$$A \qquad \varphi : \mathbb{K}^n \to \mathbb{K}^m \qquad \varphi(x) = A \cdot x$$

$$B \qquad \psi : \mathbb{K}^r \to \mathbb{K}^n \qquad \psi(y) = B \cdot y$$

$$h_o(A) = \underline{\dim \operatorname{Im} \varphi}\ , \quad h_o(B) = \dim \operatorname{Im} \psi$$

$$A \cdot B \longleftrightarrow \varphi \circ \psi \qquad \operatorname{Im}(\varphi \circ \psi) \subseteq \operatorname{Im}(\varphi)$$

$$h(A \cdot B) = \dim \operatorname{Im}(\varphi \circ \psi) \leq \dim \operatorname{Im}(\varphi) = h(A)$$

$$h(A\cdot B) = h\left((A\cdot B)^T\right) = h(B^T \cdot A^T)$$

$$\leq h(B^T) = h(B)$$

$$B = B \cdot I_m =$$

$$A \cdot B = I_n = BA \qquad = (B \cdot A) \cdot C =$$

$$A \cdot C = I_m = CA \qquad = I_m \cdot C = C$$

$$A A^{-1} = I_n = A^{-1} \cdot A$$

$$n = h(I_n) = h(A A^{-1}) \leq \min(h(A), h(A^{-1}))$$

$$\leq h(A) \leq n \implies h(A) = \underline{n}$$

$$h(A) = n \qquad \varphi : \mathbb{K}^n \to \mathbb{K}^n , \; \varphi(x) = Ax$$

$\dim \operatorname{Im} \varphi = r_0(A) = r(A) = n$

$\varphi : \mathbb{K}^n \to \mathbb{K}^n$, φ je izomorfizam.

$$\underset{\substack{\| \\ n}}{\dim \mathbb{K}^n} = \underset{\substack{\| \\ 0}}{\dim \operatorname{Ker} \varphi} + \underset{\substack{\| \\ n}}{\dim \operatorname{Im} \varphi}$$

$\exists\, \varphi^{-1} \Leftrightarrow A^{-1}$

$$\Rightarrow \quad A \cdot B = I_n \overset{?}{\Leftarrow} B \cdot A = I_n$$

$$A \mapsto \varphi \qquad \varphi : \mathbb{K}^n \to \mathbb{K}^n, \ \varphi(x) = Ax$$

$$B \mapsto \psi \qquad \psi : \mathbb{K}^n \to \mathbb{K}^n, \ \psi(y) = By$$

$$\underset{\underset{I_n}{\parallel}}{A \cdot B} \mapsto \varphi \circ \psi = id_{\mathbb{K}^n}$$

$$A^{-1}A = I_n \quad \checkmark$$

$$(B^{-1}\underbrace{A^{-1})(A}_{I_n}\cdot B) = \underbrace{B^{-1}B}_{I_n} = I_n$$

$$(A^{-1})^T A^T = (\underbrace{AA^{-1}}_{I_n})^T = (I_n)^T = I_n$$

$$A = \begin{pmatrix} r_1(A) \\ \vdots \\ r_i(A) \\ \\ r_m(A) \end{pmatrix} \qquad i \leftrightarrow j \qquad i < j \qquad I_m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} r_1(A) \\ r_j(A) \\ r_i(A) \\ r_m(A) \end{pmatrix} \qquad E = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \begin{matrix} i \\ \\ j \end{matrix}$$

$$EA = \begin{pmatrix} r_1(A) \\ \vdots \\ r_j(A) \\ r_i(A) \end{pmatrix}$$

$$\underbrace{(E_k \cdot E_{k-1} \cdots E_1)}_{A^{-1}} \cdot A = I_n$$

$\Leftarrow$ součin reg. matic je regulární.

$\Rightarrow$ A^{-1} $A = (A^{-1})^{-1}$ každá invertibilní matice je součin elementárních.

$$A \sim A_1 \sim A_2 \sim \dots A_k = B$$

$$A_1 = E_1 A \quad A_2 = E_2 A_1 \dots \dots A_k = E_k A_{k-1}$$

$$B = A_k = E_k A_{k-1} = \underbrace{(E_k \dots E_1)}_{P} A.$$

$$h(P \cdot A) \leq h(A)$$

$$h(A) = h\left(P^{-1} \cdot (\underbrace{P \cdot A}_{\sim})\right) \leq h(P \cdot A)$$

$$\varphi : \underset{\beta}{V} \to \underset{\alpha}{V} \qquad \underline{\varphi = id}$$

$$(\varphi)_{\alpha,\beta} = \left((\varphi(v_1))_\alpha, \dots, (\varphi(v_n))_\alpha \right)$$

$$P_{\alpha,\beta} = \left((v_1)_\alpha, \dots, (v_n)_\alpha \right)$$

$$(\varphi(x))_\alpha = (\varphi)_{\alpha,\beta} (x)_\beta$$

$$(x)_\alpha = P_{\alpha,\beta}(x)_\beta$$

$$x = N^T_1$$

$$(N^T_1)_\alpha = P_{\alpha,\beta} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1(P_{\alpha,\beta})$$

$(I) \Leftarrow (III)$

$(II) \Rightarrow (III)$

$(II) \Rightarrow (III)$ $\quad \alpha P(x)_\beta = \beta(x)_\beta = x$, where $P(x)_\beta = (x)_\alpha$

$$w_j = \alpha \cdot (w_j)_\alpha = \alpha \cdot n_j(P_{\alpha,\beta}) =$$

$$= n_j(\alpha \cdot P_{\alpha,\beta})$$

$$\beta = \alpha P_{\alpha,\beta}$$

$$P_{\alpha,\alpha} = \left((u_1)_\alpha , \ldots , (u_n)_\alpha \right) =$$

$$= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ldots , \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = I_n$$

$$(x)_\beta = \underline{\underline{P_{\beta,\alpha}}} (x)_\alpha \qquad \left(P_{\beta,\alpha} \right)^{-1} \cdot (x)_\beta = (x)_\alpha$$

$$\parallel P_{\alpha,\beta}$$

$$P_{\alpha,\beta} \cdot P_{\beta\gamma} = (id)_{\alpha,\beta} \cdot (id)_{\beta\gamma} =$$

$$= (id)_{\alpha,\gamma} = P_{\alpha,\gamma}$$

P neg α Pardim

$$P = P_{\alpha,\beta}$$

$$\boxed{\alpha \cdot P = \beta}$$