Příklad. 1. Nechť $\varphi : \mathbb{R}_3[x] \to \mathbb{R}^4[x]$ je lineární zobrazení zadané předpisem

$$\varphi(p) = (x^2+1)p'(x),$$

kde p' je derivace polynomu p. Najděte matici $(\varphi)_{\beta,\alpha}$ zobrazení φ v bázích $\alpha = (1, x, x^2, x^3)$ prostoru $\mathbb{R}_3[x]$ a $\beta = (x^4, x^3, x^2, x, 1)$ prostoru $\mathbb{R}_4[x]$. (Pozor na pořadí vektorů v bázi.)

$$(\varphi)_{\beta,\alpha} \overset{def}{=} \left((\varphi(1))_\beta \quad (\varphi(x))_\beta \quad (\varphi(x^2))_\beta \quad (\varphi(x^3))_\beta \right)$$

1. Určíme $\varphi(1), \varphi(x), \varphi(x^2), \varphi(x^3)$

$$\boxed{(x^k)' = k \cdot x^{k-1} \quad k \geq 0}$$

$\varphi(1) = (x^2+1)\cdot(1)' = (x^2+1)\cdot 0 = 0$

$\varphi(x) = (x^2+1)\cdot(x)' = (x^2+1)\cdot 1 = x^2+1$

$\varphi(x^2) = (x^2+1)(x^2)' = (x^2+1)\cdot 2x = 2x^3+2x$

$\varphi(x^3) = (x^2+1)\cdot(x^3)' = (x^2+1)\cdot 3\cdot x^2 = 3x^4+3x^2$

2. $(\varphi(1))_\beta = ?$, . . . , $(\varphi(x^3))_\beta = ?$

$\varphi(1) = 0 = 0\cdot x^4 + 0\cdot x^3 + 0\cdot x^2 + 0\cdot x + 0\cdot 1 \Rightarrow (\varphi(1))_\beta = \begin{array}{c} \begin{pmatrix} 0\\0\\0\\0\\0 \end{pmatrix} \end{array} \begin{array}{c} x^4\\x^3\\ \vdots \\ 1 \end{array}$

$\varphi(x) = x^2+1 = 0\cdot x^4 + 0\cdot x^3 + 1\cdot x^2 + 0\cdot x + 1\cdot 1 \Rightarrow (\varphi(x))_\beta = \begin{pmatrix} 0\\0\\1\\0\\1 \end{pmatrix} \begin{array}{c} x^4\\ \vdots \\ 1 \end{array}$

$(\varphi(x^2))_\beta = \begin{pmatrix} 0\\2\\0\\2\\0 \end{pmatrix}$ $\quad (\varphi(x^3))_\beta = \begin{pmatrix} 3\\0\\3\\0\\0 \end{pmatrix}$

3. Dosadíme do definice

$$(\varphi)_{\beta,\alpha} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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Příklad. 2. Nechť φ je zobrazení $\mathbb{R}^3$ do sebe, které je symetrií podle roviny $x_1 + 2x_2 - x_3 = 0$. Najděte matici $(\varphi)_{\alpha,\alpha} = A$ tohoto zobrazení v bázi α složené ze dvou vektorů ležících v dané rovině a třetího vektoru kolmého k rovině. Dále najděte matici $(\varphi)_{\varepsilon,\varepsilon} = B$ ve standardní bázi prostoru $\mathbb{R}^3$.

1. spočítame prvky α

$\rho: x_1 + 2x_2 - x_3 = 0$ $\qquad \vec{n} = (1, 2, -1)$

$x_3 = t \in \mathbb{R}$

$x_2 = s \in \mathbb{R}$

$x_1 = +t - 2 \cdot s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{u} + s \cdot \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{v} \rightarrow \text{vektory roviny } \rho$$

$$\boxed{\alpha = (u, v, n)}$$

2. $(\varphi)_{\alpha\alpha} \overset{def}{=} \left((\varphi(u))_\alpha \quad (\varphi(v))_\alpha \quad (\varphi(n))_\alpha \right)$

Definícia φ:

