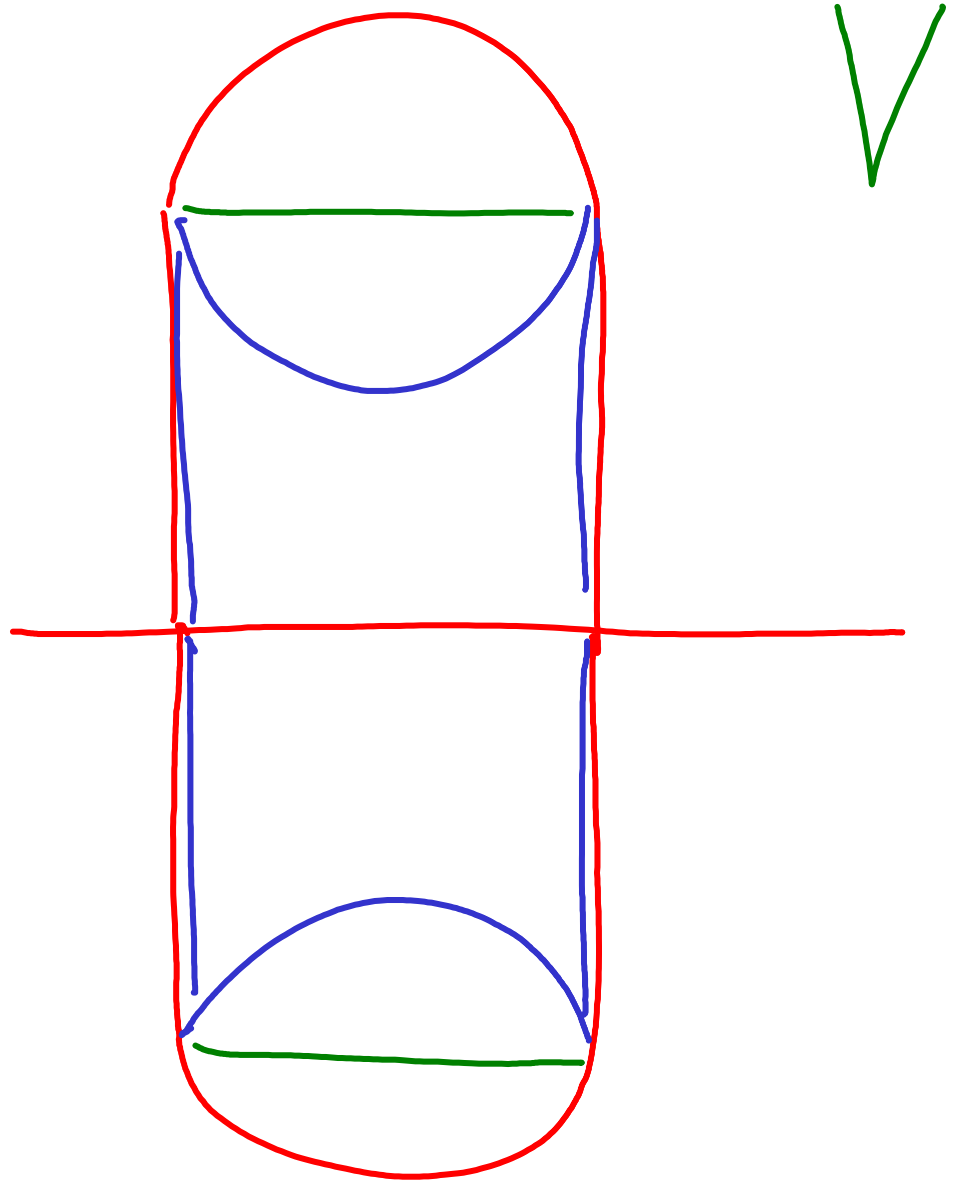
$x^2 = \sin z$ a použijeme

řídu o substituci

– minulá přednáška

$$= 12\pi \left[\frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)\right]_{-1}^{1}$$

$$= 6\pi \left\{ \left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right)\right\} = 6\pi^2$$



