Cv

$$S^p U \subseteq \otimes^p U$$

Dokažte $\rho_\sigma \circ \rho_\tau = \rho_{\tau\circ\sigma}$

$\rho_{(12)}\rho_{(23)} \cdots \rho_{(n-1,n)} = \rho_{(1,2,\dots,n)}$

- $\underline{\rho_{(23)}\rho_{(12)} \overset{?}{=} \rho_{(12)\circ(23)} = \rho_{(123)}}$

$\neq$

$\rho_{(12)}\rho_{(23)}$

na 1. pozici to, co bylo na $\sigma(1)$. pozici

Pozn. $\rho_\sigma(u_1\otimes\dots\otimes u_p) = u_{\sigma(1)}\otimes\dots\otimes u_{\sigma(p)}$

$= \rho_{(23)}\underbrace{\rho_{(12)}(u_1\otimes u_2\otimes u_3)}$

$= \underbrace{\rho_{(23)}(u_2\otimes u_1\otimes u_3)}$

$= u_2\otimes u_3\otimes u_1$

$= \rho_\sigma(u_1\otimes u_2\otimes u_3)$ pro $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3)$

$\rho_{(23)}\rho_{(12)}\, u\otimes v\otimes w$

$\rho_{(23)}\, v\otimes u\otimes w$

$v\otimes w\otimes u$

$\underline{\rho_\sigma\circ\rho_\tau}(u_1\otimes\dots\otimes u_p) \overset{\checkmark}{=} \rho_\sigma(u_{\tau(1)}\otimes\dots\otimes u_{\tau(p)})$

$\overset{???}{=} u_{\tau(\sigma(1))}\otimes\dots\otimes u_{\tau(\sigma(p))}$

$\overset{\checkmark}{=} \underline{\rho_{\tau\circ\sigma}}(u_1\otimes\dots\otimes u_p)$

Uvažme zobrazení $\qquad\qquad$ char $\mathbb{K} = 0$

$$\mathrm{Sym}: \otimes^p U \longrightarrow \otimes^p U$$

$$\mathrm{Sym} = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma \qquad \rho_\sigma(u_1 \otimes \dots \otimes u_p) = u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(p)}$$

$$\rho_\sigma \circ \rho_\tau = \rho_{\tau\circ\sigma}, \quad \rho_{id} = id$$

Př. Určete symetrizace tenzorů

$$u_1 + u_2 + 3u_3 \in \otimes^1 U, \quad \underline{u_1 \otimes u_2 + u_1 \otimes u_3 \in \otimes^2 U}, \quad \underline{u_1 \otimes u_2 \otimes u_2} \in \otimes^3 U$$

$$\mathrm{Sym}(u_1 \otimes u_2 + u_1 \otimes u_3) = \mathrm{Sym}(u_1 \otimes u_2) + \mathrm{Sym}(u_1 \otimes u_3)$$

$$= \frac{1}{2}\Big(\rho_{id}(\quad) + \rho_{(12)}(\quad)\Big)$$

$$\Sigma_3 - A_3 = \{(12), (23), (31)\}$$
$$A_3 = \{id, (123), (123)^2\}$$

$$= \frac{1}{2}(u_1 \otimes u_2 + u_1 \otimes u_3 + u_2 \otimes u_1 + u_3 \otimes u_1)$$

$$\mathrm{Sym}(u_1 \otimes u_2 \otimes u_2) = \frac{1}{6}\big(u_1 \otimes u_2 \otimes u_2 + u_2 \otimes u_2 \otimes u_1 + u_2 \otimes u_1 \otimes u_2 + u_2 \otimes u_1 \otimes u_2 + u_1 \otimes u_2 \otimes u_2 + u_2 \otimes u_2 \otimes u_1\big)$$

$$u_1 \otimes u_2 \otimes u_3 \qquad u_1 \otimes u_2 \otimes u_3 \quad u_2 \otimes u_3 \otimes u_1 \quad u_3 \otimes u_1 \otimes u_2$$

$$= \frac{1}{3}(u_1 \otimes u_2 \otimes u_2 + u_2 \otimes u_2 \otimes u_1 + u_2 \otimes u_1 \otimes u_2)$$

$$\mathrm{Sym}(u_1 + u_2 + 3u_3) = \frac{1}{1!}(\rho_{id}(u_1 + u_2 + 3u_3)) = u_1 + u_2 + 3u_3$$

$$\Sigma_1 = \{id\}$$

L. $\rho_\tau \circ \mathrm{Sym} = \mathrm{Sym}$ $\qquad \mathrm{Sym} = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma$

