Topologie

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Introduction

This text was created by writing my preparations for lectures and exercises for the subject "Topology". As a starting text, I used my notes, which I took when the subject was taught by Prof. Rosický. It was based on Pultr's book "Subspaces of Euclidean Spaces". I expanded or supplemented some parts, I drew mainly from Bredon's book "Geometry and topology". This also applies to the chapters I added.

In the text, the examples we did in the exercises are marked "cv"; I did not write sample solutions for them. I left the examples marked with "hw" as homework.

The parts of the text marked with "$*$" are more technically demanding passages that I sometimes didn't even cover in the lecture, but which I can ask about in the exam. I consider the parts marked with "$**$" to be unnecessarily difficult or special and will not ask about them. We didn't do the parts marked "nd", but I plan to cover them in the future.

1. Motivation

Topology deals with "topological spaces" – these are roughly metric spaces, we just forget the specific distances between points and only remember which points "are close". The central concept is continuity, more specifically continuous display. As a result, this means that a square is "the same" as a circle (unlike geometry). This is because there are continuous mutually inverse mappings between the square and the circle – together they specify an isomorphism.

There are other kinds of spaces – based on other kinds of views. This is, for example,

metric spaces	– isometry
differentiable varieties	– differentiable representations
algebraic varieties	– polynomial representation
PL (piecewise linear) variety	– piecewise linear representation
polyhedra	– affine mapping

Algebraic topology goes much further in non-geometry (square = circle), which declares $\mathbb{R}^n$ and the space consisting of a single point to be equal spaces, since $\mathbb{R}^n$ can be "continuously deformed" to a point.

2. Topological space

In the metric space M we define an open sphere around x with radius $\varepsilon > 0$ as

$$B_\varepsilon(x) = \{y \in M \mid \operatorname{dist}(x, y) < \varepsilon\}.$$

We say that a subset $U \subseteq M$ is *open* if for every $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. In short, we say that U contains some of its neighborhoods with every point.

Definition 2.1. A *Topology* on a set X is a system of subsets $\mathcal{X} \subseteq \mathcal{P}(X)$ satisfying the following conditions

(1) $\emptyset, X \in \mathcal{X}$,
(2) $U_i \in \mathcal{X}$, $i \in \mathcal{I} \Rightarrow \bigcup_{i \in \mathcal{I}} U_i \in \mathcal{X}$,
(3) $U_i \in \mathcal{X}$, $i \in \mathcal{I}$, $\mathcal{I}$ final $\Rightarrow \bigcap_{i \in \mathcal{I}} U_i \in \mathcal{X}$.

A *Topological space* is the set X together with the topology $\mathcal{X}$ on X . We call the elements of $\mathcal{X}$ *wiped* subsets of X.

Remark. Condition (0) follows from the other two, since $\emptyset$ is the union of the empty system of subsets and X is the intersection of the empty system.

Examples 2.2.

1. metric spaces (we can do this in more detail over time),
2. for any set X we define a *discrete topology* on X as $\mathcal{X} = \mathcal{P}(X)$ (everything is open, it is specified by the metric $\operatorname{dist}(x, y) = 1$),
3. for any set X we define a *trivial topology* on X as $\mathcal{X} = \{\emptyset, X\}$ ("nothing" is open, it is given by the pseudometric $\operatorname{dist}(x, y)$ =0),
4. for any set X we define the *topology of finite complements* on X as

$$\mathcal{X} = \{U \subseteq X \mid X \smallsetminus U \text{ final}\} \cup \{\emptyset\},$$

hw 1 5. if X is any (pre)ordered set, it is

$$\mathcal{X} = \{U \subseteq X \mid U \text{ meets } \forall x \in U\ \forall y \leq x : y \in U\}$$

topology. Conversely, if $\mathcal{X}$ is an arbitrary topology satisfying (2) even for infinite index sets $\mathcal{I}$, then there is a preorder on X specifying this topology.

We will now try to prove that the open sets specified by the metric really define the topology. The following concept will be useful for this.

Definition 2.3. A system of sets $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *subbase* of a topology $\mathcal{X}$ if $\mathcal{X}$ is the smallest topology containing $\mathcal{S}$.

A system of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *base* of the topology $\mathcal{X}$ if $\mathcal{X}$ are precisely all unions of the elements of $\mathcal{B}$. In other words,

$$\mathcal{X} = \{A \subseteq X \mid \forall x \in A\ \exists U \in \mathcal{B} \colon x \in U \subseteq A\}$$

(because then $A = \bigcup\{U \in \mathcal{B} \mid U \subseteq A\}$).

Lemma 2.4. *It holds that $\mathcal{B}$ is a basis of some topology (by definition, however, the only one), if and only if the following conditions hold*

1. $X = \bigcup \mathcal{B}$ *a*
2. *for each $U, V \in \mathcal{B}$ and $x \in U \cap V$ there exists a $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.*

cv Prove the previous lemma.

cv **Example 2.5.** Let M be a metric space. Then the system of spheres $\mathcal{B} = \{B_\varepsilon(x) \mid x \in M,\ \varepsilon > 0\}$ is the basis of the topology. (First prove that they are based on some topology, then identify it as a canonical topology on a metric space.)

Now if $\mathcal{S} \subseteq \mathcal{P}(X)$ is an arbitrary system of subsets, $\mathcal{B} = \{\text{finite intersections of elements of } \mathcal{S}\}$ satisfies the conditions of the lemma and therefore the topology generated by $\mathcal{S}$ right now

$$\mathcal{X} = \{\text{unification of finite intersections of elements } \mathcal{S}\}.$$

Definition 2.6. A subset $F \subseteq X$ is called *closed* if $X \smallsetminus F$ is open.

For example, in the space of finite complements precisely the finite sets and X are closed. For $X = \mathbb{C}$, equivalently closed sets can be described as null sets of polynomials – this example has a generalization to $\mathbb{C}^n$, see algebraic geometry.

Remark. For closed sets, "dual" axioms apply to topology axioms. Equivalently, the topology can be specified by a system of closed sets that satisfy these axioms.

Definition 2.7. *Closure* $\overline{A}$ of a subset $A \subseteq X$ is the smallest closed subset containing A, i.e.

$$\overline{A} = \bigcap_{A \subseteq F \text{ uz.}} F.$$

cv **Example 2.8.** Prove the following properties of the closure

1. $\overline{\emptyset} = \emptyset$,
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

3. $A \subseteq \overline{A}$,
4. $\overline{\overline{A}} = \overline{A}$.

Furthermore, show that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ does not hold in general.

Using a closure (or better said a closure operator), the topology can be reconstructed as follows: a subset $A \subseteq X$ is closed if and only if $A = \overline{A}$.

Remark. It is true that the topology can be equivalently specified by a closure operator (ie operator $\mathcal{P}(X) \to \mathcal{P}(X)$ satisfying axioms (1)–(4)).

Lemma 2.9. *It holds* $\overline{A} = \{x \in X \mid \forall U \text{ open}, x \in U : A \cap U \neq \emptyset\}$.

We refer to the points on the right as the *limit points* of A (and we do not need to say what the limit of the sequence is).

Proof. $x \notin \overline{A}$ holds if and only if there exists a closed $F \supseteq A$ not containing x. Going to complements, that is if and only if there exists an open $U = X \smallsetminus F$, $A \cap U = \emptyset$ and containing x. But that is exactly $x \notin RHS$. □

Definition 2.10. "Dually" we define the *interior* of A as the largest open set contained in A, i.e.

$$\mathring{A} = \bigcup_{A \supseteq U \text{ rot.}} U.$$

We say that the interior points of A are those that fit into A with some of their surroundings. We will define the surroundings more precisely later.

3. Continuous maps

Definition 3.1. A mapping $f\colon X \to Y$ between two topological spaces is called *continuous* if for every open $U \subseteq Y$ there is also $f^{-1}(U) \subseteq X$ dad

cv **Exercise 3.2.**

1. It is enough to verify the continuity for U from some (arbitrary) subbase of the topology on Y.
2. A representation of f is continuous if and only if the pattern of every closed set is closed.

——— end of 1. lecture ———

Definition 3.3. A subset $N \subseteq X$ is called a *neighborhood* of a point $x \in X$ if there exists an open set U with property $x \in U \subseteq N$.

In particular, the open neighborhood of x is the same as the open set containing x. Open sets can be characterized by neighborhood as those that are neighborhoods of all their points.

Definition 3.4. We say that the mapping $f\colon X \to Y$ is continuous at a point $x \in X$ if for every neighborhood N of the point $f(x)$ there is also $f^{-1}(N)$ around the point x.

hw 2 **Exercise 3.5.** Prove that a mapping $f\colon X \to Y$ is continuous if and only if it is continuous at every point $x \in X$.

Definition 3.6. We say that the system $\mathcal{N}$ of the neighborhood of x is a *neighborhood basis* of x if every neighborhood of x contains as a subset some element of $\mathcal{N}$. (I worked for regulars later.)

Example 3.7. In metric space, the open sphere $B_\varepsilon(x)$, $\varepsilon > 0$, centered at x forms the neighborhood of x. Alternatively, $B_{1/n}(x)$, $n \in \mathbb{N}$ forms the basis of the neighborhood of the sphere. This set is countable. Therefore, every metrizable space, i.e. a space whose topology is specified by some metric, must have a countable basis around each point – it is so-called "first countable".

nd **Exercise 3.8.**

1. Continuity at the point $x \in X$ is sufficient to verify on the neighborhoods of $f(x)$ from some (arbitrary) basis of the neighborhood.
2. A map $f: M \to N$ between metric spaces is continuous if and only if it satisfies the ε-δ-definition of continuity.

Proof. Part 1. is elementary. Part 2. follows from the fact that the open sphere $B_\delta(x)$, $\delta > 0$, forms the basis of the neighborhood of x and $B_\varepsilon(f(x))$, $\varepsilon > 0$, forms the basis of the neighborhood of $f(x)$. □

* **Exercise 3.9.** Prove that the mapping $f : X \to Y$ between preordered sets X, Y is continuous if and only if it is isotonic.

** **Lemma 3.10.** *The following conditions on the mapping $f: X \to Y$ are equivalent*

1. *f is continuous,*
2. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any subset of $B \subseteq Y$,*
3. *$f(\overline{A}) \subseteq \overline{f(A)}$ for any subset of $A \subseteq X$.*

The last condition is a generalized statement that $x_n \to x$ implies $f(x_n) \to f(x)$ (however, sequences may not be sufficient for general topological spaces).

Proof. We will first show the equivalence of 1. and 2. Since $f^{-1}(\overline{B})$ is a closed subset containing $f^{-1}(B)$, it must also contain $\overline{f^{-1}(B)}$. In the opposite direction, for closed $F \subseteq Y$ $\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$ and thus $f^{-1}(F)$ is closed.

Now we show 2. $\Rightarrow$ 3. We want $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, while obviously $A \subseteq f^{-1}(f(A))$ and thus

$$\overline{A} \subseteq \overline{f^{-1}(f(A))} \overset{2.}{\subseteq} f^{-1}(\overline{f(A)}).$$

It remains to show 3. $\Rightarrow$ 2. We want $f(\overline{f^{-1}(B)}) \subseteq \overline{B}$, while obviously $f(f^{-1}(B)) \subseteq B$ and thus

$$f(\overline{f^{-1}(B)}) \overset{3.}{\subseteq} \overline{f(f^{-1}(B))} \subseteq \overline{B}.$$ □

Definition 3.11. A mapping $f: X \to Y$ is called a *homeomorphism* if f is a bijection and both mappings f, f^{-1} are continuous.

Example 3.12.

1. The interval $(0, 1)$ is homeomorphic to $\mathbb{R}$; for example, the homeomorphism $(0, 1) \to \mathbb{R}$ is the mapping $t \mapsto \mathrm{tg}(\pi t - \pi/2)$.

2. The mapping $\mathrm{id}\colon X_{\mathrm{disc}} \to X_{\mathrm{triv}}$ is a continuous bijection, but its inverse $\mathrm{id}\colon X_{\mathrm{triv}} \to X_{\mathrm{disc}}$ is not continuous ; see next example.
3. nd Consider when the mapping $\mathrm{id}\colon (X, \mathcal{X}_0) \to (X, \mathcal{X}_1)$ is continuous.
4. The mapping $[0,1) \to S^1$, $t \mapsto e^{2\pi i t}$ is a continuous bijection, but its inverse is not continuous.
5. In fact, there is no homeomorphism $[0,1) \xrightarrow{\cong} S^1$. This is best demonstrated by finding some "invariant" that distinguishes the two spaces. In this case, for example, we can say (in time we will be able to formulate it precisely) that removing any point from S^1 does not disintegrate the space, while removing the point $t \neq 0$ from $[0,1)$ disintegrates this interval . Another such invariant is compactness.
6. For $m \neq n$ there is no homeomorphism $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$. For $n = 1$ this can be seen, similar to the previous example, using point removal and context. For higher n , "higher connectedness" is needed.

cv **Example 3.13.** Describe continuous mappings from a trivial space and continuous mappings into a discrete space.

Remark. The topology is non-algebraic (a continuous bijection is not necessarily a homeomorphism). From one of the examples, we see that the completeness of a metric space is not a topological concept, i.e. there are homeomorphic spaces, one of which fulfills this property and the other does not. On the other hand, compactness is a topological concept, later we characterize it purely in the language of open sets.

4. Subspaces, products

Definition 4.1. Let X be a topological space and let $A \subseteq X$ be its subset. We define on A the *subspace topology* as

$$\{A \cap U \mid U \subseteq X \text{ open}\}.$$

We call the set A together with the topology of the subspace *subspace* X.