$\varphi(u) = u$

$\varphi(v) = v$

$\varphi(n) = -n$

$\varphi(u) = u = 1 \cdot u + 0 \cdot v + 0 \cdot n$

$\Rightarrow (\varphi(u))_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\varphi(v) = v = 0 \cdot u + 1 \cdot v + 0 \cdot n \Rightarrow (\varphi(v))_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\varphi(n) = -n = 0 \cdot u + 0 \cdot v + (-1) \cdot n \Rightarrow (\varphi(n))_\alpha = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$$(\varphi)_{\alpha\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3. $(\varphi)_{\varepsilon,\varepsilon} \overset{def}{=} \left((\varphi(e_1))_\varepsilon \quad (\varphi(e_2))_\varepsilon \quad (\varphi(e_3))_\varepsilon \right)$

Potrebujeme zistiť $\varphi(e_1), \varphi(e_2), \varphi(e_3)$.

$$\left(\begin{array}{c|c} u & \varphi(u) \\ v & \varphi(v) \\ n & \varphi(n) \end{array} \right) \overset{\text{ERO}}{\sim} \dots \sim \left(\begin{array}{c|c} e_1 & \varphi(e_1) \\ e_2 & \varphi(e_2) \\ e_3 & \varphi(e_3) \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & 2 & -1 & -1 & -2 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & -2 & -2 & -2 & 0 \end{array} \right) \begin{array}{l} \\ 2R1 + R2 \\ R3 - R1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 1 & 2 & 2 \end{array}\right) \; R2-R3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & -2/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & \frac{2}{3} & \frac{2}{3} \end{array}\right) \begin{array}{l} R1-R3' \\ R2-2\cdot R3' \\ R3' \end{array}$$

$$\varphi(e_1) = \frac{1}{3}\underbrace{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}, \quad \varphi(e_2) = \frac{1}{3}\cdot\underbrace{\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}}, \quad \varphi(e_3) = \frac{1}{3}\cdot\underbrace{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}$$

$$\varphi(e_1) = \frac{1}{3}\cdot\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \underbrace{\frac{2}{3}}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\frac{-2}{3}}\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underbrace{\frac{1}{3}}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(\varphi(e_1))_\varepsilon = \varphi(e_1)$$

$$(\varphi(e_2))_\varepsilon = \varphi(e_2)$$

$$(\varphi(e_3))_\varepsilon = \varphi(e_3)$$

$\boxed{\forall u \in \mathbb{R}^n, \text{ potom } (u)_\varepsilon = u}$

$$(\varphi)_{\varepsilon,\varepsilon} = \frac{1}{3}\cdot\begin{pmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

Příklad. 3. Najděte matici $(\varphi)_{\alpha,\alpha}$ lineárního zobrazení $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ zadaného předpisem

$$\varphi(x) = \underbrace{\begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

v bázi $\alpha = (\underbrace{(1,1,2)^T}_{u_1}, \underbrace{(1,0,1)^T}_{u_2}, \underbrace{(1,2,2)^T}_{u_3})$.

$$\varphi(e_1) = s_1(A) \qquad \varphi(e_2) = s_2(A) \qquad \varphi(e_3) = s_3(A)$$

$$(\varphi)_{\alpha,\alpha} = \left((\varphi(u_1))_\alpha \quad (\varphi(u_2))_\alpha \quad (\varphi(u_3))_\alpha \right)$$

- $\varphi(u_1),\ \varphi(u_2),\ \varphi(u_3)$

$$\varphi(u_1) = \begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}_{\substack{\shortparallel \\ (u_1)_\varepsilon}} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}\, = \underbrace{1} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} + \underbrace{0} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \underbrace{0} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\varphi(u_2) = \begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\varphi(u_3) = \begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

- $(\varphi(u_1))_\alpha = ? \qquad (\varphi(u_2))_\alpha = ? \qquad (\varphi(u_3))_\alpha = ?$

$$(\varphi(u_1))_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (\varphi(u_2))_\alpha = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \qquad (\varphi(u_3))_\alpha = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Konečne

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Příklad. 4. Najděte matici lineárního zobrazení $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ zadaného předpisem

$$\varphi(x) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

v bázích $\alpha = (\underbrace{(1,0,1)^T}_{u_1}, \underbrace{(1,1,2)^T}_{u_2}, \underbrace{(1,-1,2)^T}_{u_3})$ a $\beta = ((1,2)^T, (2,3)^T)$.

$$(\varphi)_{\beta,\alpha} = \left(\varphi(u_1)_\beta \quad (\varphi(u_2))_\beta \quad (\varphi(u_3))_\beta \right)$$

- $\varphi(u_1), \varphi(u_2), \varphi(u_3) = ?$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{pmatrix} \cdot \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \varphi(u_1) & \varphi(u_2) & \varphi(u_3) \end{array} \\ \begin{pmatrix} 4 & 9 & 5 \\ 1 & 1 & -1 \end{pmatrix} \end{array}$$