V = objem válce

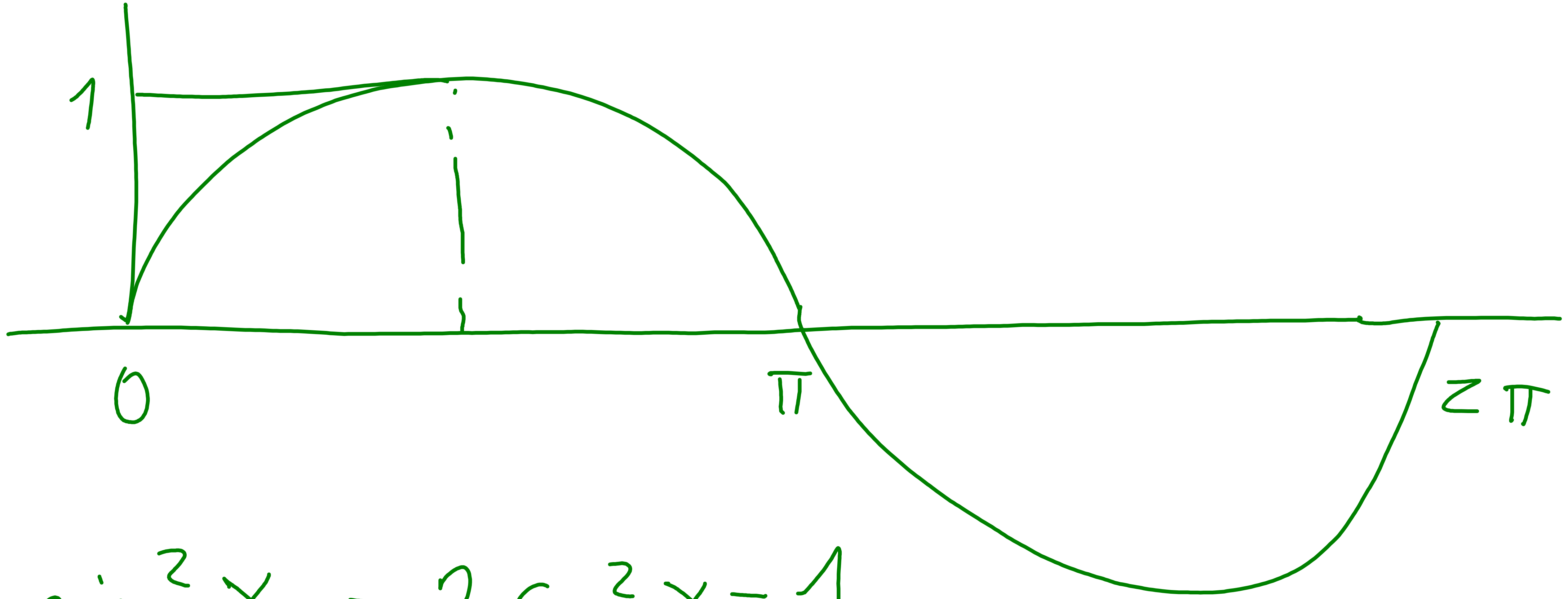
$(h(x) = 3$ rotační funkce $h)$

$$V = \pi \int_{-1}^{1} 3^2 dx = \pi 9 [x]_{-1}^{1} = 9\pi(1-(-1)) = 18\pi$$

Délka křivky

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

$f(x) = \sin x$ délka sinusoidy na intervalu $[0, \pi]$



$$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$$

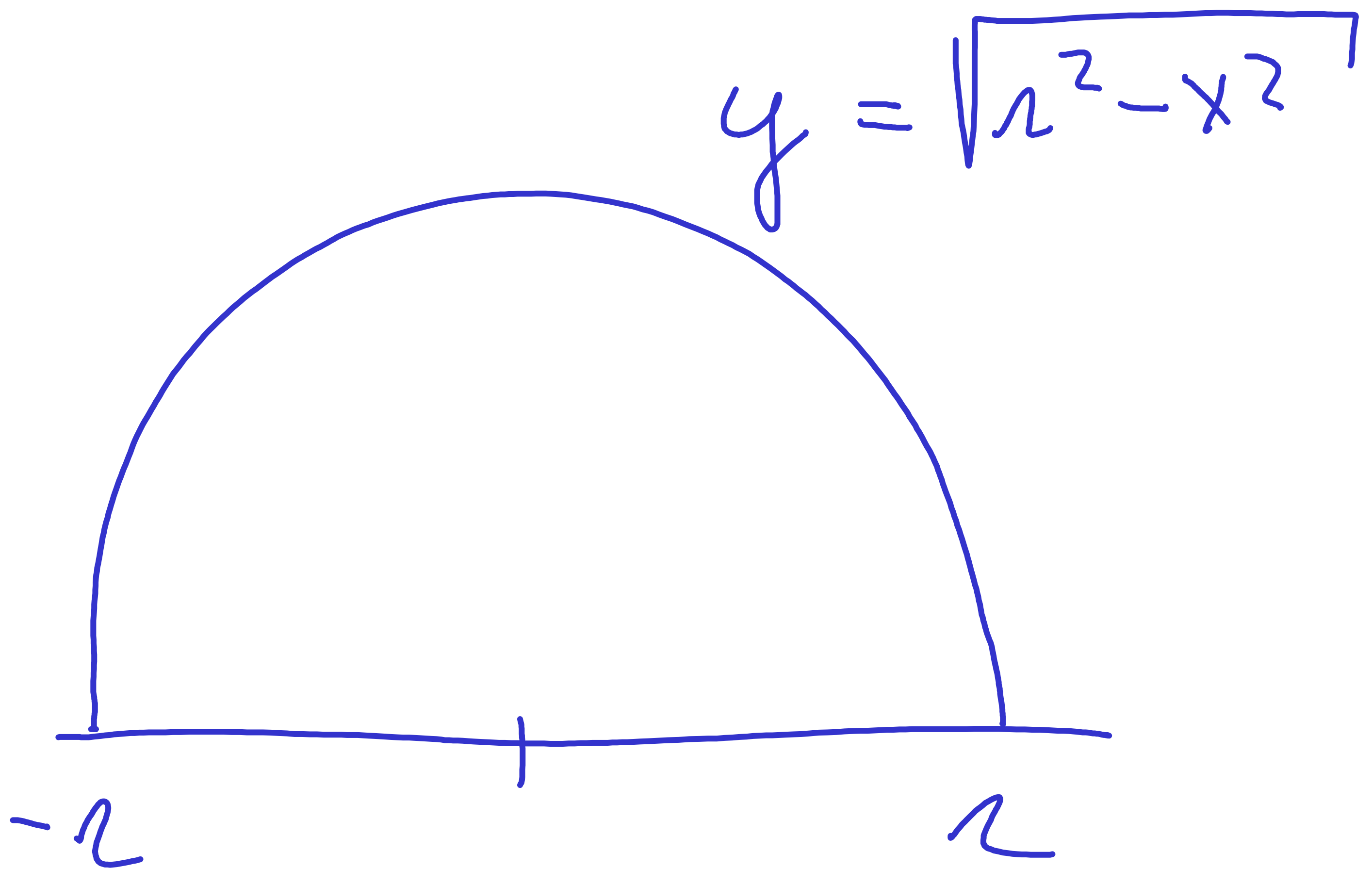
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

Uvidíme příště.