An important property of a subspace is that the embedding $i\colon A \to X$ is continuous and has the following universal property: the mapping $f : T \to A$ is continuous if and only if $if\colon T \to X$ is continuous .

$$\begin{tikzcd} T \arrow[r, dashed, "f"] \arrow[dr, "if"'] & A \arrow[d, hook, "i"] \\ & X \end{tikzcd}$$

(both follow from the fact that $A \cap U = i^{-1}(U)$). The proof can be summarized in the observation: the topology of a subspace is the smallest one for which the inclusion i is continuous.

hw 3 **Lemma 4.2.** *For the subset $B \subseteq A$ holds*

$$\mathrm{cl}_A B = A \cap \overline{B},$$

where $\mathrm{cl}_A B$ denotes the closure of B in the subspace A.

Remark. Nothing similar applies to the interior.

Definition 4.3. A system $\mathcal{A} \subseteq \mathcal{P}(X)$ of sets is called a *cover* of the space X if $\bigcup \mathcal{A} = X$.

cv **Exercise 4.4.** Let $\mathcal{U}$ be an open covering of the space X. Prove that the map $f\colon X \to Y$ is continuous if and only if every taper of $f|_U\colon U \to Y$, $U \in \mathcal{U}$, is continuous.

Similarly, prove the same for a *finite* closed coverage of $\mathcal{F}$.

hw 4 **Exercise 4.5.** Prove that the square is homeomorphic to the circle.

Definition 4.6. Let X, Y be topological spaces. We define on $X \times Y$ the *product topology* generated by the basis

$$\{U \times V \mid U \in \mathcal{X},\, V \in \mathcal{Y}\}.$$

We call the set $X \times Y$ together with the product topology the *product* of the topological spaces X, Y.

An important property of the product is that the projections $p\colon X \times Y \to X$, $q\colon X \times Y \to Y$ are continuous and have the following universal property: the mapping $f = (g, h)$ $colT \to X \times Y$ is continuous if and only if its components $pf = g\colon T \to X$ and $qf = h\colon T \to Y$ are continuous.

$$\begin{tikzcd}
 & & X \\
T \arrow[r, dashed, "f"] \arrow[urr, bend left, "pf"] \arrow[drr, bend right, "qf"'] & X \times Y \arrow[ur, "p"] \arrow[dr, "q"'] & \\
 & & Y
\end{tikzcd}
\qquad \equiv \qquad
\begin{tikzcd}
 & & X \\
T \arrow[r, dashed, "{(g,h)}"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "h"'] & X \times Y \arrow[ur, "p"] \arrow[dr, "q"'] & \\
 & & Y
\end{tikzcd}$$

(both follow from the fact that $U \times V = p^{-1}(U) \cap q^{-1}(V)$).

A little more complicated is the product of infinitely many topological spaces, where the clue to the correct definition is just the previous universal property and its proof. Let's mark

$$p_j\colon \prod_{i \in \mathcal{I}} X_i \to X_j$$

projection onto the j-th component.

Definition 4.7. Let X_i, $i \in \mathcal{I}$, be topological spaces. We define on $\prod_{i \in \mathcal{I}} X_i$ the *product topology* generated by the subbase

$$\{p_j^{-1}(U) \mid j \in \mathcal{I},\, U \subseteq \mathcal{X}_j \text{ open}\}.$$

We call the set $\prod_{i \in \mathcal{I}} X_i$ together with the product topology *product* of the topological spaces X_i, $i \in \mathcal{I}$.

cv **Exercise 4.8.** Prove that the product $\prod_{i \in \mathcal{I}} F_i$ of closed sets $F_i \subseteq X_i$ is closed.

——————— end of 2. lecture ———————

5. Separability Axioms

Remark. There is a separability axiom T_0.

Definition 5.1. A topological space X is called T_1if for every two points $x, y \in X$, $x \neq y$ there exists an open neighborhood $U \ni x$ disjoint with y, i.e. $y \notin U$.

Lemma 5.2. *A topological space X is T_1if and only if all its one-point subsets are closed.*

Proof. "$\Leftarrow$": It is enough to choose $U = X \smallsetminus \{y\}$ in the definition. "$\Rightarrow$": Let $y \in X$. Then for any $x \neq y$ there exists an open $U_x \ni x$ not containing y. Therefore, $\bigcup_{x \neq y} U_x = X \smallsetminus \{y\}$ is open and $\{y\}$ is therefore closed. □

Definition 5.3. A topological space X is called *T_2 (Hausdorff)* if for every two points $x, y \in X$, $x \neq y$ there exist disjoint open neighborhoods $U \ni x$, $V \ni y$, i.e. $U \cap V = \emptyset$.

cv **Example 5.4.** The space of finite complements is T_1but not Hausdorff (unless the support set is finite).

Lemma 5.5. *A topological space X is Hausdorff if and only if $\Delta_X \subseteq X \times X$ is a closed subset. Here $\Delta_X = \{(x, x) \mid x \in X\}$ is the "diagonal".*

Proof. "$\Rightarrow$": We show that $X \times X \smallsetminus \Delta_X$ is open. Let $(x, y) \in X \times X \smallsetminus \Delta_X$, i.e. $x \neq y$. By definition, $U \ni x$, $V \ni y$ exist disjoint open. Then $(x, y) \in U \times V \subseteq X \times X \smallsetminus \Delta_X$, where $U \times V$ is basic open.

"$\Leftarrow$": By analogy; let $x, y \in X$, $x \neq y$, i.e. $(x, y) \in X \times X \smallsetminus \Delta_X$. Since $X \times X \smallsetminus \Delta_X$ is open, there exists a basic open subset $U \times V$ with property $(x, y) \in U \times V \subseteq X \times X \smallsetminus \Delta_X$. Therefore, $x \in U$, $y \in V$, and $U \cap V = \emptyset$. □

nd Alternatively, one can prove "$\Rightarrow$" using $\Delta_X = \{(x, y) \mid p(x, y) = q(x, y)\}$.

Corollary 5.6. *Let $f, g\colon X \to Y$ be two continuous mappings and Y be Hausdorff. Then*

$$\{x \in X \mid f(x) = g(x)\}$$

is a closed subset of X.

Proof. The representation $(f, g)\colon X \to Y \times Y$ is continuous, whereas

$$\{x \in X \mid f(x) = g(x)\} = (f, g)^{-1}(\Delta_Y).$$

□

cv **Example 5.7.** The orthogoal group $\mathrm{O}(n) \subseteq \mathrm{GL}(n)$ is closed. (similarly $\mathrm{SL}(n)$)

Theorem 5.8.

1. *Subspaces of Hausdorff spaces are Hausdorff.*
2. *Products of Hausdorff spaces are Hausdorff.*

Proof. Let $x, y \in A$ be separated in X by open sets U, V. Then $A \cap U$, $A \cap V$ are open sets in A separating x from y.

Let $(x_i), (y_i) \in \prod_{i \in \mathcal{I}} X_i$ be different points. Then there exists an index $j \in \mathcal{I}$ such that $x_j \neq y_j$. Since X_j is Hausdorff, there exist $U \ni x_j$, $V \ni y_j$, $U \cap V = \emptyset$. Then $p_j^{-1}(U)$, $p_j^{-1}(V)$ separate (x_i) from (y_i). □

Definition 5.9. A T_1-space X is called *T_3 (regular)* if for each of its points $x \in X$ and a closed subset $F \subseteq X$ not containing x there are open disjoint neighborhoods $U \ni x$, $V \supseteq F$, i.e. $U \cap V = \emptyset$.

Lemma 5.10. *The topological space X is regular if and only if for every point $x \in X$ the closed neighborhood x forms a basis of the neighborhood, i.e. for every neighborhood $N \ni x$ there exists a closed neighborhood $F \ni x$ satisfying $N \supseteq F$.*

Proof. "$\Rightarrow$": For every open neighborhood $W \ni x$, it is enough to find a closed subneighborhood. By definition, one can separate x from $X \smallsetminus W$, i.e. $x \in U$, $X \smallsetminus W \subseteq V$, $U \cap V = \emptyset$. In other words, $x \in U \subseteq X \smallsetminus V \subseteq W$, so $X \smallsetminus V$ is a closed subregion of x.

"$\Leftarrow$": Let $x \notin F$, i.e. $x \in X \smallsetminus F$ form an open neighborhood. By assumption, there exists $x \in G \subseteq X \smallsetminus F$, where G is a closed neighborhood of x, i.e. $x \in U \subseteq G$, $F \subseteq X \smallsetminus G = V$. $\square$

Example 5.11. Every metric space M is regular – closed spheres form the basis of the neighborhood of every point. In a moment we will prove an even stronger claim in another way.

Theorem 5.12.

1. *Subspaces of regular spaces are regular.*
2. *Products of regular spaces are regular.*

Proof. Let $F \subseteq A$ be closed not containing $x \in A$. Then $A \cap \overline{F} = F$, so $x \notin \overline{F}$ and can be separated in X by U, V; in A they can then be separated by $A \cap U$, $A \cap V$. An alternative proof leads via the previous lemma.

Let $(x_i) \in \prod_{i \in \mathcal{I}} X_i$, $(x_i) \in U$ be an open neighborhood. Then there exist $j_1, \ldots, j_n \in \mathcal{I}$ and open sets $U_k \subseteq X_{j_k}$ such that

$$(x_i) \in p_{j_1}^{-1}(U_1) \cap \cdots \cap p_{j_n}^{-1}(U_n) \subseteq U.$$

Let $x_{j_k} \in F_k \subseteq U_k$ be closed subregions. Then

$$(x_i) \in \underbrace{p_{j_1}^{-1}(F_1) \cap \cdots \cap p_{j_n}^{-1}(F_n)}_{\text{closed neighborhood}} \subseteq U. \qquad \square$$

Definition 5.13. A T_1-space X is called *T_4 (normal)* if for each of its two disjoint closed subsets $F, G \subseteq X$ there exist open disjoint neighborhoods $U \supseteq F$, $V \supseteq G$.

The analogy of the previous theorem does not hold – see the proof: if $F, G \subseteq A$ are disjoint closed subsets, it is not necessarily true that $\overline{F}$, $\overline{G}$ are disjoint (except for the case where A is closed). So it should not be hard to believe that there are normal spaces whose subspaces and products are not normal.

Example 5.14. Every metric space M is normal. This is because for any $A \subseteq M$ the function $\operatorname{dist}(A, -)\colon M \to \mathbb{R}$ is continuous (does not shorten distances). If we now put

$$f(x) = \frac{\operatorname{dist}(F, x)}{\operatorname{dist}(F, x) + \operatorname{dist}(G, x)},$$

is this function defined and continuous everywhere, $\operatorname{im} f \subseteq [0, 1]$. Meanwhile, $f(x) = 0$ on F and $f(x) = 1$ on G, so one can choose

$$U = f^{-1}[0, 1/2), \quad V = f^{-1}(1/2, 1).$$

The phenomenon from the previous example is separation using continuous functions. We will return to it later.

6. Compact Spaces

Definition 6.1. A topological space X is called *compact* if a finite subcovering can be selected from any of its openopencoverings.

In what follows, we will very often use the following interpretation of the compactness of the subspace $A \subseteq X$. If $\mathcal{U}$ is a system of open sets in X such that $A \subseteq \bigcup \mathcal{U}$, then there exist finitely many $U_1, \dots, U_n \in \mathcal{U}$ such that $A \subseteq U_1 \cup \dots \cup U_n$.

cv **Example 6.2.** $\mathbb{R}$ is not compact.

cv **Example 6.3.** (a, b) is not compact (no coverage found).

Theorem 6.4. *The closed interval* $[a, b]$ *is compact.*

Remark. The previous theorem uses the completeness of real numbers – it does not hold over $\mathbb{Q}$. The interval $[a, b]_{\mathbb{Q}} = \mathbb{Q} \cap [a, b]$ is not compact: let $c \in (a, b)$ be irrational. Then

$$[a, b]_{\mathbb{Q}} = \bigcup_{n \in \mathbb{N}} [a, c - 1/n)_{\mathbb{Q}} \cup (c + 1/n, b]_{\mathbb{Q}}.$$

Proof. Let $\mathcal{U}$ be an open covering of $[a, b]$. Let's consider

$$T = \{t \in [a, b] \mid \text{the interval } [a, t] \text{ can be covered by finitely many elements } \mathcal{U}\}.$$

Obviously $a \in T$ and hence $T \neq \emptyset$. So we can put $t_0 = \sup T$.

We first show that $t_0 \in T$. Indeed, there exists $U \in \mathcal{U}$ such that $t_0 \in U$ and therefore there exists some $t_1 < t_0$ such that the entire interval $[t_1, t_0] \subseteq U$. Since $t_1 \in T$, $[a, t_1] \subseteq U_1 \cup \dots \cup U_n$ holds. Therefore, $[a, t_0] \subseteq U_1 \cup \dots \cup U_n \cup U$.

Now we show by argument that $t_0 = b$. If $t_0 < b$, we get $t_0 \in U \in \mathcal{U}$ again, and T also contains some $t_1 \in U$, $t_1 > t_0$. This is a dispute with $t_0 = \sup T$. $\square$

Theorem 6.5. *A closed subspace of a compact space is compact.*

Proof. Let $F \subseteq X$ be closed and $\mathcal{U}$ be some system of open sets with property $\bigcup \mathcal{U} \supseteq F$. Then $\mathcal{V} = \mathcal{U} \cup \{X \smallsetminus F\}$ is an open covering of X. Due to the compactness of X , it is

$$X = U_1 \cup \dots \cup U_n \cup (X \smallsetminus F)$$

and therefore $F \subseteq U_1 \cup \dots \cup U_n$. $\square$

——————————— end of 3. lecture ———————————

Theorem 6.6. *A compact subspace of a Hausdorff space is closed.*

Proof. Let $C \subseteq X$ be compact, $x \notin C$. We want to find some $U \ni x$, $U \cap C = \emptyset$. Let $y \in C$. Then there exist disjoint $U_y \ni x$, $V_y \ni y$. The system $\{V_y \mid y \in C\}$ forms an open covering of C. For compactness, a finite subcover $C \subseteq V_{y_1} \cup \dots \cup V_{y_n}$ can be selected from it. Then

$$x \in U_{y_1} \cap \dots \cap U_{y_n} = U$$

is an open neighborhood of x and $U \cap C = \emptyset$ because $U \cap V_{y_k} \subseteq U_{y_k} \cap V_{y_k} = \emptyset$. $\square$

Corollary 6.7. *In a compact Hausdorff space, closed sets are precisely compact.*

Theorem 6.8 (about the product)**.** *The product $X \times Y$ of two compact spaces X, Y is compact.*

We will prove the theorem later.