$\varphi(u_1) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\varphi(u_2) = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$

$\varphi(u_3) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

- $(\varphi(u_1))_\beta, (\varphi(u_2))_\beta, (\varphi(u_3))_\beta$

(A) $\varphi(u_1) = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \underline{a_{11}} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \underline{a_{12}} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow (\varphi(u_1))_\beta = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}$

(B) $\varphi(u_2) = \begin{pmatrix} 9 \\ 1 \end{pmatrix} = a_{21} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_{22} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow (\varphi(u_2))_\beta = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$

(C) $\varphi(u_3) = \begin{pmatrix} 5 \\ -1 \end{pmatrix} = a_{31} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_{32} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow (\varphi(u_3))_\beta = \begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix}$

(A) $\begin{array}{c} \begin{array}{cc} a_{11} & a_{12} \end{array} \\ \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 3 & 1 \end{array}\right) \end{array}$ (B) $\begin{array}{c} \begin{array}{cc} a_{21} & a_{22} \end{array} \\ \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 2 & 3 & 1 \end{array}\right) \end{array}$ (C) $\begin{array}{c} \begin{array}{cc} a_{31} & a_{32} \end{array} \\ \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 3 & -1 \end{array}\right) \end{array}$

$$\underline{\left(\begin{array}{cc|ccc} 1 & 2 & 4 & 9 & 5 \\ 2 & 3 & 1 & 1 & -1 \end{array}\right)} \sim \text{ERO} \sim \left(\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & \end{array}\right)$$

$$\left(\begin{array}{cc|ccc} 1 & 2 & 4 & 9 & 5 \\ 2 & 3 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|ccc} 1 & 2 & 4 & 9 & 5 \\ 0 & -1 & -7 & -17 & -11 \end{array}\right) \; -2 \cdot R1 + R2$$

$$\sim \left(\begin{array}{cc|ccc} 1 & 0 & -10 & -25 & -17 \\ 0 & 1 & 7 & 17 & 11 \end{array}\right) \; 2 \cdot R2 + R1$$

(A) $\begin{array}{c} \begin{array}{cc} a_{11} & a_{12} \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 7 \end{array}\right) \end{array}$ (B) $\begin{array}{c} \begin{array}{cc} a_{21} & a_{22} \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & -25 \\ 0 & 1 & 17 \end{array}\right) \end{array}$ (C) $\begin{array}{c} \begin{array}{cc} a_{31} & a_{32} \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & -17 \\ 0 & 1 & 11 \end{array}\right) \end{array}$

$a_{12} = 7$ $a_{22} = 17$ $a_{31} = -17$

$a_{11} = -10$ $a_{21} = -25$ $a_{32} = 11$

$(\varphi(u_1))_\alpha = \begin{pmatrix} -10 \\ 7 \end{pmatrix}$ $(\varphi(u_2))_\alpha = \begin{pmatrix} -25 \\ 17 \end{pmatrix}$ $(\varphi(u_3))_\alpha = \begin{pmatrix} -17 \\ 11 \end{pmatrix}$

$a_{21} = -25$ $\qquad$ $a_{32} = 11$

$\left(\varphi(u_1)\right)_\beta = \begin{pmatrix} -10 \\ 7 \end{pmatrix}$ $\qquad$ $\left(\varphi(u_2)\right)_\beta = \begin{pmatrix} -25 \\ 17 \end{pmatrix}$ $\qquad$ $\left(\varphi(u_3)\right)_\beta = \begin{pmatrix} -17 \\ 11 \end{pmatrix}$

$$(\varphi)_{\beta,\alpha} = \begin{pmatrix} -10 & -25 & -17 \\ 7 & 17 & 11 \end{pmatrix}.$$

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Príklad. 5. Najděte matici $(\mathrm{id})_{\beta,\alpha}$ identického zobrazení zobrazení $\mathrm{id}(: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$, kde $\alpha = (\underbrace{x^2+x+1}_{p_1}, \underbrace{x+2}_{p_2}, \underbrace{x^2-x}_{p_3})$ a $\beta = (\underbrace{x^2+1}_{q_1}, \underbrace{x^2-x-1}_{q_2}, \underbrace{x+1}_{q_3})$. Najděte rovněž matice $(\mathrm{id})_{\alpha,\alpha}$ a $(\mathrm{id})_{\beta,\beta}$.