Povrch pláště rotačního tělesa

Funkce $f(x) \geq 0$ na $[a,b]$. Rotací grafu kolem osy x vznikne těleso. Obsah pláště tohoto tělesa spočítáme takto:

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1 + f'^2(x)}\, dx$$



Příklad: Povrch koule o poloměru r

$$S = 2\pi \int_{-r}^{r} \underbrace{\sqrt{r^2 - x^2}}_{f(x)} \cdot \underbrace{\sqrt{\frac{r^2}{(r^2 - x^2)}}}_{\sqrt{1 + (f'(x))^2}}\, dx = 2\pi \int_{-r}^{r} r\, dx = 2\pi r \int_{-r}^{r} 1\, dx$$

$$f'(x) = \frac{1}{2} \frac{(-2x)}{\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$= 2\pi r \left[x \right]_{-r}^{r} = 2\pi r (r - (-r)) =$$

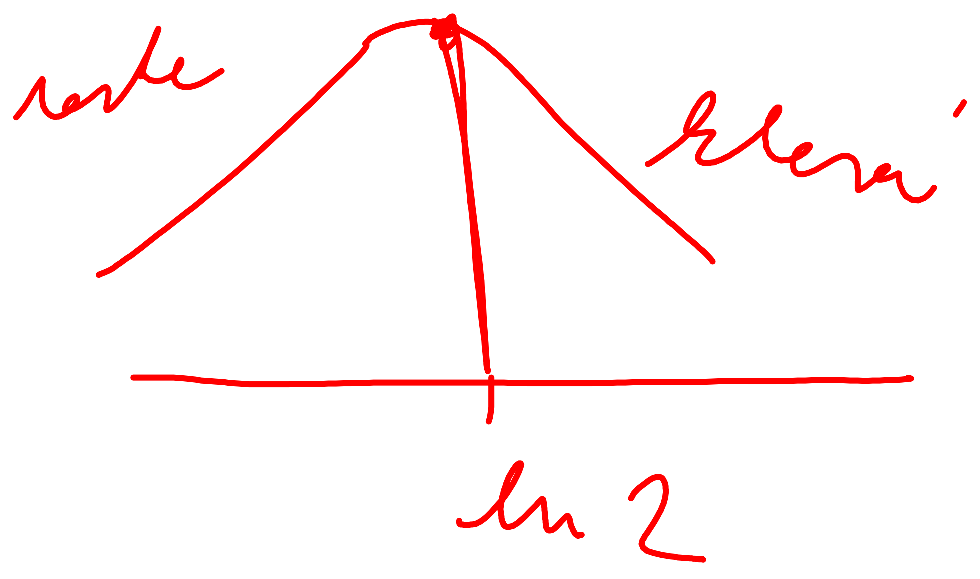
$$= 4\pi r^2$$

= povrch koule