Corollary 6.9. *A subset $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. "$\Rightarrow$": Closedness follows from the Hausdorffness of $\mathbb{R}^n$, boundedness follows from the coverage of $B_k(0)$, $k \in \mathbb{N}$.

"$\Leftarrow$": From the boundedness $A \subseteq [-k, k]^n$, while the cube $[-k, k]^n$ is compact by the product theorem. Therefore, even its closed subset A is compact. $\square$

Theorem 6.10. *A continuous image of a compact space is compact.*

Proof. Let f be a continuous mapping $f\colon X \to Y$ from a compact space X and let $\mathcal{U}$ be an open covering of $f(X)$. Then $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open covering of X and thus $X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)$, or $f(X) \subseteq U_1 \cup \cdots \cup U_n$. $\square$

Theorem 6.11. *A continuous bijection $f\colon X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.*

Proof. It suffices to show that f^{-1} is continuous, i.e. that for a closed $F \subseteq X$, $f(F) \subseteq Y$ is also closed. However, F is compact, so $f(F)$ is also compact and therefore closed. $\square$

Definition 6.12. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let us denote the projection $p\colon X \to X/{\sim}$. Let us define the *quotient (identification) topology* on the decomposition of $X/{\sim}$ as

$$\{V \subseteq X/{\sim} \mid p^{-1}(V) \subseteq X \text{ open}\}.$$

We call the decomposition of $X/{\sim}$ together with the quotient topology the *quotient* of X according to the relation $\sim$.

A fundamental property of the quotient is that the mapping $f\colon X/{\sim} \to Y$ is continuous if and only if $fp\colon X \to Y$ is continuous,

$$\begin{tikzcd} X \arrow[r, "fp"] \arrow[d, "p"'] & Y \\ X/{\sim} \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

In connection with the previous sentence, some quotients can be described very concretely.

Examples 6.13.

cv 1. Describe the topology of the quotient $\mathbb{R}/\mathbb{Q}$ of the group $\mathbb{R}$ by its subgroup $\mathbb{Q}$. It's not even T_1 even though $\mathbb{R}$ is even T_4.

cv 2. $([0,1] \times S^{n-1})/(\{0\} \times S^{n-1}) \cong D^n$; here $X/A = X/{\sim}$, where $a \sim a'$ for any $a, a' \in A$. (The display required is "polar coordinates".)

cv 3. $D^n/S^{n-1} \cong S^n$. (The necessary representation is given by orbiting around S^n along major circles, for which there is a simple formula.)

cv 4. $S^1 \times S^1 \cong [0,1]^2/\sim$ ($\cong \mathbb{R}^2/\mathbb{Z}^2$ – this is the group quotient).

5. Figure illustrating other areas as quotients of polygons; a note on hyperbolic tiling.

** 6. $SS^{n-1} \cong S^n$

** 7. A line with double origin $\mathbb{R} \times \{-1,1\}/\sim$ where $(x,-1) \sim (x,1)$ at any time $x \neq 0$ – is not Hausdorff, although it is "locally Hausdorff".

nd 8. Each retract is both a quotient and a subspace.

hw 5 9. $\mathbb{R}^2/\sim$, $x \sim y \Leftrightarrow |x| = |y|$, is homeomorphic to $\mathbb{R}_{\geq 0}$

Now we can prove the product theorem. Let us first state the lemma.

Lemma 6.14 (tube lemma)**.** *Let X be a compact space, Y any, $W \subseteq X \times Y$ an open set containing $X \times \{y\}$. Then there exists an open neighborhood $V \ni y$ such that $X \times V \subseteq W$.*

Proof. From the definition of the product topology, for each (x,y) there exist open neighborhoods $U_x \ni x$ and $V_x \ni y$ such that $U_x \times V_x \subseteq W$. A finite subcover $U_{x_1}, \ldots, U_{x_n}$ can be selected from the coverage $\{U_x \mid x \in X\}$. Let us set $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then

$$U_{x_i} \times V \subseteq U_{x_i} \times V_{x_i} \subseteq W$$

and thus also $X \times V = \bigcup U_{x_i} \times V \subseteq W$. □

hw 6 **Example 6.15.** Prove that the lemma is equivalent to the following statement: the projection $X \times Y \to Y$ is *closed*, i.e. the image of a closed set is closed.

Proof of the Product Theorem. Let $\mathcal{W}$ be the open coverage of $X \times Y$. For each $y \in Y$, consider a subspace $X \times \{y\}$ that is homeomorphic to X and therefore compact. Since $\mathcal{W}$ is its open coverage, $W_{1,y}, \ldots, W_{n,y} \in \mathcal{U}$ covering $X \times \{y\}$ can be selected. By the lemma, $W_{1,y} \cup \cdots \cup W_{n,y}$ contains a subset of the form $X \times V_y$. Thus, we see that it suffices to cover Y with finitely many V_y since each $X \times V_y$ is covered with finitely many elements of $\mathcal{W}$. But since $\{V_y \mid y \in Y\}$ is an open coverage of Y, this follows from the compactness of Y. □

Our next goal will be the proof of Tikhonov's theorem about infinite products of compact spaces. We can first prove the so-called Alexander's lemma.

Lemma 6.16 (Alexander)**.** *Let X be a topological space. If there exists a subbase $\mathcal{S}$ such that a finite subcover can be selected from every open coverage $\mathcal{U} \subseteq \mathcal{S}$, then X is compact.*

Proof. The proof is based on the axiom of choice, specifically on the principle of maximality, or as it is called in Czech. Assume that X is not compact and choose a maximal open covering
hw 7 $\mathcal{U}$ that has no finite subcovering (the assumptions of Zorn's lemma are easily verified).

Let $x \in X$ be an arbitrary point. Since $\mathcal{U}$ is a coverage, there exists $x \in U \in \mathcal{U}$. Since $\mathcal{S}$ is a subbase, then there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that

$$x \in S_1 \cap \cdots \cap S_n \subseteq U.$$

We now show by argument that some S_i is an element of $\mathcal{U}$. If $S_i \notin \mathcal{U}$, by the maximality of $\mathcal{U}$ there exists a finite $\mathcal{U}_i \subseteq \mathcal{U}$ such that $\{S_i\} \cup \mathcal{U}_i$ is a covering, i.e. $\mathcal{U}_i$ covers $X \smallsetminus S_i$. But then $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ covers

$$(X \smallsetminus S_1) \cup \cdots \cup (X \smallsetminus S_n) = X \smallsetminus (S_1 \cap \cdots \cap S_n) \supseteq X \smallsetminus U$$

and thus $\{U\} \cup \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ covers X , which is a dispute.

Denoting the corresponding $S_i \in \mathcal{U}$ as S_x, we have $x \in S_x \in \mathcal{S} \cap \mathcal{U}$. But since $\{S_x \mid x \in X\} \subseteq \mathcal{S}$ is an open coverage of the elements of $\mathcal{S}$, a finite subcover can be selected from it by assumption. But this will also be the final sub-covering of $\mathcal{U}$, dispute. □

Theorem 6.17 (Tikhonov)**.** *The product of any number of compact spaces is compact.*

Proof. Let $X = \prod_{i\in\mathcal{I}} X_i$ where X_i is compact. We show that the subbasis

$$\{p_j^{-1}(U_j) \mid j \in \mathcal{I},\, U_j \subseteq X_j \text{ open}\}$$

satisfies the conditions of Alexander's lemma. Let $\mathcal{U}$ be an open covering of subbasic sets. Let us define $\mathcal{U}_j$ as the set of those open $U_j \subseteq X_j$ such that $p_j^{-1}(U_j) \in \mathcal{U}$. Assume that no $\mathcal{U}_j$ is covered. Then there exists, for each $j \in \mathcal{I}$, a point $x_j \in X_j$ such that $x_j \notin \bigcup \mathcal{U}_j$. But then the point with components $(x_j)_{j\in\mathcal{I}}$ does not lie in $\bigcup \mathcal{U}$, which contradicts the fact that $\mathcal{U}$ is a cover.

Therefore, there is some $\mathcal{U}_j$ covering, and due to compactness, a finite subcovering $U_1, \ldots, U_n$ can be selected from it. Then apparently $p_j^{-1}(U_1), \ldots, p_j^{-1}(U_n)$ is a finite subcovering of $\mathcal{U}$. □

——— end of 4. lecture ———

Theorem 6.18. *A compact Hausdorff space is normal.*

Proof. Let $F \subseteq X$ be closed, $x \notin F$. For any $y \in F$, there exist $U_y \ni x$, $V_y \ni y$ open disjoint. Since F is compact, there are finitely many $y_1, \ldots, y_n \in F$ such that

$$F \subseteq V_{y_1} \cup \cdots \cup V_{y_n}, \quad x \in U_{y_1} \cap \cdots \cap U_{y_n};$$

these are open and disjoint.

hw 8 The implication of $T_3 \Rightarrow T_4$ is for homework. □

A *mass point* of a sequence (x_n) is a point x such that for every open neighborhood $U \ni x$ there are infinitely many terms $x_n \in U$. Our goal will now be to show that a metric space is compact if and only if every sequence has a mass point. Let's call this property sequential compactness for now. Let us define the diameter $\operatorname{diam} A = \sup\{\operatorname{dist}(x, y) \mid x, y \in A\}$.

Theorem 6.19 (Lebesgue's lemma)**.** *Let $\mathcal{U}$ be an open covering of a (sequentially) compact metric space M. Then there exists $\varepsilon > 0$ such that every subset $A \subseteq M$ of diameter $\operatorname{diam} A \leq \varepsilon$ lies in some $U \in \mathcal{U}$.*

We call the number from the theorem the *Lebesgue number* coverage of $\mathcal{U}$.

Proof. Obviously, it suffices to find ε such that every closed sphere of radius ε lies in some $U \in \mathcal{U}$. Assume that no such ε exists and choose, for each $n \in \mathbb{N}$, a sphere $B_{1/n}(x_n)$ that does not fit into any $U \in \mathcal{U}$. Passing to the subsequence, we can assume that $x_n \to x$. Since $\mathcal{U}$ is an open coverage, there exists $\delta > 0$ such that $B_{2\delta}(x) \subseteq U \in \mathcal{U}$. For $n \gg 0$, $\operatorname{dist}(x_n, x) < \delta$ and $1/n < \delta$ and therefore $B_{1/n}(x_n) \subseteq B_{2\delta}(x) \subseteq U$, dispute. □

Theorem 6.20. *A metric space is compact if and only if it is sequentially compact.*

Proof. The "⇒" direction is simple. Let x_n be a sequence that has no mass point. Then for each $x \in M$ there exists some ball $B_{\varepsilon_x}(x)$ containing only a finite number of terms of the sequence. By choosing a finite subcovering, we get that there are only finitely many points of the sequence in the entire space, disputes.

For the opposite direction "⇐", let $\varepsilon > 0$ be the Lebesgue number $\mathcal{U}$. Since every sphere of radius ε fits into some $U \in \mathcal{U}$, it suffices to cover M with finitely many spheres of radius ε. Let us successively choose a sequence of points that are at least ε away from each other. Such a sequence must necessarily be finite, since no subsequence of it is Cauchy and therefore cannot converge; let us denote it by $x_1, \dots, x_n$. Then $M = B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)$. □

? *Question.* Say something about networks, their convergence and the equivalence of compactness with the existence of a convergent subnet? Say something about filters, ultrafilters and compactness equivalence with every ultrafilter converging (to at least one point)?

7. Connectedness

Classically, the empty topological space is considered connected, but for various reasons it is better not to consider it connected. We avoid this dilemma by limiting ourselves to non-empty spaces.

Definition 7.1. Let X be a nonempty topological space. We say that X is *continuous* if the only subsets of $A \subseteq X$ that are both open and closed are $\emptyset$ and X.

Subsets from the definition (i.e., those that are both open and closed) are called *ambiguous*, in English clopen. If U is dual and $V = X \setminus U$ is its complement, then the entire space X is a disjoint set of $X = U \sqcup V$. In the section on separation axioms, we used disjoint open sets to separate subsets of X, and this decomposition then corresponds to the fact that the entire space consists of two separate parts.

cv **Lemma 7.2.** *A nonempty space X is continuous if and only if every continuous mapping $\chi\colon X \to \{0, 1\}$ is constant.*

Theorem 7.3. *The closed interval $[a, b]$ is continuous.*

Remark. This interval property depends, like compactness, on the completeness of the real numbers. Specifically, $[a, b]_\mathbb{Q}$ is not continuous: we choose an arbitrary irrational c with property $a < c < b$, then $[a, b]_\mathbb{Q} = [a, c)_\mathbb{Q}\ cup(c, b]_\mathbb{Q}$.

Proof. Assume that $U \subseteq [a, b]$ is dual, $\emptyset, X \neq U$. By possibly passing to the complement, we can assume that $a \in U$. Let's mark

$$T = \{t \in [a, b] \mid [a, t] \subseteq U\}$$

We want $b \in T$. Obviously $a \in T$ and therefore $t_0 = \sup T$ exists. We get $t_0 \in T$ from the closedness of U, and $t_0 + \varepsilon \in T$ from the openness, except for the case $t_0 = b$. This is a conflict with $U \neq X$.