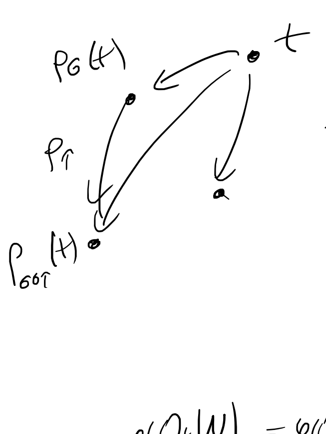
D. $\rho_\tau \left(\frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma(t)\right) =$ $\qquad \sigma \longmapsto \sigma \circ \tau$

$\qquad \sigma \circ \tau^{-1} \longleftarrow\!\!| \ \sigma$

$$= \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\tau \circ \rho_\sigma(t) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_{\sigma\circ\tau}(t)$$

$$= \underbrace{\frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma(t)}_{\mathrm{Sym}}$$

$$= \mathrm{Sym}(t).$$



$$A \underset{(\,)^T}{\overset{(\,)^T}{\rightleftarrows}} A^T$$

L. $\mathrm{Sym} \circ \rho_\tau = \mathrm{Sym}$

D. stejně

$$u_1 \otimes u_2 \longmapsto \frac{1}{2}(u_1 \otimes u_2 + u_2 \otimes u_1)$$

$$\varphi(0+W) = \varphi(0) + \overbrace{\varphi(W)}^{0} = 0$$

D. stejný

$$u_1\otimes u_2 \longmapsto \tfrac{1}{2}(u_1\otimes u_2 + u_2\otimes u_1)$$

$$\begin{tikzcd} \otimes^p U \arrow[r, "Sym"] \arrow[d] & \otimes^p U \\ S^p U \arrow[ur, dashed, "sym"'] & \end{tikzcd}$$

kvocient podle

podpr. $W = [\rho_\tau(t) - t]$

$$\varphi(0+W) = \varphi(0) + \varphi(W) = 0$$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "pr"'] & X \\ V/W \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \quad \exists! \Leftrightarrow \underline{\underline{\varphi(W)=0}}$$

$W = 0 + W \qquad \varphi(v+W) = \varphi(v)$

$Sym\, W \overset{?}{=} 0$ tj. $Sym(\rho_\tau(t) - t) \overset{?}{=} 0$

$Sym(\rho_\tau(t)) - Sym(t) \overset{?}{=} 0$ ✓ podle Lemmatu

$$sym(\overset{u_1\cdot u_2}{u_1 \vee u_2}) = Sym(u_1\otimes u_2) = \tfrac{1}{2}(u_1\otimes u_2 + u_2\otimes u_1)$$

$$pr(u_1\otimes u_2) = pr(u_2\otimes u_1) = pr\Big(\underbrace{\tfrac{1}{2}(u_1\otimes u_2 + u_2\otimes u_1)}\Big)$$

spec. repr. --- je „symetrický"

$\rho_\tau(t) = t$

$$\begin{array}{c} \otimes^p U \\ \downarrow \end{array}$$

Věta. $sym: S^p U \overset{\cong}{\underset{pr}{\rightleftarrows}}$ {symetrické tenzory, tj. $t \in \otimes^p U$ splňující $\rho_\tau(t) = t \quad \forall \tau$}

= kanonická volba reprezentanta

Cv. antisymetrizace $u_1 + u_2 + 3u_3 \in \otimes^1 U$, $u_1 \otimes u_2 - u_2 \otimes u_3 + u_3 \otimes u_3 \in \otimes^2 U$

$$Alt : \otimes^p U \to \otimes^p U$$

$$Alt = \frac{1}{p!} \sum_{\sigma \in \Sigma} \operatorname{sign} \sigma \cdot \rho_\sigma$$

$p = 2$: $Alt = \frac{1}{2}(id - \rho_{(12)})$

$$Alt(u_1 \otimes u_2 - u_2 \otimes u_3 + u_3 \otimes u_3) = \frac{1}{2}\big(u_1 \otimes u_2 - u_2 \otimes u_3 + \underline{u_3 \otimes u_3} - u_2 \otimes u_1 + u_3 \otimes u_2 - \underline{u_3 \otimes u_3}\big)$$

$$= \underbrace{\frac{1}{2}(u_1 \otimes u_2 - u_2 \otimes u_1)}_{Alt(u_1 \otimes u_2)} - \underbrace{\frac{1}{2}(u_2 \otimes u_3 - u_3 \otimes u_2)}_{Alt(u_2 \otimes u_3)}$$