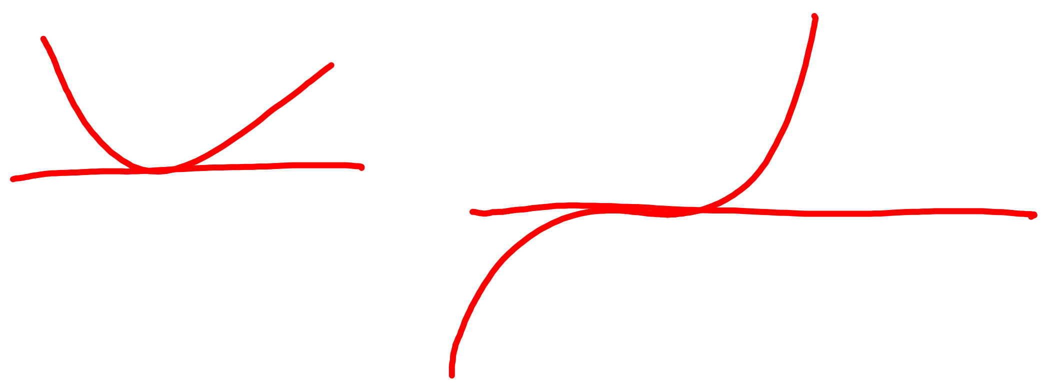
7. úloha

$$C(t) = e^{-t} - e^{-2t} \quad ?$$

$$C'(t) = 0 \quad \ldots\ldots \quad t = \ln 2 \ ?$$

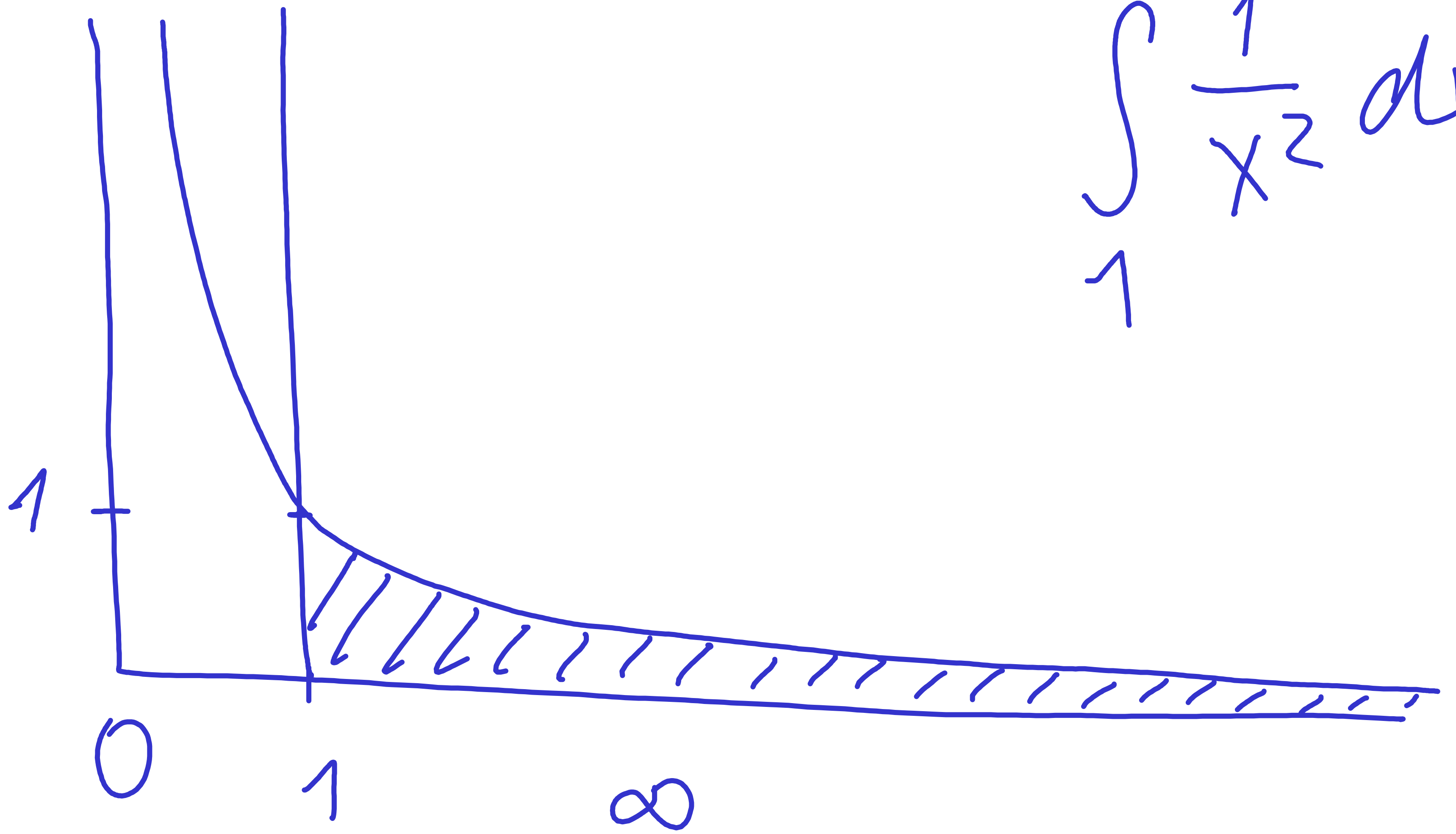


$f' > 0 \mid f' < 0$



Nevlastní integrál

$f(x) = \frac{1}{x^2}$



K velké číslo

$$\int_1^K \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^K = -K^{-1} - (-1) = 1 - \frac{1}{K}$$

$$\lim_{K \to \infty} 1 - \frac{1}{K} = 1$$

$F(x) = -x^{-1}$ je prim. funkce k $\frac{1}{x^2}$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{x \to \infty} F(x) - F(1) = 1$$

nevlastní integrál.

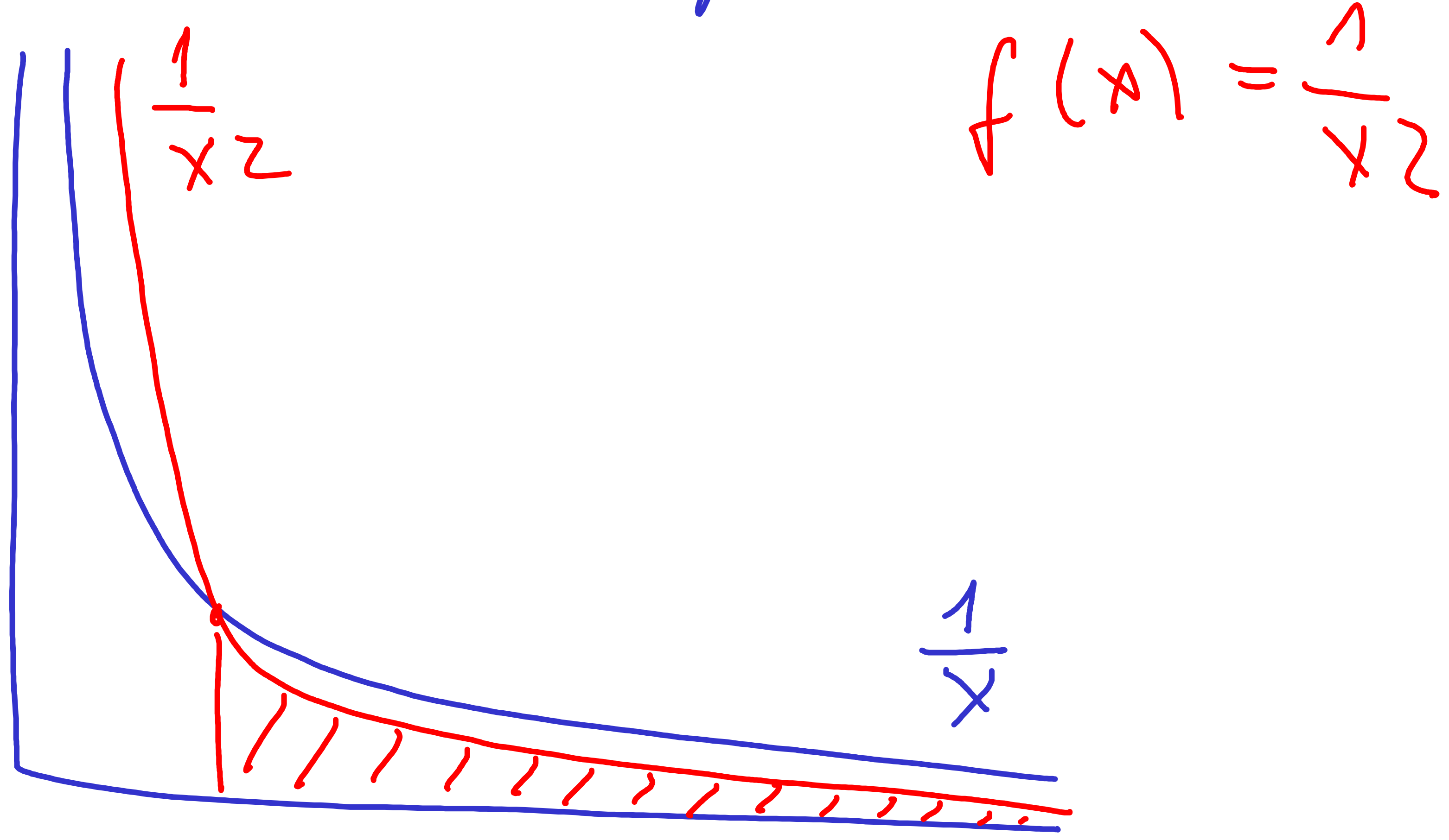
Jestliže limita existuje konečná, říkáme, že integrál konverguje.