(Otherwise: if the continuous mapping $\chi\colon X \to \{0, 1\}$ were to take on the values 0 and 1, it would have to take on the values $1/2$ as well, dispute.) □

Theorem 7.4. *A continuous image of a continuous space is continuous.*

Proof. Let $f\colon X \to Y$ be a continuous mapping where X is continuous. Let $\chi\colon f(X) \to \{0,1\}$ be an arbitrary continuous function. Then $\chi f\colon X \to \{0,1\}$ is also continuous and therefore constant. Since $f\colon X \to f(X)$ is surjective, χ is also constant.

(Otherwise: if $U \subseteq f(X)$ is dual, $f^{-1}(U) \subseteq X$ is also dual.) □

Theorem 7.5. *The closure of a continuous subset is continuous.*

Proof. If $\chi\colon \overline{A} \to \{0,1\}$ is a continuous function, then its taper to A is constant, say $\chi|_A = 0$. Meanwhile, $\chi^{-1}(0)$ is a closed set containing A and therefore $\chi^{-1} = \overline{A}$, so χ is constant. □

Theorem 7.6. *Let $\mathcal{M}$ be a system of connected subsets in X such that $A \cap B \neq \emptyset$ for every two $A, B \in \mathcal{M}$. Then $\bigcup \mathcal{M}$ is continuous.*

Proof. Let $f\colon \bigcup \mathcal{M} \to \{0,1\}$ be a continuous mapping. Then its narrowing to every $A \in \mathcal{M}$ is constant, due to the connection of A. At the same time, this constant value must be the same for all $A \in \mathcal{M}$, due to the non-emptiness of the intersections. So f is constant. □

Corollary 7.7. *The real axis $\mathbb{R} = \bigcup_{n\in\mathbb{N}}[-n,n]$ is continuous.*

cv **Example 7.8.** The intervals $[a,b]$, $[a,b)$, (a,b) are continuous – prove. Further prove that they are not homeomorphic by removing points.

$**$ **Example 7.9.** Prove that space

$$\{(x, \sin\tfrac{\pi}{x}) \mid x > 0\} \cup \{(0,y) \mid -1 \le y \le 1\}$$

is continuous but not arc-continuous.

Definition 7.10. A *Component* of a nonempty space X is a maximal continuous subset.

Theorem 7.11. *Any topological space is a disjoint union of its components. These components are closed.*

Proof. Let $x \in X$ and consider the system $\mathcal{M} = \{A \subseteq X \mid A \text{ continuous}, x \in A\}$. Then $\bigcup \mathcal{M}$ is a continuous subset containing x, obviously maximal. If two different components had a non-empty intersection, their union would be continuous, which would be a contradiction of maximality. Closedness follows from the fact that the closure of a continuous subset is continuous and from maximality. □

Definition 7.12. We say that a topological space X is *totally discontinuous* if its components are one-point.

hw 9 **Example 7.13.** Prove that $\mathbb{Q}$ is totally disjoint.

The set of dual sets together with the inclusion, $(\mathrm{Ob}(X), \subseteq)$, forms a Boolean algebra (the dual sets are closed for finite union, finite intersections, and complements). This way we get all Boolean algebras (except for isomorphism). After narrowing down to compact Hausdorff totally disjoint spaces, so-called Stone spaces, we get a unique correspondence, so-called *Stone's duality*.

nd Conversely, let B be a Boolean algebra. Consider the set $S(B)$ of all homomorphisms of

Boolean algebras $\varphi\colon B \to 2$.[1] Thus, we can equip $S(B) \subseteq 2^B$ with a subspace topology, where 2^B has a product topology (2 has a discrete topology). As a subspace of the product of Hausdorff totally disjoints, is Hausdorff and totally disjoint. We now show that $S(B)$ is closed, i.e., compact. Let $\varphi\colon B \to 2$ not be a homomorphism of Boolean algebras. For example, suppose $\varphi(b) = 0 = \varphi(c)$ but $\varphi(b \wedge c) = 1$. If we denote the projection $2^B \to 2$ onto the bth component as p_b, then $\varphi \in p_b^{-1}(0) \cap p_c^{-1}(0)\ capp_{b\wedge c}^{-1}(1)$, with no mapping from this open neighborhood lying in $S(B)$.

nd **Theorem 7.14** (Stone)**.** *The above constructions specify the (contravariant) equivalence between Stone spaces and Boolean algebras.*

nd Contravariance means that homomorphisms of algebras $B \to B'$ correspond to continuous mappings $S(B') \to S(B)$.

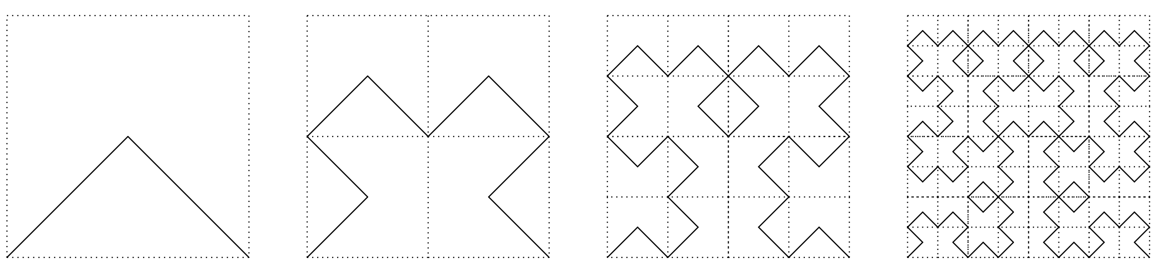
** *Proof.* The homeomorphism $X \to S(\mathrm{Ob}(X))$ sends the point $x \in X$ to the mapping $U \mapsto (x \in U)$, where $x \in U$ needs to be understood as logical value, i.e. element 2.

The isomorphism $B \to \mathrm{Ob}(S(B))$ sends an element $b \in B$ to the dual set

$$p_b^{-1}(1) = \{\varphi \in S(B) \mid \varphi(b) = 1\} \subseteq S(B). \qquad \square$$

end of 5. lecture

Example 7.15 (Path filling a square)**.** Let's start with the path shown in the first figure, which we traverse at a constant speed, denote it by γ_1. In the next steps, we replace all sections of γ_n that look like γ_1 with corresponding sections that look like γ_2. All paths are traversed at constant speed.



γ_1 γ_2 γ_3 γ_4

Let us set $\gamma = \lim_{n\to\infty} \gamma_n$. Since $\gamma_{n+1}(t)$ and $\gamma_n(t)$ lie in the same square of side $(1/2)^{n-1}$, this sequence is uniformly convergent and therefore γ continuous. It remains to show that it is surjective. Let x be any point of the square and write it as the intersection of the sequence of squares with sides $(1/2)^{n-1}$ shown in the figures. In each such square lies some point $\gamma_n(t_n)$ and therefore $x = \lim_{n\to\infty} \gamma_n(t_n)$. Passing to the convergent subsequence, we can assume $t_n \to t$ and then $\gamma(t) = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \gamma_n(t_n) = x$ from uniform convergence.

(Otherwise $0, x_1x_2x_3x_4\ldots \mapsto (0, x_1x_3\ldots; 0, x_2x_4\ldots)$.)

Theorem 7.16. *The product of two continuous spaces is continuous.*

[1] An alternative characterization of such homomorphisms is using $\varphi^{-1}(0)$ – these are maximal ideals – or using $\varphi^{-1}(1)$ – these are ultrafilters. (A filter in B is an upwardly closed set $F \subseteq B$, closed on finite suprems. An ultrafilter is a maximal filter not containing 0.)

Proof. Let $f\colon X \times Y \to \{0,1\}$ be a continuous mapping and let (x,y) and (x',y') be two points of $X \times Y$. Then $f(x,y) = f(x',y)$ from the relation $X \times \{y\} \cong X$ and $f(x',y) = f(x',y')$ from the context $\{x'\} \times Y \cong Y$. Thus, f is constant. $\square$

Example 7.17. The spaces $\mathbb{R}$ and $\mathbb{R}^n$, where $n > 1$, are not homeomorphic – again using point removal. For this, it is necessary to prove that $\mathbb{R}^n \smallsetminus \{0\}$ is continuous. It can be written as the union of continuous sets of the form $\mathbb{R}^i \times \mathbb{R}_\pm \times \mathbb{R}^j$, where $\mathbb{R}_\pm$ is either a set of positive or negative numbers and $i+1+j = n$. Although these sets do not have non-empty intersections, this only happens for the pairs $\mathbb{R}^i \times \mathbb{R}_+ \times \mathbb{R}^j$ and $\mathbb{R}^i \times \mathbb{R}_- \times \mathbb{R}^j$ and those have a non-empty intersection with any other set from the system $(n > 1)$.

Theorem 7.18. *Any product of continuous spaces is continuous.*

Proof. Again, let $f : \prod_{i\in\mathcal{I}} X_i \to \{0,1\}$ and let $x = (x_i) \in f^{-1}(0)$. Since f is continuous, it takes the value 0 also on some neighborhood $p_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap p_{j_n}^{-1}(U_{j_n})$ of the point x. In particular, f takes on the value 0 at all points of y satisfying $x_{j_1} = y_{j_1}, \ldots, x_{j_n} = y_{j_n}$. In the previous proof, we showed that f has the same values at points differing in finitely many components. Together, f is constant. $\square$

From now on, we will denote $I = [0,1]$.

Definition 7.19. A *Path* in X is a continuous mapping $\gamma\colon I \to X$. We say that γ connects the points $\gamma(0)$, $\gamma(1)$.

Definition 7.20. A non-empty space X is called *path-continuous* (traditionally *arc-continuous*) if every two of its points can be connected by a path.

Theorem 7.21. *Any path-continuous space is continuous.*

Proof. If γ_x is a path connecting some selected point $x_0 \in X$ to a point x , then $X = \bigcup_{x\in X} \operatorname{im}\gamma_x$, where each $\operatorname{im}\gamma_x$ is continuous (as the image of I) and all intersect at x_0 . $\square$

In the opposite direction, the theorem does not always apply, but only under certain limiting conditions. We say that a space X is *locally path-continuous* if a path-continuous neighborhood forms a basis of the neighborhood at every point, i.e. if for every neighborhood $N \ni x$ there exists a path-continuous neighborhood O satisfying $N \supseteq O \ni x$.

cv **Example 7.22.** Open subsets of Euclidean spaces are locally path-connected. General subsets need not be locally path-connected (see Example 7.9).

Lemma 7.23. *If x can be connected to y and y can be connected to z by a path, then x can also be connected to z by a path.*

cv *Proof.* Let γ be the path connecting x to y and let δ be the path connecting y to z. Let's put

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2 \\ \delta(2t-1) & 1/2 \le t \le 1 \end{cases}$$

Since the intervals $[0,1/2]$ and $[1/2,1]$ form a finite closed coverage and $\gamma * \delta$ is continuous on each, it is continuous on all of $[0,1]$. Meanwhile, $(\gamma*\delta)(0) = \gamma(0) = x$ and $(\gamma*\delta)(1) = \delta(1) = z$. $\square$

Theorem 7.24. *If X is both continuous and locally path-connected, then it is path-connected.*

Proof. Let $x \in X$ and consider $C(x) = \{y \in X \mid x \text{ can be connected by path s } y\}$. According to the local path connection assumption, $C(x)$ is open – whenever x can be connected to y and O is any path-connected neighborhood of y, then x can be connected to any point O . Since X is a disjoint union of $C(x)$, where $x \in X$, the complement of $C(x)$ is also open and therefore $C(x)$ is dual. At the same time, $x \in C(x)$, so $C(x) = X$ follows from the connection, and therefore x can be connected to every point of X. □

hw 10 *Remark.* For a general space X , the set $C(x)$ is called a path component from the previous proof, and similarly, it can be shown that for a locally path-connected X , its components coincide with the path components.

? *Question.* Mention locally contiguous spaces, that their components are open? Say something about continua (compact, continuous metric spaces) and the Hahn–Mazurkiewicz theorem (continuous images of an interval are precisely locally continuous continua)?

8. Locally Compact Spaces

Definition 8.1. A Hausdorff space X is called *locally compact* (Hausdorff) if a compact neighborhood forms the basis of the neighborhood of every point, i.e. if every neighborhood $N \ni x$ contains a compact subneighborhood $N \supseteq C \ni x$.

? *Question.* Consistently say locally compact *Hausdorff.*

Example 8.2.

1. Every compact Hausdorff space is locally compact because it is regular (the uclosed-neighborhood is a basis of the neighborhood) and closed=compact.
2. Every Euclidean space is locally compact – closed spheres form the basis of the neighborhood and are compact.
3. Discrete spaces are locally compact.
4. The rational numbers $\mathbb{Q}$ are not locally compact – no neighborhood is compact.

Probably skip the next one, the proof is pretty useless; otherwise, of course, a picture will help - I used round=open, square=closed watering cans.

** **Theorem 8.3.** *If every point of the Hausdorff space X has some compact neighborhood, then it is locally compact.*

Proof. Let C be a compact neighborhood of x and let N be any of its neighborhoods. Then $C \cap N$ is a neighborhood of x in the compact Hausdorff space C. Therefore, there exists a compact subneighborhood $x \in D \subseteq C \cap N$. Since D is a neighborhood of x in C and C is a neighborhood of x in X , D is also a neighborhood of x in X . □

Theorem 8.4. *The product of two locally compact spaces is locally compact.*

Proof. Obviously, it suffices to find a compact neighborhood of $(x, y) \in X \times Y$ that fits into $U \times V \ni (x, y)$. Let $U \supseteq C \ni x$ and $V \supseteq D \ni y$. Then $C \times D$ is the sought-after compact neighborhood. □

Construction 8.5 (one-point compactification)**.** Let X be a locally compact space and $\infty \notin X$. Let's put $X^+ = X \cup \{\infty\}$. Let us define a topology on X^+ as follows: $U \subseteq X^+$ is openif and only

- $\infty \notin U$ and U is open in X or
- $\infty \in U$ and $X \setminus U$ is compact.