$$(\mathrm{id})_{\beta,\alpha} \overset{def}{=} \left((\mathrm{id}(p_1))_\beta \quad (\mathrm{id}(p_2))_\beta \quad (\mathrm{id}(p_3))_\beta \right) = \left((p_1)_\beta \quad (p_2)_\beta \quad (p_3)_\beta \right)$$

definícia: $\mathrm{id}(p) = p \quad \forall p \in \mathbb{R}_2[x]$

$(p_1)_\beta = ?$ $\qquad p_1 = \_\_ \cdot q_1 + \_\_ \cdot q_2 + \_\_ \cdot q_3$

(A) $x^2+x+1 = a_{11} \cdot (x^2+1) + a_{12} \cdot (x^2-x-1) + a_{13} \cdot (x+1)$

$x+2 = a_{21} \cdot (x^2+1) + a_{22} \cdot (x^2-x-1) + a_{23} \cdot (x+1)$

$x^2-x = a_{31} \cdot (x^2+1) + a_{32} (x^2-x-1) + a_{33} (x+1)$

Postupujte obdobne ako v 4.

(A)

$x^0: \quad 1 = a_{11} + (-1) \cdot a_{12} + a_{13}$

$x^1: \quad 1 = a_{11} \cdot 0 + (-1) \cdot a_{12} + a_{13}$

$x^2: \quad 1 = a_{11} + a_{12} + 0 \cdot a_{13}$

B

$2 = a_{21} - a_{22} + a_{23}$

$1 = a_{21} \cdot 0 - a_{22} + a_{23}$

$0 = a_{21} + a_{22}$

(C)

$0 = a_{31} - a_{32} + a_{33}$

$-1 = - a_{32} + a_{33}$

$1 = a_{31} + a_{32}$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 & -1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{array}\right) \ R3 - R1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right) \ 2.R2 + R3 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right) \ \begin{array}{l} R1 + R2' \\ R2 - R3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$

$(p_1)_\beta \quad (p_2)_\beta \quad (p_3)_\beta$

$$(\mathrm{id})_{\beta,\alpha} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

· $(id)_{\alpha,\alpha} = \left((p_1)_\alpha \quad (p_2)_\alpha \quad (p_3)_\alpha \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$p_1 = \underbrace{1} \cdot p_1 + \underbrace{0} \cdot p_2 + \underbrace{0} p_3 \Rightarrow (p_1)_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$p_2 = 0 \cdot p_1 + 1 \cdot p_2 + 0 \cdot p_3 \Rightarrow (p_2)_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$p_3 = 0 \cdot p_1 + 0 \cdot p_2 + 1 \cdot p_3 \Rightarrow (p_3)_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

· $(id)_{\beta,\beta} = \left((q_1)_\beta \quad (q_2)_\beta \quad (q_3)_\beta \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Příklad. 6. Nechť lineární zobrazení $\varphi : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}_2[x]$ má v bázích

$$\alpha = \left(\underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}_{A_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{A_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{A_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}}_{A_4} \right)$$

a

$$\beta = (\underbrace{x^2 - 2x + 3}_{q_1}, \underbrace{x + 2}_{q_2}, \underbrace{2x^2 - 1}_{q_3})$$

matici

$$(\varphi)_{\beta,\alpha} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 0 & 1 \\ 2 & 1 & -1 & 3 \end{pmatrix}.$$

Najděte předpis

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \dots.$$

$\cdot\ \varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ?$

$\mathcal{E}$ = štandardná báza $\mathrm{Mat}_{2\times 2}(\mathbb{R})$

$$\mathcal{E} = \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_4} \right)$$

$2 \cdot 2 = 4$

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi(a \cdot E_1 + b \cdot E_2 + c \cdot E_3 + d \cdot E_4)$$

φ je lineárne zobrazenie

$$= a \cdot \varphi(E_1) + b \cdot \varphi(E_2) + c \cdot \varphi(E_3) + d \cdot \varphi(E_4)$$

Potrebujeme zistiť $\varphi(E_1), \varphi(E_2), \varphi(E_3), \varphi(E_4) = ?$

Toto zistíme pomocou:

$$(\varphi)_{\beta,\mathcal{E}} = \left((\varphi(E_1))_\beta \ \ (\varphi(E_2))_\beta \ \ (\varphi(E_3))_\beta \ \ (\varphi(E_4))_\beta \right).$$