$$g(x) = \frac{1}{x}$$

$$\int_1^\infty \frac{1}{x}\,dx = \Big[\ln x\Big]_1^\infty$$

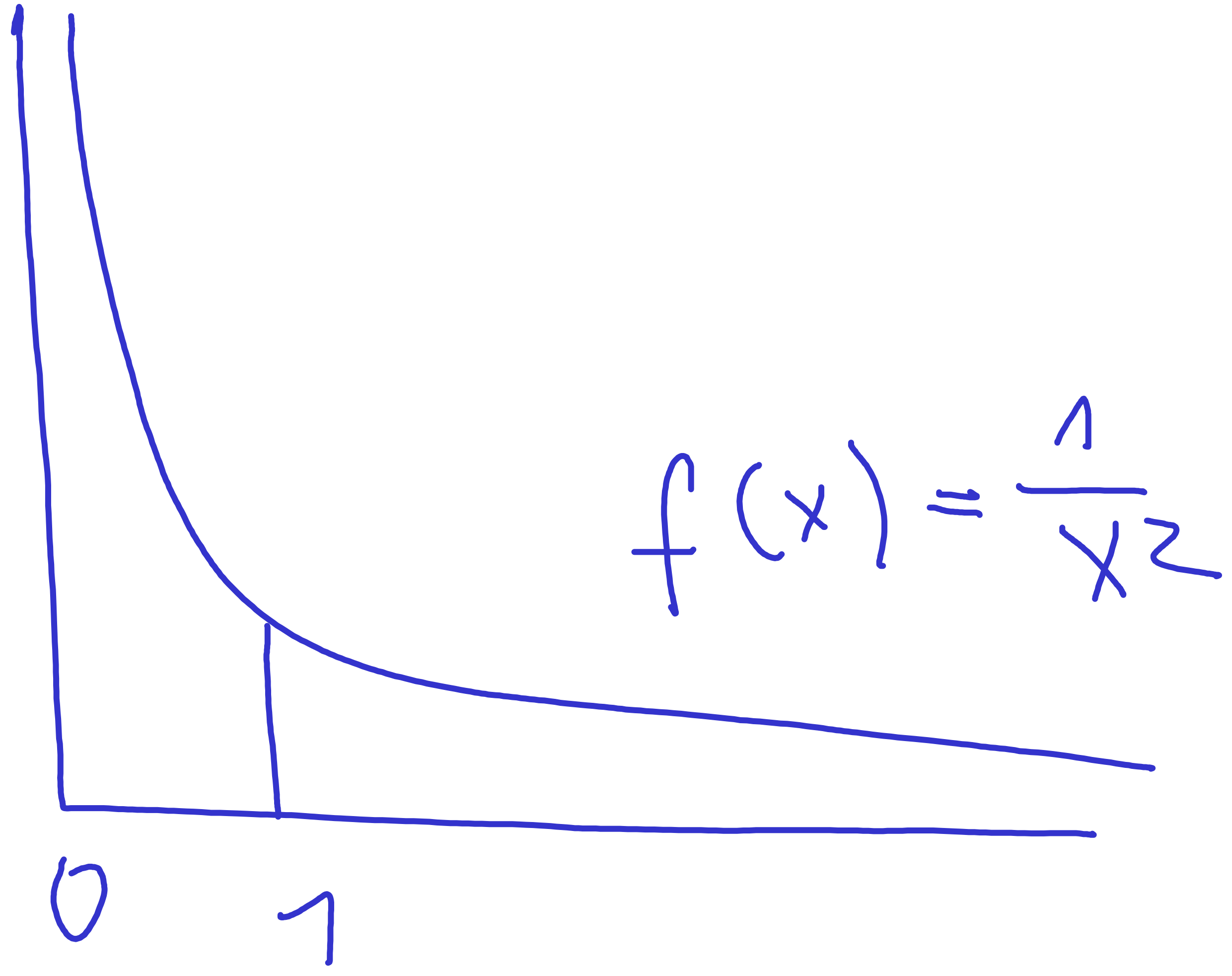
$$= \lim_{x\to\infty} \ln x - \ln 1$$

$$= \infty - 0 = \infty$$



Integrál $\displaystyle\int_1^\infty \frac{dx}{x}$ diverguje.