This space is called a *one-point compactification* of the space X.

hw 11 **Example 8.6.** Prove that this is indeed a topology (for this you will need to prove that the final union of compact subsets is compact).

Theorem 8.7. *The space X^+ is a compact Hausdorff space of which X is a subspace.*

Proof. Obviously, all "tracks" $X \cap U$ of open $U \subseteq X^+$ are open (this requires Hausdorffness), so indeed X is a subspace of X^+.

Let $\mathcal{U}$ be an arbitrary open covering of X^+. Then some $U_0 \in \mathcal{U}$ contains ∞ and from openness $X \setminus U_0$ is compact. Therefore, finitely many $U_1, \ldots, U_n$ covering $X \setminus X_0$ can be selected from $\mathcal{U}$. Together, $U_0, U_1, \ldots, U_n$ cover X^+.

The points of X can be separated by open subsets in X. So it remains to separate $x \in X$ and ∞. Due to local compactness, there exists a compact neighborhood $C \ni x$. From the definition of the neighborhood of $C \supseteq U \ni x$ and then $U \ni x$ and $X^+ \setminus C \ni \infty$, the separating open sets are sought. □

cv **Example 8.8.** The topology on X^+ is uniquely determined by the above requirements.

cv **Example 8.9.** Describe the one-point compactification of $\mathbb{R}^n$. (describe the stereographic projection $S^n \to \mathbb{R}^n$, $x \mapsto \frac{x-e_0}{x_0-1}$, show that it is continuous together with its inverse $v \mapsto \frac{-1+|v|^2}{1+|v|^2} \cdot e_0 + \frac{2}{1+|v|^2} \cdot v$ – both are given by the formula (my universal reasoning) – and use the previous example). The formula for the inversion can be left behind HW, with the fact that I would deduce for them that it must be of the form $e_0 + k(v - e_0)$.

hw 12 **Example 8.10.** Describe the one-point compactification of a compact Hausdorff space.

end of 6. lecture

Proposition 8.11. *A continuous bijection $f : X \to Y$ between locally compact spaces is a homeomorphism if and only if it is "regular" (proper, i.e., the pattern of a compact set is compact).*

Proof. By the definition of topology on one-point compactification, $f^+ : X^+ \to Y^+$ is continuous if and only if f is regular. But then it is a continuous bijection between compact Hausdorff spaces and thus a homeomorphism. Its contraction f must then also be a homeomorphism. □

This can (maybe) be applied to the example of the stereographic projection $S^n \setminus \{e_0\} \to \mathbb{R}^n$ – but one needs to prove regularity.

Theorem 8.12. *Locally compact spaces are just open subspaces of compact Hausdorff spaces.*

Proof. "⇒": every locally compact space X is an open subspace of X^+.

"⇐": every compact Hausdorff space is locally compact and these are obviously closed on open subspaces. □

Remark. It also holds that a closed subset of a locally compact subspace X is locally compact: compact neighborhoods of x in F can be obtained as intersections of F with compact neighborhoods of x in X .

** Even more generally, a subset $A \subseteq X$ is locally compact if and only if it is the intersection of an open and a closed subset (specifically, A is open in $\overline{A}$).

9. Real Functions

Definition 9.1. A *Compactification* of a space X is an embedding of $X \hookrightarrow K$ of the space X into some compact Hausdorff space K as a subspace such that $\overline{X} = K$ holds.

A basic example is the one-point compactification mentioned above. The opposite extreme is the so-called Stone-Čech compactification, which adds as many points as possible. This compactification works for any completely regular spaces – we'll deal with those in this chapter.

Definition 9.2. Let X be a T_1 topological space. We say that X is $T_{3\frac{1}{2}}$ *(completely regular)* if for each of its points $x \in X$ and a closed subset $F \subseteq X$ not containing x there exists a continuous function $f\ colX \to [0,1]$ such that $f(x) = 0$ and $f|_F = 1$.

? *Question.* Exercise: every locally compact Hausdorff space is completely regular (this also follows from one-point compactification).

Example 9.3. Every metric space is completely regular. We even proved that it is "completely normal", i.e., that any two closed sets can be separated by a function. We will see in a moment that this condition is equivalent to normality.

Theorem 9.4. *Completely regular spaces are closed to subspaces and products.*

Proof. If $A \subseteq X$ is an arbitrary subspace and $x \in A$, $F \subseteq A$ is closed in A and does not contain x, then x and $\overline{F}$ are also disjoint in X. Therefore, there exists $f : X \to [0,1]$ separating x from $\overline{F}$. Its narrowing to A separates x from F.

Now let X_i be completely regular, $i \in \mathcal{I}$, and consider the product $X = \prod_{i\in\mathcal{I}} X_i$. If $x \in X$ and $F \subseteq X$ is closed not containing x, then $x \in X \smallsetminus F$ (open) and by the definition of the product topology

$$x \in p_{j_1}^{-1}(U_1) \cap \cdots \cap p_{j_n}^{-1}(U_n) \subseteq X \smallsetminus F$$

for some open $U_k \subseteq X_{j_k}$. By passing to the complements $F_k = X_{j_k} \smallsetminus U_k$ we get

$$p_{j_1}^{-1}(F_1) \cup \cdots \cup p_{j_n}^{-1}(F_n) \supseteq F$$

and the closed set on the left still does not contain x. Therefore, it is enough to separate it from x. Let us choose continuous functions $f_k : X_{j_k} \to [0,1]$ separating $p_{j_k}(x)$ from F_k and set

$$f(x) = \max\{f_1(p_{j_1}(x)), \ldots, f_n(p_{j_n}(x))\}.$$

□

cv **Exercise 9.5.** Prove that for continuous functions $f, g : X \to \mathbb{R}$, $\max\{f, g\}$ is also continuous.

Theorem 9.6. *Completely regular spaces are precisely the subspaces of the cube* $[0,1]^S$.

Proof. Every subspace $[0,1]^S$ is completely regular according to the previous theorem.

On the contrary, let X be completely regular and let us set $S = \{f : X \to [0,1] \mid f \text{ is continuous}\}$. We will write the components $t \in [0,1]^S$ as $t_f = p_f(t)$. Let us define the mapping $h\colon X \to [0,1]^S$ in terms of its components

$$\begin{tikzcd} X \arrow[r, dashed, "h"] \arrow[dr, "f"'] & {[0,1]^S} \arrow[d, "p_f"] \\ & {[0,1]} \end{tikzcd}$$

thus $h(x) = (f(x))_{f\in S}$. According to the universal product property, h is continuous. Next, we show that is injective and, finally, that it is a homeomorphism to its image.

Let $x, y \in X$ be two distinct points and let $f : X \to [0,1]$ be a continuous function separating x from y . Then $(h(x))_f = 0$ and $(h(y))_f = 1$, therefore $h(x) \neq h(y)$. It remains to show that the image of the closed set $F \subseteq X$ is closed in $h(X)$. Therefore, let's choose an arbitrary point $h(x) \notin h(F)$ and look for its neighborhood disjoint with $h(F)$. By injectivity, $x \notin F$ holds and therefore there exists $f : X \to [0,1]$ separating x from F . But then $(h(x))_f = 0$, while $p_f|_{h(F)} = 1$. Therefore, $(p_f)^{-1}[0,1)$ is the searched open neighborhood of $h(x)$ disjoint with $h(F)$. □

Corollary 9.7. *A topological space is compactified if and only if it is completely regular.*

Proof. If X has a compactification, it is a subspace of the compact Hausdorff space, which we proved to be normal. We will see in a moment that $\mathrm{T}_4 \Rightarrow \mathrm{T}_{3\frac{1}{2}}$(Uryshohn's theorem).

Conversely, let X be completely regular. Then the mapping $h : X \to \overline{h(X)}$ from the previous proof is a compactification. □

Definition 9.8. To embed $h\colon X \to [0,1]^S$ from the previous proof, let us set $\beta(X) = \overline{h(X)}$. This is a compactification of the space X and it is called the *Stone-Čech compactification*.

Remark. The Stone-Cech compactification has the following universal property: if $X \hookrightarrow K$ is an arbitrary compactification, then there exists a single continuous extension $\beta(X) \to K$ (it simply extends to the mapping $\beta(X) \to \beta(K) \cong K$). For this reason, this is the "biggest" possible compactification.

nd *Remark.* The continuous functions $f : \beta(X) \to \mathbb{R}$ are in unique correspondence with the *bounded* continuous functions $X \to \mathbb{R}$. Because of the compactness of $\beta(X)$, $f(\beta(X))$ is bounded and more so $f(X)$. Conversely, if $g\colon X \to \mathbb{R}$ is an arbitrary bounded function, then we can understand it as $g\colon X \to [a,b]$ and get $f\colon \beta(X) \to \beta([a,b]) \cong [a,b]$.

? *Question.* If X is a discrete topological space, then $\beta(X)$ should be the set of all ultrafilters on X (it is clear that each ultrafilter determines one point of $\beta(X)$; but why can't two ultrafilters enter the same point?). This is related to Stone's duality, see elsewhere.

∗ **Theorem 9.9.** *A completely regular topological space with a countable topology basis is metrizable.*

Proof. Analyzing the proof of the cubing theorem, one can easily arrive at the following observation. Let $S_0 \subseteq S = \{f\colon X \to I \text{ continuous}\}$. Then the mapping $h_0\colon X \to I^{S_0}$ with components $h_0 = (f)_{f\in S_0}$ is an embedding if for every closed set F and a point $x \notin F$ there exists $f \in S_0$ such that $f(x) = 0$, $f|_F = 1$.

Next, we find a countable set S_0 with this property. Then $h_0\colon X \hookrightarrow I^\omega$ and there exists a metric on I^ω

$$\mathrm{dist}(x,y) = \sum \tfrac{1}{2^n}|x_n - y_n|.$$

hw 13 Prove that this metric specifies on I^ω the product topology $I^\omega = \prod_{n=1}^\infty I$. The metric on $X \cong h_0(X)$ is then obtained by narrowing the metric on I^ω.

It remains to find S_0. Let $U \subseteq V$ be basic open sets. If there is a continuous $f\colon X \to I$ with property $f|_U = 0$, $f|_{X \smallsetminus V} = 1$, then we choose one and denote it $F_{U,V}$. Let's put

$$S_0 = \{f_{U,V} \mid U \subseteq V \text{ basic such that } f_{U,V} \text{ exists}\}.$$

The condition needs to be verified. Let $x \notin F$, so $x \in X \smallsetminus F$. By the definition of a topology base, there exists a base V with property $x \in V \subseteq X \smallsetminus F$. Due to complete regularity, there exists $f\colon X \to I$ such that $f(x) = 0$ and $f|_{X \smallsetminus V} = 1$. By appropriate reparameterization

$$\varphi(t) = \begin{cases} 0 & t \leq \frac{1}{2} \\ 2t-1 & t \geq \frac{1}{2} \end{cases}$$

we get a function φf which is zero on the neighborhood $f^{-1}[0,\frac{1}{2}) \ni x$. Again, there exists a basis U with property $f^{-1}[0,\frac{1}{2}) \supseteq U \ni x$. Therefore, the function $f_{U,V}$ exists; may not be equal to φf , but nevertheless any $f_{U,V}$ separates x from F as we require. □

nd In particular, a compact Hausdorff space is metrizable if and only if it has a countable topology basis. Indeed, we saw in the proof of Theorem 6.20 that every compact metric space X can be covered by a finite system $\mathcal{B}_n$ of spheres of radius $1/n$ for every $n \in \mathbb{N}$. It then simply turns out that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ forms the basis of the topology: if U is open and $x \in U$, then one can find $B_{2/n}(x) \subseteq U$. In $\mathcal{B}_n$ then one can find a ball containing x and it will necessarily lie in $B_{2/n}(x)$.

* **Theorem 9.10** (Urysohn)**.** *Let X be a normal space and let $F, G \subseteq X$ be its two disjoint closed subsets. Then there exists a continuous $f\colon X \to [0,1]$ such that $f|_F = 0$ and $f|_G = 1$.*

Proof. Let us set $F_0 = F$, $U_1 = X \smallsetminus G$. So we are looking for a function f with the properties $f|_{U_0} = 0$, $f|_{X \smallsetminus U_1} = 1$. By normality, there exist $F_0 \subseteq U_{1/2} \subseteq F_{1/2} \subseteq U_1$ (since $X \smallsetminus F_{1/2}$ is a neighborhood of $X \smallsetminus U_1$ disjoint with $U_{1/2}$).

In the next step we get

$$F_0 \subseteq U_{1/4} \subseteq F_{1/4} \subseteq U_{1/2} \subseteq F_{1/2} \subseteq U_{3/4} \subseteq F_{3/4} \subseteq U_1.$$

and inductively then the system of open sets $U_{m/2^n}$ and closed sets $F_{m/2^n}$ satisfying $U_r \subseteq F_r$ and for $r < s$ also $F_r \subseteq U_s$.