$u_3 \wedge u_3 = 0 \longleftrightarrow 0 = Alt(u_3 \otimes u_3)$

$$alt : \Lambda^p U \underset{\longleftarrow}{\xrightarrow{\cong}} \{\text{antisym. tenzory } t : \rho_\tau(t) = \operatorname{sign} \tau \cdot t\}$$

$$0 = u \wedge u \longleftrightarrow Alt(u \otimes u) = 0$$

$\mathrm{Hom}(S^pU, \mathbb{k}) \cong \mathrm{Lin}_p(U, \dots, U; \mathbb{k})$ sym

$\mathrm{Hom}(\Lambda^pU, \mathbb{k}) \cong \mathrm{Lin}_p(U, \dots, U; \mathbb{k})$ alt

co něco mezi ?

Př. $\omega \in \mathrm{Lin}_3(U, U, U; V)$ splňuje

$$\omega(u, v, w) = \omega(v, u, w) \quad (1)$$
$$\omega(u, v, w) = -\omega(u, w, v) \quad (2)$$

dokažte

$\Rightarrow \omega = 0$

Dk. $\underline{\omega(u,v,w)} \overset{(1)}{=} \omega(v,u,w) \overset{(2)}{=} -\omega(v,w,u) \overset{(1)}{=} -\omega(w,v,u) \overset{(2)}{=} +\omega(w,u,v)$

$\overset{(1)}{=} +\omega(u,w,v) \overset{(2)}{=} -\underline{\omega(u,v,w)}$

$\Rightarrow \omega(u,v,w) = 0 \Rightarrow \omega = 0$

Def. $\mathrm{Vol}(u,v,w) = \langle u\times v, \underset{u}{\underset{\uparrow}{w}} \rangle$ $\leftarrow$ určuje sk. součin $u\times v$ s čímkoliv $\Rightarrow$ určuje $u\times v$

Věta. Vektorový součin má následující vlastnosti:

- $u\times v$ je kolmý na u,v
- $u\times v \neq 0$, právě když u,v jsou lineárně nezávislé a pak
- $(u,v,u\times v)$ je kladná báze
- $|u\times v| = |u|\cdot|v|\cdot \sin\measuredangle(u,v)$ $\leftarrow$ neorientovaný úhel $\in [0,\pi]$

Cv. Dokažte!

- $\langle u\times v, u\rangle = \mathrm{Vol}(u,v,u) = 0$ ✓ (antisym)
- u,v lin. nez. $\Rightarrow \exists\, w$ t.ž. (u,v,w) je báze

  $\mathrm{Vol}(u,v,w) = \langle u\times v, w\rangle \neq 0 \Rightarrow u\times v \neq 0$

  u,v lin. záv. $\Rightarrow \underset{0}{\underset{\|}{\mathrm{Vol}(u,v,w)}} = \langle u\times v, w\rangle \quad \forall w$

  $\Rightarrow u\times v = 0$

- $(u,v,u\times v)$ kladná $\Leftrightarrow \mathrm{Vol}(u,v,u\times v) > 0$ ✓

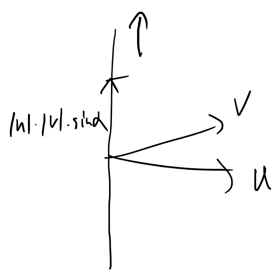
  $\mathrm{Vol}(u,v,u\times v) = \langle u\times v, u\times v\rangle = |u\times v|^2$

- $|u\times v|^4 = \mathrm{Vol}(u,v,u\times v)^2 = \det\begin{pmatrix} |u|^2 & \langle u,v\rangle & 0 \\ \langle u,v\rangle & |v|^2 & 0 \\ 0 & 0 & |u\times v|^2 \end{pmatrix}$

  $= \big(|u|^2|v|^2 - \underbrace{\langle u,v\rangle^2}_{(|u|\cdot|v|\cdot\cos\alpha)^2}\big)\cdot|u\times v|^2$