$$\int_1^\infty \frac{1}{x^\alpha}\,dx \qquad \begin{array}{ll} \alpha > 1 & \text{konverguje} \\ \alpha \le 1 & \text{diverguje} \end{array}$$



$$\int_0^1 \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_0^1 = (-1) - \lim_{x \to 0_+}\left(-\frac{1}{x}\right) =$$

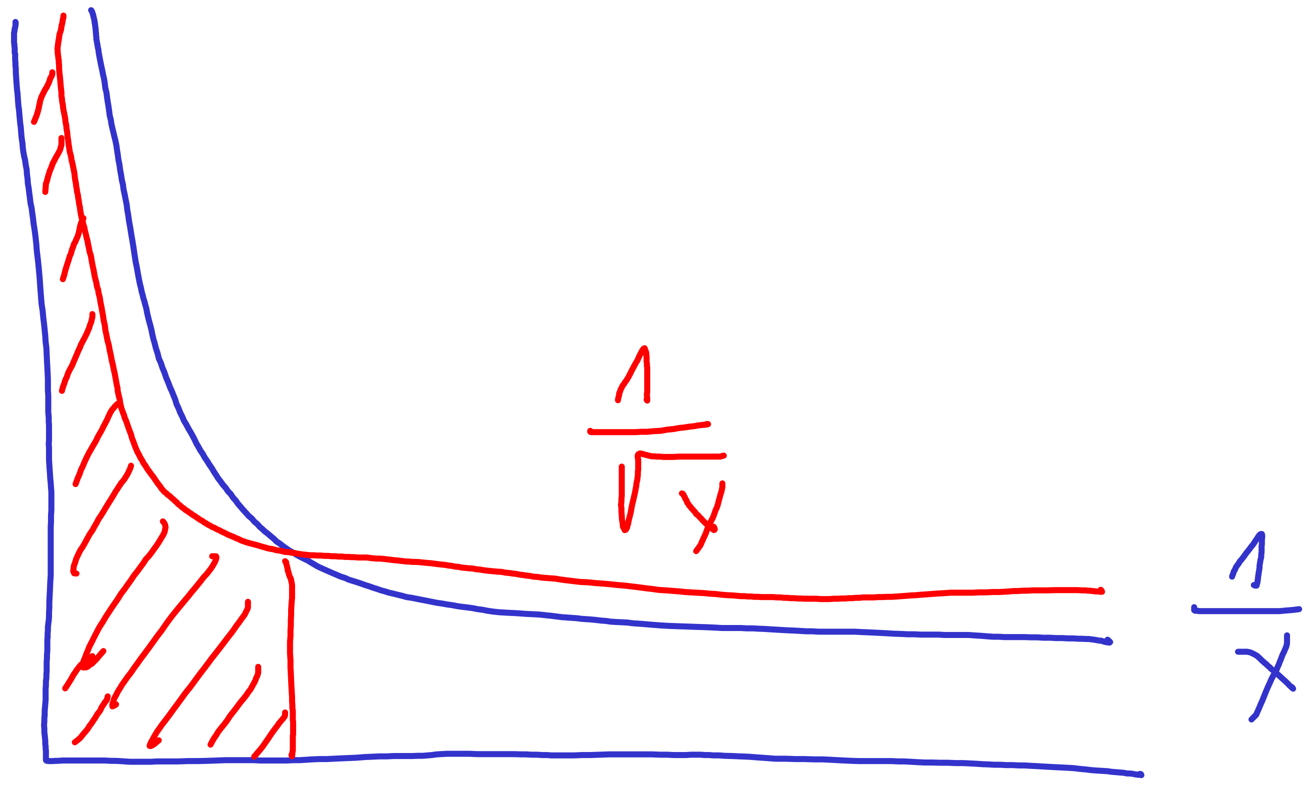
$$= (-1) + \lim_{x \to 0_+} \frac{1}{x}$$