Let $f(x) = \inf\{r = m/2^n \in [0,1] \mid x \in U_r\}$, where $f(x) = 1$ if x does not lie in no U_r. In particular, then, $f|_{X \smallsetminus U_1} = 1$; apparently also $f|_{F_0} = 0$. The continuity of the display f follows from the easily verifiable formula $f^{-1}(a,b) = \bigcup_{a<r<s<b}(U_s \smallsetminus F_r)$. □

** **Theorem 9.11** (Tietze)**.** *Let $F \subseteq X$ be a closed subset of the normal space X. Then any $g\colon F \to [0,1]$ can be extended to $f\colon X \to [0,1]$.*

Proof. A bit more symmetrical is the case of a function with values in the interval $[-1,1]$. Let us define successive approximations of the expansion of f, where $f = \lim_n f_n$. Consider the closed sets $F = g^{-1}[-1,-1/3]$ and $G = g^{-1}[1/3,1]$ and choose an arbitrary function $f_1\colon X\ to[-1/3,1/3]$ such that $f_1|_F = -1/3$ and $f_1|_G = 1/3$. Then $|g(x) - f_1(x)| \leq 2/3$. In the next step, we similarly try to approximate the function

$$g - f_1\colon F \to [-2/3, 2/3]$$

and again we manage to find $f_2\colon X \to [-2/3^2, 2/3^2]$ such that $|(g(x)-f_1(x))-f_2(x)| \leq 2^2/3^2$... In general, then $f_n\colon X \to [-2^{n-1}/3^n, 2^{n-1}/3^n]$ and $|g(x) - f_1(x) - \ldots - f_n(x)| \leq 2^n/3^n$. Since the sequence of partial sums $f_1 + \cdots + f_n$ is uniformly convergent, the sum $f_1 + f_2 + \cdots$ is a continuous function and by estimation it coincides with g on F. □

** Similarly, the display can be extended to $\mathbb{R}$. This simply follows from the previous two theorems and the homeomorphism $\mathbb{R} \cong (-1,1)$, thanks to which it suffices to extend $g\colon A \to (-1,1)$ to X. Let $f\colon X \to [-1,1]$ be the extension from Tietze's theorem. Then $f^{-1}(\{-1,1\})$ is a closed set disjoint with A and thus a function $\lambda\colon X \to [0,1]$ can be found such that $\lambda = 0$ on $f^{-1}(\{-1,1\})$ and $\lambda = 1$ on A. The sought extension is then $\lambda \cdot f$.

10. Homotopy, fundamental group, covering

Definition 10.1. Let X, Y be topological spaces, $f_0, f_1\colon X \to Y$ continuous mappings. We say that f_0, f_1 are *homotopic* if there exists a continuous ordering $h\colon [0,1] \times X \to Y$ such that $h(0,x) = f_0(x)$, $h(1,x) = f_1(x)$. We mark $f_0 \sim f_1$, or $h\colon f_0 \sim f_1$. The mapping h is called a *homotopy* between f_0 and f_1.

cv **Example 10.2.** Every two mappings $f_0, f_1\colon X \to \mathbb{R}^n$ are homotopic. The same is true for any convex subset of $\mathbb{R}^n$.

Example 10.3 (proof later)**.** The embedding $S^1 \to \mathbb{R}^2 \smallsetminus \{0\}$ is not homotopy with no constant mapping. In the following, we will more or less show that the homotopy classes of the mapping $S^1 \to \mathbb{R}^2 \smallsetminus \{0\}$ are in bijection with $\mathbb{Z}$, while the number corresponding to the mapping f is the so-called winding number f, i.e. number of rotations of f around the origin.

Theorem 10.4. *A homotopy is an equivalence relation on a set of continuous mappings.*

Proof. For the reflexivity of $f \sim f$, it suffices to take the homotopy $(t,x) \mapsto f(x)$ ("constant homotopy"). For the symmetry of $h\colon f_0 \sim f_1 \Rightarrow \overline{h}\colon f_1 \sim f_0$, just take $\overline{h}(t,x) = h(1-t,x)$. If $h\colon f_0 \sim f_1$ and $k\colon f_1 \sim f_2$, then

$$(h * k)(t,x) = \begin{cases} h(2t,x), & t \leq \frac{1}{2} \\ k(2t-1,x), & \frac{1}{2} \leq t \end{cases}$$

is a homotopy of $f_0 \sim f_2$. Its continuity follows from the fact that $[0,\frac{1}{2}] \times X$ and $[\frac{1}{2},1] \times X$ form a finite closed coverage of $I \times X$. □

Theorem 10.5. *If $f_0 \sim f_1\colon X \to Y$ and $g_0 \sim g_1\colon Y \to Z$, then also $g_0 f_0 \sim g_1 f_1\colon X \to Z$.*

Proof. Let h be a homotopy of $f_0 \sim f_1$ and let k be a homotopy of $g_0 \sim g_1$. Then $k(t, h(t,x))$ is a homotopy between $k(0,h(0,x)) = g_0 f_0(x)$ and $k(1,h(1,x)) = g_1 f_1(x)$. □

end of 7. lecture

Definition 10.6. Spaces X, Y are called *homotopically equivalent* if there exist continuous mappings $f\colon X \to Y$, $g\colon Y \to X$ such that $gf \sim \mathrm{id}_X$, $fg \sim \mathrm{id}_Y$. We denote by $X \simeq Y$. The mappings f and g are called *homotopic equivalences.*

cv **Example 10.7.** $\mathbb{R}^n \simeq \{*\}$ holds, i.e. $\mathbb{R}^n$ is downloadable; $\mathbb{R}^n \smallsetminus \{0\} \simeq S^{n-1}$.

Fundamental group (Poincaré). Let $x_0, x_1, x_2 \in X$ be fixed points. As in the proof of transitivity, let us define for the path γ from x_0 to x_1 and the path δ from x_1 to x_2 their *binding*

$$\beta * \gamma(t) = \begin{cases} \beta(2t), & t \in [0, \frac{1}{2}] \\ \gamma(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

This operation is "almost associative", specifically associative up to homotopy. We define the *path homotopy* between γ_0 and γ_1 as the homotopy $h\colon [0,1] \times [0,1] \to X$ satisfying

$$h(0,x) = \gamma_0(x), \quad h(1,x) = \gamma_1(x), \quad h(t,0) = x_0, \quad h(t,1) = x_1.$$

(In other words, all paths between γ_0 and γ_1 start and end at the same points x_0, x_1.)

A special case of paths are loops in x_0, i.e. paths from x_0 to x_0. In such a case, the path homotopy is called *loop homotopy.* We define

$$\pi_1(X, x_0) = \{\text{loops in } x_0\}/\text{loop homotopy}.$$

On $\pi_1(X, x_0)$ we define the operation $[\beta] \cdot [\gamma] = [\beta * \gamma]$.

nd *Remark.* Equivalently, loops can be thought of as continuous mappings $S^1 \cong I/\{0,1\} \to X$ that send $1 \mapsto x_0$. The homotopies of the loops then correspond to the homotopies of the mapping $S^1 \to X$, which send $1 \mapsto x_0$ all the time. We are talking about mappings and homotopies of spaces with a chosen point $(S^1, 1) \to (X, x_0)$.

Theorem 10.8. *The above operation is well defined and specifies a group structure on $\pi_1(X, x_0)$.*

Proof. If h is a homotopy of loops $\beta_0 \sim \beta_1$ and k is a homotopy of loops $\gamma_0 \sim \gamma_1$, then

$$(h * k)(t,s) = \begin{cases} h(2t, s), & t \le \frac{1}{2} \\ k(2t-1, s), & \frac{1}{2} \le t \end{cases}$$

is a homotopy of loops $\beta_0 * \gamma_0 \sim \beta_1 * \gamma_1$.

Associativity: let us define $\alpha * \beta * \gamma$, a loop that goes through all three loops α, β, γ three times faster. Both $\alpha * (\beta * \gamma)$, $(\alpha * \beta) * \gamma$ are some reparameterizations, namely

$$\alpha * (\beta * \gamma) = (\alpha * \beta * \gamma) \circ \varphi_r, \quad (\alpha * \beta) * \gamma = (\alpha * \beta * \gamma) \circ \varphi_l,$$

where $\varphi_r, \varphi_l\colon I \to I$ are concrete continuous mappings, see the figure. Since I is convex, we have $\varphi_r \sim \varphi_l$ and therefore also $\alpha * (\beta * \gamma) \sim (\alpha * \beta) * \gamma$.

hw 14 Why is this a homotopy of loops?

The unit is the constant path $\varepsilon(t) = x_0$, again $\varepsilon * \gamma$ and $\gamma * \varepsilon$ are "reparameterizations" of the γ loop. The inversion is given by the loop $\overline{\gamma}(t) = \gamma(1-t)$.

hw 15 Show that this is indeed an inversion. $\square$

Example 10.9. It holds $\pi_1(\mathbb{R}^n, 0) = \{e\}$.

cv **Example 10.10.** If $x_0, x_1 \in X$ can be connected by a path, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. This isomorphism depends on the choice of path – identify it for the loop (ie, for $x_0 = x_1$).

cv **Example 10.11.** Prove $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Definition 10.12. *Disjoint union* of topological spaces X, Y is a set

$$X \sqcup Y = (\{0\} \times X) \cup (\{1\} \times Y)$$

together with the topology

$$\{W \subseteq X \sqcup Y \mid X \cap W \subseteq X \text{ open}, Y \cap W \subseteq Y \text{ open}\}$$

Remark. Alternatively, the topology is given by $\{U \sqcup V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$.

? *Question.* Use a different notation for disjoint union, such as $X + Y$?

Denoting the inclusions $i: X \to X \sqcup Y$, $j: Y \to X \sqcup Y$, the disjoint union has the following universal property

$$\begin{tikzcd}
X \arrow[dr, "i"] \arrow[drr, bend left, "fi"] & & \\
& X \sqcup Y \arrow[r, dashed, "f"] & Z \\
Y \arrow[ur, "j"] \arrow[urr, bend right, "fj"'] & &
\end{tikzcd}$$

i.e., the mapping $f: X \sqcup Y \to Z$ is continuous if and only if both tapers $fi: X \to Z$, $fj: Y \to Z$ are continuous. This is because $X \cap U = i^{-1}(U)$, $Y \cap U = j^{-1}(U)$. We also denote the representation f by $f = [g, h]$, where $g = fi$, $h = fj$ are its narrowings to the subspaces X, Y.

cv **Example 10.13.** Prove that a topological space is discontinuous if and only if it is homeomorphic to the disjoint union of two nonempty spaces. If $Z = X \sqcup Y$ is a set, then Z has disjoint union topology if and only if X, Y are dual.

Remark. If X, Y have a metric, let's redefine it to a metric on $X \sqcup Y$ using $\operatorname{dist}(x, y) = 1$. This imposes on $X \sqcup Y$ a disjoint union topology.

Definition 10.14. Let X_i, $i \in \mathcal{I}$, be topological spaces. We define disjoint union as $\bigsqcup_{i \in \mathcal{I}} X_i = \bigcup_{i \in \mathcal{I}} \{i\} \times X_i$ along with the topology

$$\{U \subseteq \bigsqcup_{i \in \mathcal{I}} X_i \mid \forall i \in \mathcal{I} : X_i \cap U \subseteq X_i \text{ open}\}$$

The topological space $\bigsqcup_{i \in \mathcal{I}} X_i$ again has the universal property, and again every X_i is an open and closed subspace.

Definition 10.15. A continuous map $p: Y \to X$ is called a *cover* if for every $x \in X$ there exists an open neighborhood $U \ni x$ and a homeomorphism $\varphi: \bigsqcup_{i \in \mathcal{I}} U \xrightarrow{\cong} p^{-1}(U)$ such that it commutes

$$\begin{tikzcd}
\bigsqcup_{i \in \mathcal{I}} U \arrow[rr, "\varphi"] \arrow[dr, "{[\mathrm{id}]}"'] & & p^{-1}(U) \arrow[dl, "p"] \\
& U &
\end{tikzcd}$$

where $[\mathrm{id}] : \bigsqcup_{i\in\mathcal{I}} U \to U$ is the mapping that is the identity on each summation, that is, $p\varphi(i,x) = x$.

cv **Example 10.16.** Prove that $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi it}$ is a covering.

Theorem 10.17 (raising roads)**.** *Let $p\colon Y \to X$ be a covering and $\gamma\colon I \to X$ a path. For each $y_0 \in p^{-1}(\gamma(0))$ there is a unique path $\widetilde{\gamma}\colon I \to Y$ such that $\widetilde{\gamma}(0) = y_0$, $p\widetilde{\gamma} = \gamma$. We call it the* lifting *path γ starting at y_0.*

Proof. From the definition of covering, X is covered by open sets U such that $p^{-1}(U) \cong \bigsqcup U$; denote this coverage by $\mathcal{U}$. Next, consider $\gamma^{-1}(\mathcal{U}) = \{\gamma^{-1}(U) \mid U \in \mathcal{U}\}$, the open coverage of I. By Lebesgue's lemma, there exists $n \in \mathbb{N}$ such that every interval $[\frac{k-1}{n}, \frac{k}{n}]$ lies in some $\gamma^{-1}(U) \in \gamma^{-1}(\mathcal{U})$, i.e. $\gamma[\frac{k-1}{n}, \frac{k}{n}] \subseteq U$. Let us define $\widetilde{\gamma}$ successively on the intervals $[\frac{0}{n}, \frac{1}{n}], \dots, [\frac{n-1}{n}, \frac{n}{n}]$. Let the homeomorphism $\varphi\colon \bigsqcup_{i\in\mathcal{I}} U \cong p^{-1}(U)$ be $\varphi^{-1}(y_0) \in \{i\}\ timesU$. Since $\widetilde{\gamma}[\frac{0}{n}, \frac{1}{n}]$ is continuous and must contain y_0, it must lie in $\varphi(\{i\} \times U)$.

hw 16 to prove the last statement

Since $p\colon \varphi(\{i\} \times U) \xrightarrow{\cong} U$ is a homeomorphism, we are forced to put

$$\widetilde{\gamma}(t) = (p|_{\varphi(\{i\}\times U)})^{-1}\gamma(t).$$

This determines the narrowing of $\widetilde{\gamma}$ to $[\frac{0}{n}, \frac{1}{n}]$ and, in particular, $\widetilde{\gamma}(\frac{1}{n})$, the starting point of the lifting of $\gamma|_{[\frac{1}{n},1]}$. ☐

Theorem 10.18 (homotopy lifting)**.** *Let $p\colon Y \to X$ be a covering, $h\colon I\times P \to X$ a homotopy, and $\widetilde{h}_0\colon P \to Y$ (a partial veduntion) such that $p(\widetilde{h}_0(z)) = h(0,z)$. Then there exists a unique homotopy $\widetilde{h}\colon I \times P \to Y$ such that $\widetilde{h}(0,z) = \widetilde{h}_0(z)$, $p\widetilde{h}(t,z) = h(t,z)$. We call it the* lifting *homotopy of h starting at $\widetilde{h}_0$.*

end of 8. lecture

** *Proof.* By the preceding theorem, for each $z \in P$ there exists a single continuous lift $h(-,z)$ starting at $\widetilde{h}_0(z)$, denote it by $\widetilde{h}(-,z)$. This proves the uniqueness of $\widetilde{h}$, it remains to show its continuity.