  $= |u|^2|v|^2\sin^2\alpha\cdot|u\times v|^2$

$|u\times v|^2 = |u|^2|v|^2\sin^2\alpha \qquad /\ ()^{1/2}$



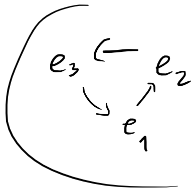
$\overbrace{e_1\ e_2}^{e_3}$ $\quad e_1$ nebo e_2

- $(u\times v)\times w = \underbrace{\langle u,w\rangle v - \langle v,w\rangle u}$

$(e_1\times e_2)\times e_1 = e_3\times e_1 = e_2 \qquad \langle e_1,e_1\rangle e_2 - \langle e_2,e_1\rangle e_1 = e_2$

$(e_1 \times e_2) \times e_1 = e_3 \times e_1 = e_2$ $\qquad$ $\underbrace{\langle e_1, e_1 \rangle}_{1} e_2 - \underbrace{\langle e_2, e_1 \rangle}_{0} e_1 = e_2$

$(e_1 \times e_2) \times e_2 = e_3 \times e_2 = -e_1$ $\qquad$ $\langle e_1, e_2 \rangle e_1 - \langle e_2, e_2 \rangle e_1 = -e_1$



$(u \times v) \times w \neq u \times (v \times w)$

$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$ $\qquad$ Jacobiho identita

↳ plyne snadno z $(u \times v) \times w = \langle u, w \rangle v - \langle v, w \rangle u$

→ $\mathbb{R}^3$ s vekt. souč. je Lieova algebra.

U v.p. nad $\mathbb{C}$ s bazí $(e_1, \dots, e_n)$

$\Rightarrow U$ $\qquad \mathbb{R}$ $\qquad (e_1, \dots, e_n, ie_1, \dots, ie_n)$

chceme: $\forall u \in U : \exists! a^1, \dots, a^n, b^1, \dots, b^n \in \mathbb{R}:$ $u = a^1 e_1 + \dots + a^n e_n + b_1 ie_1 + \dots + b_n ie_n$

$u = (a^1 + b^1 i) e_1 + \dots + (a^n + b^n i) e_n$

$$(\bar{e}_1 \dots \bar{e}_n) = (e_1 \dots e_n) \begin{pmatrix} 1 & & z \\ & \ddots & \\ & & 1 \end{pmatrix} \text{ nebo } \underbrace{\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \ddots \; 1 \end{pmatrix}}_{P} \text{ nebo } \begin{pmatrix} 1 & & \\ & z & \\ & & \ddots \; 1 \end{pmatrix}$$

$\bar{e}_j = e_k$ $\qquad e_j \leftrightarrow e_k$

$\bar{e}_k = e_j$

$\Rightarrow (\bar{e}_1, \dots, \bar{e}_n, i\bar{e}_1, \dots, i\bar{e}_n)$, $(e_1, \dots, e_n, ie_1, \dots, ie_n)$ jsou souhlasně or.

$$\Rightarrow (\bar{e}_1, \dots, \bar{e}_n, i\bar{e}_1, \dots, i\bar{e}_n) = (e_1, \dots, e_n, ie_1, \dots, ie_n) \begin{array}{c|cc} & n & n \\ \hline n & P & 0 \\ n & 0 & P \end{array}$$

$$\det \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = (\underbrace{\det P}_{-1})^2 = 1$$

$$\begin{array}{c} \\ j \rightarrow \\ k \rightarrow \end{array} \begin{pmatrix} 1 & & & & \\ & 1 & 0 & & a^j_k = 1 \\ & & 1 & 1 & \\ & a^k_j = 1 & & 0 & \\ & & & & 1 \end{pmatrix}$$
$\qquad\qquad \uparrow j \qquad \uparrow k$

$\sum \operatorname{sign} \sigma \cdot a_1^{\sigma(1)} \cdots a_n^{\sigma(n)}$

$= \operatorname{sign}(jk)\, a_1^1 \cdots$

$= (-1)\, 1 \cdots 1$

$= -1$

$$\boxed{\det(A \cdot P) = \det A \cdot (-1)}$$

$\begin{pmatrix} 1 & & \\ & z & \\ & & \ddots \; 1 \end{pmatrix}$ pouze pro $n = 1$ $\qquad (z)$

$\bar{e}_1 = z \cdot e_1 = (a + bi) e_1$
$= a e_1 + b\, ie_1$

$i\bar{e}_1 = \dots$

$$(\bar{e}_1, i\bar{e}_1) = \underline{(e_1, ie_1)} \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{\det = a^2 + b^2 > 0}$$