$$= -1 + \infty = \infty$$

nevlastní integrál diverguje

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1$$

$$= \left[2\sqrt{x}\right]_0^1 = 2 - 0 = 2$$

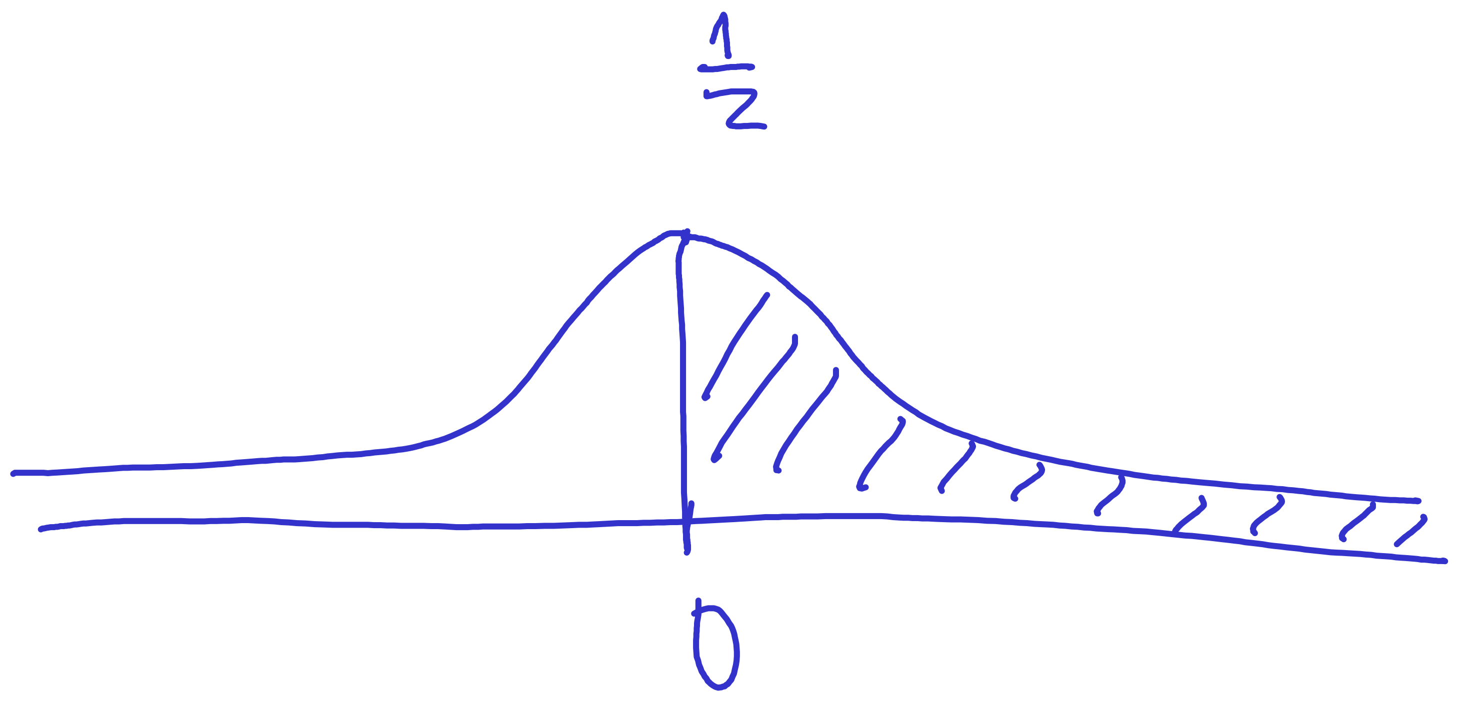


Tento integrál konverguje.

<u>Příklady</u>

(A) $$\int_0^\infty \frac{1}{x^2+1}\,dx$$

$$= \left[\operatorname{arctg} x\right]_0^\infty = \lim_{x\to\infty} \operatorname{arctg} x - \operatorname{arctg} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$



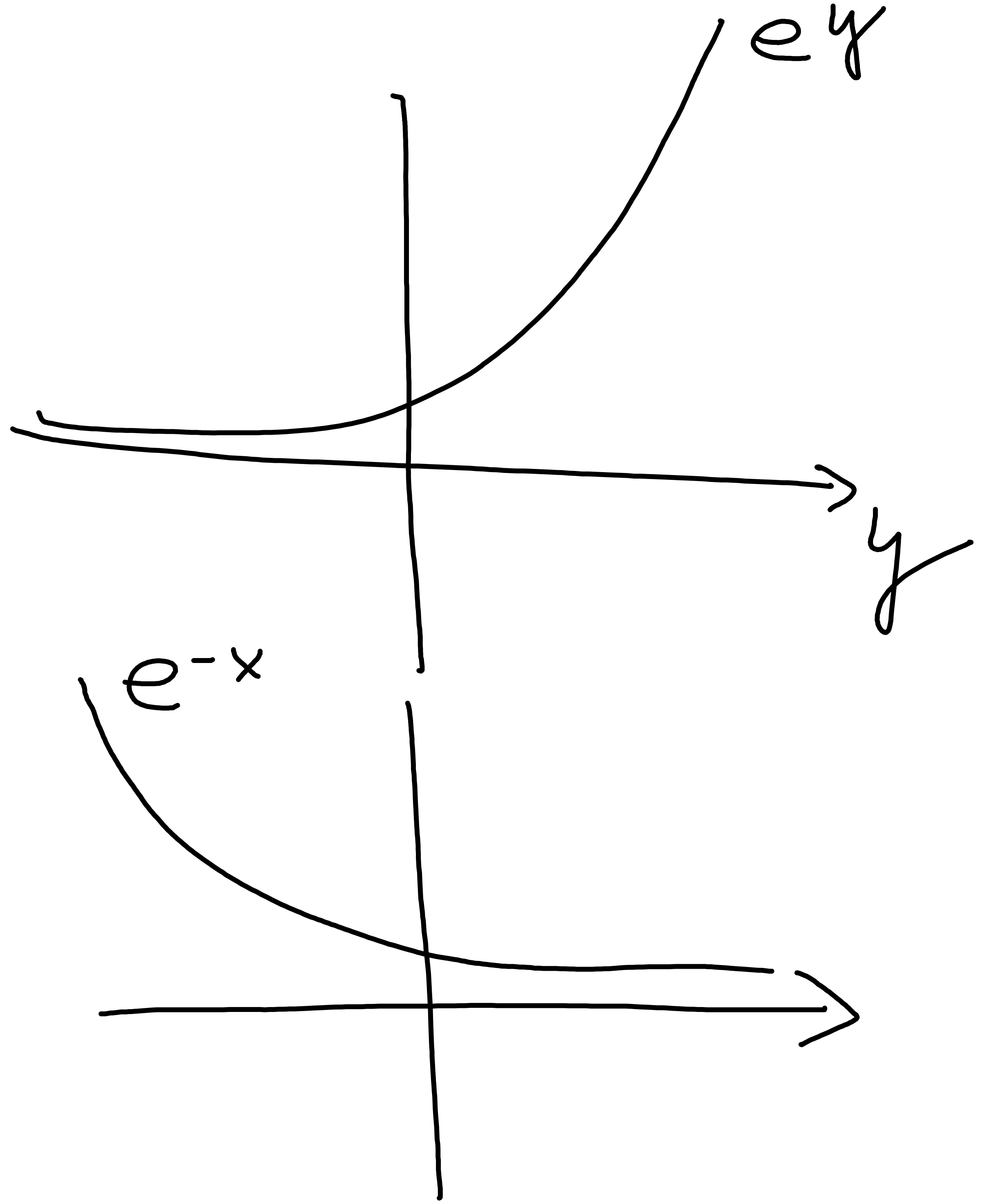
(B) $$\int_0^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\infty} = \lim_{x\to\infty}\left(-e^{-x}\right) - \left(-e^{-0}\right) =$$