Túto maticu spočítame pomocou $(\varphi)_{\beta,\alpha}$.

$$(\varphi)_{\beta,\mathcal{E}} = \underbrace{(\varphi)_{\beta,\alpha}}_{\text{v zadaní ✓}} \cdot (id)_{\alpha,\mathcal{E}}$$

? potrebujeme spočítať

$(id)_{\alpha,\mathcal{E}}$ = matica prechodu medzi bázami $\mathcal{E}$ a α

$$(id)_{\alpha,\mathcal{E}} = \left[(id)_{\mathcal{E},\alpha} \right]^{-1} \quad \text{(z prednášky)}$$

Použijeme $id_{\mathcal{E},\alpha}$, lebo sa jednoducho spočíta

$$(id_{\mathcal{E},\alpha}) = \left((A_1)_\mathcal{E} \ \ (A_2)_\mathcal{E} \ \ (A_3)_\mathcal{E} \ \ (A_4)_\mathcal{E} \right)$$

$$A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \underline{1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \underline{2} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \underline{0} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \underline{1} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(A_1)_\mathcal{E} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$(A_1)_{\mathcal{E}} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad (A_2)_{\mathcal{E}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (A_3)_{\mathcal{E}} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \quad (A_4)_{\mathcal{E}} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\underline{(id)_{\mathcal{E},\alpha}} = \underline{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$$

Počítame $\quad (id)_{\alpha,\mathcal{E}} = \left[(id)_{\mathcal{E},\alpha} \right]^{-1}$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) -2\cdot R1 + R2$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & -2 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R2 - R3' \\ -2\cdot R4' + R3 \\ R4 - R1 - R2 \end{array}$$

$$\left[(id)_{\mathcal{E},\alpha} \right]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 \\ -2 & 2 & 1 & -2 \\ 1 & -1 & 0 & 1 \end{pmatrix} = \underline{id_{\alpha,\mathcal{E}}}$$

- $$\underline{(\varphi)_{\beta,\mathcal{E}}} = \underline{(\varphi)_{\beta,\alpha}} \cdot \underline{(id)_{\alpha,\mathcal{E}}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 0 & 1 \\ 2 & 1 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 \\ -2 & 2 & 1 & -2 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & -3 & -2 & 5 \\ 7 & -6 & -2 & 7 \end{pmatrix} = B$$

z definície $(\varphi)_{\beta,\mathcal{E}}$:

$$s_1(B) = (\varphi(E_1))_\beta$$
$$s_2(B) = (\varphi(E_2))_\beta$$
$$s_3(B) = (\varphi(E_3))_\beta$$
$$s_4(B) = (\varphi(E_4))_\beta$$

$\Rightarrow$

$$\varphi(E_1) = -1\cdot q_1 + 0\cdot q_2 + 7\cdot q_3$$
$$= -(x^2 - 2x + 3) + 7\cdot(2x^2 - 1)$$
$$= \underline{13x^2 - 10 + 2x}$$

$$\varphi(E_2) = 0\cdot q_1 - 3\cdot q_2 - 6\cdot q_3$$
$$= -3\cdot(x+2) - 6\cdot(2x^2 - 1)$$

Cieľ bol $\underbrace{\varphi(E_1), \ldots, \varphi(E_4)}$ = ?

$$= -3\cdot(x+2) - 6\cdot(2x^2-1)$$
$$= \underline{-12x^2-3x}$$
$$\varphi(E_3) = 1\cdot q_1 - 2q_2 - 2\cdot q_3$$
$$= (x^2-2x+3) - 2\cdot(x+2) - 2\cdot(2x^2-1)$$
$$= \underline{-3x^2-4x+1}$$
$$\varphi(E_4) = 2\cdot q_1 + 5\cdot q_2 + 7\cdot q_3$$
$$= 2\cdot(x^2-2x+3) + 5\cdot(x+2) + 7\cdot(2x^2-1)$$
$$= \underline{16x^2+x+9}$$

$$\underbrace{\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \varphi(a\cdot E_1 + b\cdot E_2 + c\cdot E_3 + d\cdot E_4)$$
$$= a\cdot\varphi(E_1) + b\cdot\varphi(E_2) + c\,\varphi(E_3) + d\cdot\varphi(E_4)$$
$$= a\cdot(11x^2+2x-10) + b\cdot(-12x^2-3x) + c\cdot(-3x^2-4x+1) + d\cdot(16x^2+x+9).$$