Let $z_0 \in P$ be fixed and choose $U_1, \dots, U_n \subseteq X$ open such that $h(-,z_0)[\frac{k-1}{n}, \frac{k}{n}] \subseteq U_k$. Let $\varphi_k\colon p^{-1}(U_k) \cong \bigsqcup U_k$ be a local trivialization and i_k an index such that

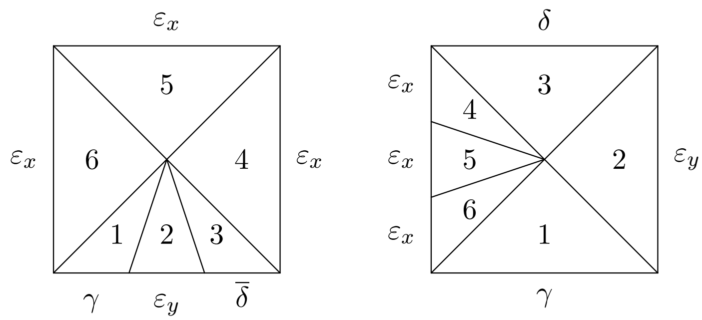
$$\varphi_k\widetilde{h}(-,z_0)[\tfrac{k-1}{n}, \tfrac{k}{n}] \subseteq \{i_k\} \times U_k.$$

It is simply shown that the same relations hold on some neighborhood $N \ni z_0$ (using the tube lemma). But this means that $\varphi_k\widetilde{h}(t,z) = (i_k, h(t,z))$. Since φ_k is a homeomorphism, $\widetilde{h}|_{[\frac{k-1}{n},\frac{k}{n}]\times N}$ is continuous. Due to the fact that the intervals are closed and finitely many, $\widetilde{h}|_{I\times N}$ is also continuous and thus also $\widetilde{h}$. ☐

Definition 10.19. A topological space X is called *simply connected* (1-connected) if it is path connected and $\pi_1(X, x_0) = \{e\}$ for every/some $x_0 \in X$.

Lemma 10.20. *If X is simply continuous and $x, y \in X$, then there exists a path from x to y, unique except for path homotopy.*

Proof. Let γ, δ be two paths from x to y. Then $\gamma * \varepsilon_y * \overline{\delta}$ is a loop in x. Due to the simple connection, it is homotopy to trivial loops. This homotopy can be "rearranged" into a homotopy between γ and δ, see



□

Theorem 10.21. *Let $p\colon Y \to X$ be a covering where Y is simply continuous. Then there is a bijection between $\pi_1(X, x_0)$ and $p^{-1}(x_0)$.*

Proof. Let us fix $y_0 \in p^{-1}(x_0)$. Let's define a view

$$p^{-1}(x_0) \to \pi_1(X, x_0), \quad y \mapsto [p\gamma_y],$$

where γ_y is an arbitrary path from y_0 to y. By definition, it is unique up to the homotopy of paths and thus $p\gamma_y$ is unique up to the homotopy of loops ($p(y_0) = x_0 = p(y)$).

The inverse view is

$$\pi_1(X, x_0) \to p^{-1}(x_0), \quad [\gamma] \mapsto \widetilde{\gamma}(1),$$

where $\widetilde{\gamma}$ is the lift of γ starting at y_0. Let $[\gamma] = [\delta]$, i.e., $\gamma \sim \delta$ as loops, and let h be some such homotopy of loops. By the homotopy lifting theorem, there is a single lifting $\widetilde{h}$ starting at $\widetilde{h}(0, s) = y_0$, so there must be $\widetilde{h}(t, 0) = \widetilde{\gamma}(t)$, $\widetilde{h}(t, 1) = \widetilde{\delta}(t)$ and therefore $\widetilde{h}(1, s)$ is a path from $\widetilde{\gamma}(1)$ to $\widetilde{\delta}(1)$ lying in $p^{-1}(x_0)$. However, since $p^{-1}(x_0)$ is discrete (from the local trivialization of the coverage), this path must be constant and $\widetilde{\gamma}(1) = \widetilde{\delta}(1)$; therefore the display is well defined.

It remains to verify that the above mappings are mutually inverse. But this is clear from the appropriate choice of data with which they are defined. □

Definition 10.22. Let $p : Y \to X$ be any continuous mapping, $y_0 \in Y$ any point, and $x_0 = p(y_0) \in X$ its image. We define *induced display*

$$p_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [p\gamma].$$

Since $p(\gamma * \delta) = (p\gamma) * (p\delta)$, this is a group homomorphism.

hw 17 **Example 10.23.** Prove that for the covering $p\colon Y \to X$ the induced mapping p_* is injective.

** **Theorem 10.24.** *Let $p\colon Y \to X$ be a covering, Y path-continuous. Then there is a bijection between $p_*\pi_1(Y, y_0)\backslash\pi_1(X, x_0)$ and $p^{-1}(x_0)$ (recall that the quotient $H\backslash G$ is the set of classes Hg, $g \in G$).*

cv **Example 10.25.** Compute the fundamental group of the circle $\pi_1(S^1, 1)$ and describe the representatives of all homotopy classes. (By the previous theorem, it is in a bijection with $\mathbb{Z}$; prove that it is actually a group isomorphism. The representatives are given by the mappings $z \mapsto z^n$.)

** **Example 10.26.** The fundamental group $S^1 \vee S^1$, its infinite covering, and an infinitely generated free subgroup of a free group on two generators.

Let $f\colon X \to X$ be the mapping of X into itself. We say that $x \in X$ is a *fixed point* of f if $f(x) = x$.

Theorem 10.27 (Brouwer's theorem in 2 dimension)**.** *Every continuous mapping $f\colon D^2 \to D^2$ has a fixed point.*

Proof. We reduce the proof to the following statement: there is no retraction of D^2 onto S^1, i.e. a continuous mapping $r\colon D^2 \to S^1$ such that $r|_{S^1} = \mathrm{id}$. If r existed, we would get

$$\begin{array}{ccccc} S^1 & \hookrightarrow & D^2 & \longrightarrow & S^1 \\ \\ \pi_1(S^1,1) & \longrightarrow & \pi_1(D^2,1) & \longrightarrow & \pi_1(S^1,1) \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{Z} & & \{e\} & & \mathbb{Z} \end{array}$$

whereas, according to the definition of retraction, the composition is equal to id, and thus $\mathrm{id}_{\mathbb{Z}}$ would factor over $\{e\}$, which is not possible.

It remains to show how Brouwer's theorem follows from the non-existence of retraction – again a controversy. If there were a continuous mapping $f\colon D^2 \to D^2$ without a fixed point, we would make a retraction r from it such that $r(x)$ is the intersection of S^1 with the (open) half-line drawn from $f(x)$ by the point x. One can easily derive a formula for r that proves that it is a continuous mapping. $\square$

Theorem 10.28 (Fundamental theorem of algebra)**.** *Every non-constant polynomial over $\mathbb{C}$ has a root.*

Proof. The basic idea of the proof is that one can count the number of roots (counted according to their multiplicity) inside a given circle. We will show it first on a trivial example of the polynomial $f_n(z) = z^n$. Let us deal with the loop $\gamma(t) = Re^{2\pi it}$, which bounds a circle of radius R. The composition $f_n\gamma\colon I \to \mathbb{R}^2 \smallsetminus \{0\}$ is given by the formula $f_n(\gamma(t)) = Re^{2\pi int}$ and it circles the origin exactly n times; therefore, in the group $\pi_1(\mathbb{R}^2 \smallsetminus \{0\}, 1) \cong \mathbb{Z}$ represents element n. The main idea of the proof will then be that the same is true for any polynomial g of degree n – if all its roots lie inside a circle of radius R, then $g\gamma$ is a loop representing the element n.

Let $g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$. We first limit the possible roots of this polynomial. Let $R > |a_{n-1}| + \cdots + |a_1| + |a_0|$, $R \geq 1$. Then for $|z| = R$ it holds

$$\begin{aligned} |g(z)| &\geq |z^n| - |a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \\ &\geq R^n - \big(|a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0|\big) \\ &\geq R^{n-1}\big(R - (|a_{n-1}| + \cdots + |a_1| + |a_0|)\big) > 0 \end{aligned}$$

and thus g has all its roots inside a circle of radius R.

For the same reason, for $t \in I$ the polynomial $z^n + t(a_{n-1}z^{n-1} + \cdots + a_1z + a_0)$ has roots only inside the circle of radius R . We consider again the loop $\gamma(t) = Re^{2\pi it}$. Then the above homotopy of polynomials determines the homotopy $g\gamma \sim f_n\gamma$ of loops in $\mathbb{R}^2 \smallsetminus \{0\}$. We calculated that the second loop represents the element $n \in \mathbb{Z} \cong \pi_1(\mathbb{R}^2 \smallsetminus \{0\}, 1)$ and the same is therefore true for $g\gamma$.

Now suppose that g has no roots. Then any constant-loop homotopy $h\colon \gamma \sim \varepsilon$ specifies the homotopy of loops $g\gamma \sim \varepsilon$ in $\mathbb{R}^2 \smallsetminus \{0\}$. But this is only possible for $n = 0$. $\square$

Remark. In the proof, we "counted" only the roots inside a sufficiently large circle. Similarly, one can count the roots of g inside any region bounded by the curve $\gamma\colon I \to \mathbb{C}$ as the homotopy class of the loop $g\gamma$ in $\mathbb{R}^2 \smallsetminus \{0\}$. The "continuous dependence" of the roots on the polynomial also follows from this principle. If z_0 is a root of the polynomial g and g' is a polynomial close to g, then g' has a root close to z_0.

——— end of 9. lecture ———

cv **Example 10.29.** Prove that $\pi_1(S^n) = \{e\}$ for every $n > 1$. (Hint: cut each path into connecting paths that lie in the complement of the north/south pole and change each by homotopy to a path with "small image".)

cv **Example 10.30.** Let us consider $\mathcal{P}_n = \mathcal{P}(\mathbb{R}^{n+1})$ the topology of the quotient $S^n/\sim$, $x \sim -x$. Prove that the mapping $S^n \to \mathcal{P}_n$, $x \mapsto [x]$, is a covering and compute $\pi_1(P_n)$, $n > 1$. (Hint: if the open set U lies inside one hemisphere, then for the canonical projection $p\colon S^n \to \mathcal{P}_n$ it holds that $p\colon U \xrightarrow{\cong} p(U)$; due is that p is open.)

hw 18 **Example 10.31.** Generalize the previous to "properly discontinuous" actions (see Bredon, but the name does not fit the classical definition), i.e. actions satisfying: for each point $x \in X$ there exists a neighborhood $U \ni x$ such that $gU \cap U = \emptyset$ for $g \neq e$. In that case, the projection $X \to G\backslash X$ is a covering, and if X is simply continuous, then $\pi_1(G\backslash X) \cong G$.

hw 19 **Example 10.32.** Describe the fundamental group of the Klein bottle as $G\backslash\mathbb{R}^2$.

∗∗ **Example 10.33.** Prove that $[S^1, X] \cong \pi_1(X)/\mathrm{conj}$ holds for a path-connected space X.

11. Simplicial complexes, Brouwer's theorem, dimension invariance

Next, we prove Brouwer's theorem in the general dimension.

Theorem 11.1 (Brouwer's theorem)**.** *Every continuous mapping $D^n \to D^n$ has a fixed point.*

First, let's think about what it says in the 1 dimension. We have $D^1 = [-1, 1]$ and thus we claim that every mapping $f\colon [-1, 1] \to [-1, 1]$ has a fixed point. But this follows from the fact that $f(-1) \geq -1$, $f(1) \leq 1$ and therefore somewhere in the interval $[-1, 1]$ must be $f(x) = x$.

We will prove Brouwer's theorem combinatorially. Therefore, we first need to replace the disk D^n with some combinatorial object. We will use the following theorem for this, in which $\partial X = \overline{X} \smallsetminus \mathring{X}$ is the boundary of X.

Theorem 11.2. *Let $X \subseteq \mathbb{R}^n$ be a compact convex subset with nonempty interior. Then there exists a homeomorphism $h\colon D^n \to X$ such that $h(S^{n-1}) = \partial X$.*

Proof. First, we can achieve that the origin of 0 is an interior point of X by possibly shifting X, which is a homeomorphism.

Let us define the representation $d\colon S^{n-1} \to \mathbb{R}_+$ as

$$d(v) = \max\{t \in \mathbb{R}_+ \mid tv \in X\}.$$

Since 0 is an interior point, the above set is nonempty and, due to compactness, also bounded and closed; therefore the maximum element exists.

An important step will be to show the continuity of the display d. Then we define

$$h'\colon I \times S^{n-1} \to X, \quad h'(t, v) = td(v)v;$$

this apparently sends $\{0\} \times S^{n-1}$ to 0 and thus induces the display

$$h\colon D^n \cong (I \times S^{n-1})/(\{0\} \times S^{n-1}) \to X,$$

which is continuous and by definition also a bijection between compact Hausdorff spaces. This is the desired homeomorphism.

We will first show that $tv \in \mathring{X}$ holds for $t \in \mathbb{R}_+$ and $v \in S^{n-1}$ if and only if $t < d(v)$. This exactly corresponds to the condition $h(S^{n-1}) = \partial X$. Obviously, for an interior point tv, $t < d(v)$. Conversely, an isosceles centered at $d(v)v$ converting 0 to tv sends some sphere $B_\varepsilon(0) \subseteq X$ to some sphere $B_{\varepsilon'}(tv) \subseteq X$ and therefore tv is internal.