$$= 0 + e^{0} = 1$$

(C) $$\int_0^{\infty} \sin x\,dx = \left[-\cos x\right]_0^{\infty} =$$

$$= \underbrace{\lim_{x\to\infty}\left(-\cos x\right)}_{\text{neexistuje}} - \left(-\cos 0\right)$$

Integrál neexistuje.



Příklad:

Zjistěte, zda konverguje $\int_0^1 x \ln x \, dx$

$$\int_0^1 \underset{u'(x)}{x}\, \underset{v(x)}{\ln x}\, dx \overset{\text{per partes}}{=} \left[\frac{x^2}{2}\ln x\right]_0^1 - \int_0^1 \frac{x^2}{2}\frac{1}{x}\,dx = \left[\frac{x^2}{2}\ln x\right]_0^1 - \left[\frac{x^2}{4}\right]_0^1$$

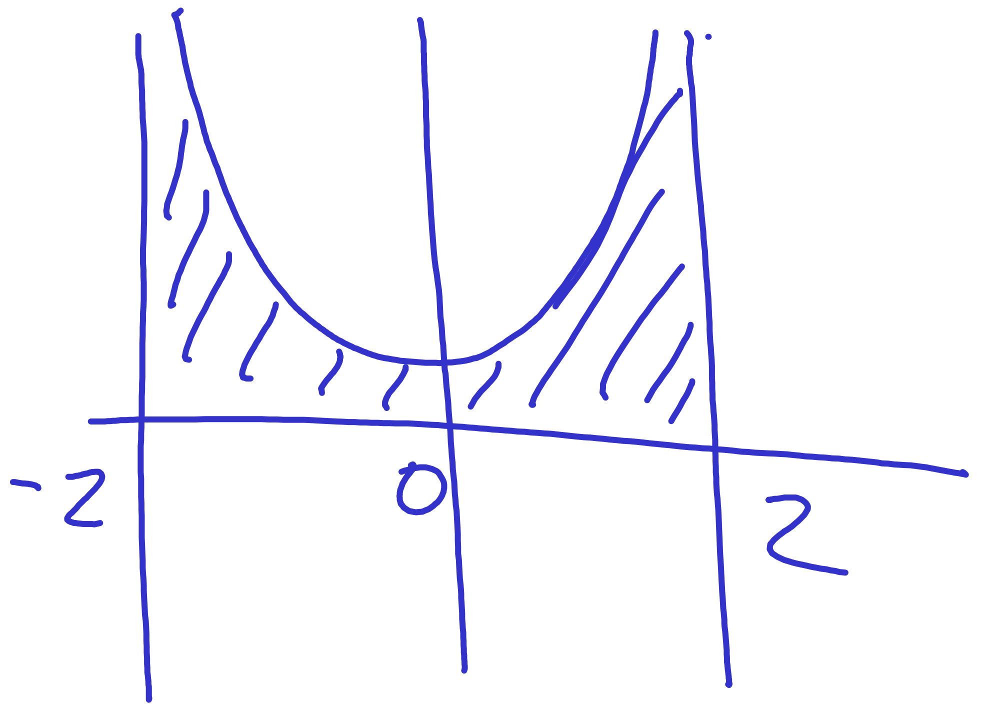
$u(x) = \frac{x^2}{2}$ $\quad v'(x) = \frac{1}{x}$

$$= \underbrace{\frac{1^2}{2}\ln 1}_{0} - \underbrace{\lim_{x\to 0+}\frac{x^2}{2}\ln x}_{=0} - \left(\frac{1}{4} - 0\right) = \underline{\underline{\frac{1}{4}}}$$

$$\lim_{x\to 0+} x^2 \ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x^2}} \overset{l'H}{=} \lim_{x\to 0+} \frac{\frac{1}{x}}{(-2)\frac{1}{x^3}} =$$

$$= \lim_{x\to 0+}\left(-\frac{x^2}{2}\right) = 0 \Rightarrow \lim_{x\to 0+} \frac{x^2}{2}\ln x = 0.$$

<u>Příklad</u> Zjistěte, zda konverguje $\int_{-2}^{2} \frac{1}{4-x^2}\,dx = \int_{-2}^{0} + \int_{0}^{2}$



$$\int_0^2 \frac{1}{4-x^2}\,dx = \int_0^2 \left(\frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x}\right)dx$$

$$= \left[-\ln|2-x|\right]_0^2 + \left[\ln|2+x|\right]_0^2 =$$

$$= \underbrace{\lim_{x \to 2_-}\left(-\ln(2-x)\right)}_{+\infty} - \left(-\ln 2\right) + \ln 4 - \ln 2 =$$

$$= +\infty + \ln 2 + \ln 4 - \ln 2 = \infty + 2\ln 2 = \infty$$

Integrál diverguje.