Now we prove the continuity of the mapping d. Let $v_n \to v$ be a convergent sequence. Since $d(S^{n-1})$ is bounded, it suffices to prove that every convergent subsequence of $d(v_n)$ converges to $d(v)$. Assume for simplicity that $d(v_n)$ itself converges to some t. Because

$$d(v_n)v_n \to tv$$

and the sequence on the left lies in X, it must also be $tv \in X$ and therefore $t \leq d(v)$. Now suppose that $t < d(v)$. Then tv is an interior point of X and therefore also $d(v_n)v_n$ is an interior point for $n \gg 0$. According to the previous paragraph, however, these are border points, a dispute. □

We now describe our combinatorial D^n disk model.

We say that the points $A_0, \ldots, A_k \in \mathbb{R}^n$ are *affine independent* if $A_1 - A_0, \ldots, A_k - A_0$ are linearly independent vectors. A *simplex* of dimension k (also k-simplex) is called a convex hull

$$s = [A_0, \ldots, A_k] = \{\xi_0 A_0 + \cdots \xi_k A_k \mid \xi_i \geq 0,\, \xi_0 + \cdots + \xi_k = 1\},$$

of affinely independent points $A_0, \ldots, A_k$.

Example 11.3. A standard simplex Δ^k of dimension k is defined as a convex hull $\Delta^k = [e_0, \ldots, e_k]$ of standard basis vectors $\mathbb{R}^{n+1}$. If $s = [A_0, \ldots, A_k]$ is any other k-dimensional simplex, then there exists a single affine mapping $\Delta^k \to s$ sending e_i to A_i and it is homeomorphism. Thus, any two simplexes of dimension k are homeomorphic.

According to the previous theorem, any simplex of dimension n is homeomorphic to D^n – since every two simplexes of dimension n are homeomorphic, this follows from the case of the convex hull of the n-tice of affinely independent points in $\mathbb{R}^n$ (it has a non-empty interior).

The *Wall* of the simplex s is $[A_{i_0}, \ldots, A_{i_\ell}]$, where $0 \leq i_0, \ldots, i_\ell \leq k$ (and we can assume that these indexes are mutually different and ordered). The combinatorial *interior* of s is

$$\operatorname{int}_c s = \{\xi_0 A_0 + \cdots \xi_k A_k \mid \xi_i > 0,\, \xi_0 + \cdots + \xi_k = 1\}$$

and the combinatorial *boundary* then $\partial_c s = s \smallsetminus \operatorname{int}_c s$.

Simplicial complex K in $\mathbb{R}^n$ is a finite set of simplexes such that

- every simplex face of K lies in K,
- if s, t are two simplexes from K, then $s \cap t$ is their common face.

The *body* of the complex K is the set $|K| = \bigcup K = \bigcup_{s \in K} s$. A subset $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if there exists a simplicial complex K such that $P = |K|$. The simplicial complex K is then called the *triangulation* of the polyhedron P.

Example 11.4. A square has a triangulation consisting of two triangles. The cube is more complicated – it has a triangulation consisting of six tetrahedra.

Subdivision L of the complex K is a simplicial complex such that $|L| = |K|$ and every simplex $s \in L$ lies in some simplex $t \in K$, i.e. $s \subseteq t$. *Barycentric subdivision* $\operatorname{sd} K$ is defined as follows: $\operatorname{sd}[A_0, \dots, A_k]$ is given by all faces of k-simplexes

$$[A_{i_0}, \tfrac{1}{2}(A_{i_0} + A_{i_1}), \dots, \tfrac{1}{k+1}(A_{i_0} + \dots + A_{i_k})],$$

where $i_0, \dots, i_k$ is an arbitrary permutation of $0, \dots, k$; the barycentric subdivision of $\operatorname{sd} K$ is the union of all $\operatorname{sd} s$, $s \in K$.

The *Fineness* of triangulation K is $\mu(K) = \max\{\operatorname{diam} s \mid s \in K\} = \max\{\operatorname{dist}(A, B) \mid [A, B] \in K\}$, i.e. the largest diameter of the simplex K or, equivalently, the largest distance of points connected by an edge.

Lemma 11.5. $\mu(\operatorname{sd} K) \le \frac{\dim K}{\dim K+1} \mu(K)$.

Proof. In the proof, we will use the following observation several times: the greatest distance of a point from a point of a simplex is always realized in some vertex of this simplex.[2]

It is apparently enough to prove the statement for the simplex $s = [A_0, \dots, A_k]$. Let t be the simplex of its barycentric subdivision. The diameter of t is the maximum length of an edge between its vertices. If this edge does not contain $\frac{1}{k+1}(A_0 + \dots + A_k)$, it is a simplex barycentric subdivision of some wall s and we can use induction with respect to dimension k.

The maximum distance $\frac{1}{k+1}(A_0 + \dots + A_k)$ from the vertices of the barycentric subdivision occurs for some vertex A_i and thus

$$\mu(\operatorname{sd} s) \le \max\{\operatorname{dist}(A_i, \tfrac{1}{k+1}(A_0 + \dots + A_k)) \mid i = 0, \dots, k\},$$

where $\operatorname{dist}(A_i, \frac{1}{k+1}(A_0 + \dots + A_k)) = \frac{k}{k+1} \operatorname{dist}(A_i, \frac{1}{k}(A_1 + \dots \widehat{A_i} \dots + A_k))$ and this is bounded by $\frac{k}{k+1}$ times the distance A_i from one of the vertices A_j, $j \ne i$. □

Theorem 11.6 (Sperner's lemma)**.** *Let T be the triangulation of the simplex $[A_0, \dots, A_n]$ and let φ be the mapping of the set of vertices T into the set $\{0, \dots, n\}$ such that for $B \in [A_{i_0}, \dots, A_{i_k}]$ is $\varphi(B) \in \{i_0, \dots, i_k\}$. Then the number of simplexes $[B_0, \dots, B_n] \in T$ such that $\varphi\{B_0, \dots, B_n\} = \{0, \dots, n\}$ is odd.*

———————— end of 10. lecture ————————

[2] The greatest distance of a point X from s is the same as the greatest distance of X' from s, where X' is the projection of X into the linear subspace L generated by s. By the proof of the previous theorem (s has a nonempty interior inside L), this is maximal for points from the boundary. Next, induction is used.

Proof. By induction with respect to n. Let's mark

S_0 set of n-simplexes whose vertices are marked by exactly all $0, \dots, n-1$, $p_0 = \#S_0$;
S_1 set of n-simplexes whose vertices are marked by exactly all $0, \dots, n$, $p_1 = \#S_1$;
R the set of $(n-1)$-simplexes whose vertices are marked by exactly all $0, \dots, n-1$, $x = \#R$.

Let us count the number of (s, r) pairs in two ways, where s is an n-simplex and r is its face of dimension $n-1$ with property $r \in R$.

Let us first deal with how many ways s can be chosen. The latter obviously lies either in S_0 or in S_1. In the first case, the wall r can be selected in exactly two ways, in the second case, in exactly one way. Therefore, the number of (s, r) pairs is equal to $2p_0 + p_1$.

Now let's divide the same number according to the options to choose the simplex r. This can be chosen in exactly x ways, while each such simplex lies in exactly two[3] n-simplexes s with the exception of those r that lie on the boundary of the simplex $[A_0, \dots, A_n]$. Thanks to the condition on marking the vertices in the walls, r then lies in the wall $[A_0, \dots, A_{n-1}]$ and according to the induction assumption, the number y of such simplexes is odd. So the number of pairs is equal

$$2p_0 + p_1 = 2x - y$$

and thus $p_1 = 2(x - p_0) - y$ is odd. □

Proof of Brouwer's Theorem. According to the previously performed reduction, it suffices to show the non-existence of the retraction D^n on S^{n-1}. Thanks to $\Delta^n \cong D^n$, converting $\partial_c \Delta^n$ to S^{n-1}, then it suffices to show the non-existence of the retraction $r \colon \Delta^n \to \partial_c \Delta^n$. From the possible retraction, we will construct a vertex label $\operatorname{sd}^N \Delta^n$, $N \gg 0$, which will contradict Sperner's lemma. The label $\varphi(B)$ of the vertex $B \in \operatorname{sd}^N \Delta^n$ is given by the index i of any e_i for which in the expression

$$r(B) = \xi_0 e_0 + \cdots + \xi_n e_n$$

is ξ_i the largest of all (there may be more of them – in that case we choose any of them; the vertex e_i is closest to the point $r(B)$). The idea of the proof is that, with sufficiently fine triangulation, the images of simplexes will be small and cannot have vertices marked by all $0, \dots, n$.

We now define the open coverage $U_0, \dots, U_n$ of the boundary $\partial_c \Delta^n$ by the prescription

$$U_i = \{\xi_0 e_0 + \cdots \xi_n e_n \mid \xi_i < \tfrac{1}{n+1}\}$$

Since ξ_j are non-negative and their sum is 1, the only point Δ^n not belonging to $U_0 \cup \cdots \cup U_n$ is the barycenter $\frac{1}{n+1}(e_0 + \cdots + e_n)$, which, however, does not lie on the boundary. At the same time, it is also obvious that for $r(B) \in U_i$, e_i is not the closest vertex to $r(B)$. By Lebesgue's lemma, there exists $\varepsilon > 0$ such that every subset of mean ε lies in some of $r^{-1}(U_0), \dots, r^{-1}(U_n)$. Let us choose $N \gg 0$ such that $\mu(\operatorname{sd}^N \Delta^n) \le \varepsilon$. Then, for $s \in \operatorname{sd}^N \Delta^n$,

[3] Formally, this follows from the fact that every n-simplex s having r as a wall lies in exactly one of the half-spaces determined by r. Thus, if there is only one such s, r cannot lie inside $[A_0, \dots, A_n]$. If two such n-simplexes s, s' lie in the same half-space, they intersect at some interior point; therefore, this possibility cannot occur in the simplicial complex.

$r(s) \subseteq U_i$ will hold for some i and, in particular, all vertices of s will be evaluated by numbers from the set $\{0, \ldots, i-1, i+1, \ldots, n\}$. To dispute Sperner's lemma, it is sufficient to verify the boundary condition. So let $B \in \operatorname{sd}^N \Delta^n$ be a vertex lying in the wall $[e_{i_0}, \ldots, e_{i_k}]$. Then $r(B) = B = \xi_0 e_0 + \cdots + \xi_n e_n$ with coefficients $\xi_j = 0$ for all $j \notin \{i_0, \ldots, i_k\}$; in particular $\varphi(B) \in \{i_0, \ldots, i_k\}$. □

Theorem 11.7 (on dimension invariance)**.** *If $\mathbb{R}^n \cong \mathbb{R}^m$ are homeomorphic, then $n = m$.*

Let's start with a simple reduction. If $\mathbb{R}^n \cong \mathbb{R}^m$, then the one-point compactifications, $S^n \cong S^m$, will also be homeomorphic. In what follows, we will show that spheres of different dimensions are not even homotopically equivalent.

We say that a mapping $f : X \to Y$ is *trivial* if it is homotopy to a constant mapping. Otherwise, we say it is *substantial.*

Theorem 11.8. *Let Y be a topological space. The continuous mapping $f\colon S^n \to Y$ is trivial just as it can be extended continuously to D^{n+1}.*

cv *Proof.* If f can be extended to $g\colon D^{n+1} \to Y$, the homotopy of f with constant mapping is $h(t, x) = g(tx)$; namely $h(0, x) = g(0) = \text{const}$ and $h(1, x) = g(x) = f(x)$.

Conversely, let $h : I \times S^n \to Y$ be a homotopy between the constant mapping and f. Since $h(0, x)$ is independent of x, we obtain from the universal property of the quotient a continuous representation

$$h'\colon D^{n+1} \cong (I \times S^n)/\!\sim\, \to Y,$$

where $(0, x) \sim (0, x')$. That's the extension you're looking for. □

Theorem 11.9. *Every homeomorphism $f\colon S^n \to Y$ is essential.*

Proof. This is a direct consequence of Brouwer's theorem and the previous theorem. A possible extension of $g\colon D^{n+1} \to Y$ would give a retraction

$$D^{n+1} \xrightarrow{g} Y \xrightarrow{f^{-1}} S^n. \qquad \square$$

In particular, no sphere S^n is contractible, i.e., homotopically equivalent to a one-point space. This is equivalent to the fact that identity is irrelevant.

∗ **Example 11.10.** Prove that every homotopy equivalence $f\colon S^n \to Y$ is essential.

In what follows, we show that every mapping $f\colon S^n \to S^m$, $n < m$, is trivial. According to the previous, then it cannot be a homeomorphism, which proves the dimension invariance theorem.

Definition 11.11. Let K, L be two simplicial complexes. We say that the representation $f : |K| \to |L|$ is *simplicial* with respect to the triangulations K, L, if for any simplex $s = [A_0, \ldots, A_k]\ inK$ holds $[f(A_0), \ldots, f(A_k)] \in L$ and on s f is affine, i.e. holds $f(\xi_0 A_0 + \cdots + \xi_k A_k) = \ xi_0 f(A_0) + \cdots + \xi_k f(A_k)$.

Let us emphasize that the vertices $f(A_0), \ldots, f(A_k)$ do not have to be different, thus we get a simplicial representation of the triangle on the segment. This doesn't quite correspond to the combinatorial definition of a simplicial complex as a set of simplexes of different dimensions that satisfy something – one would expect a simplicial representation to send k-simplexes to

k-simplexes. However, a degree of generality of the definition is needed – otherwise there would be no simplicial mapping of a nontrivial polyhedron to a point.[4]

Theorem 11.12. *(on simplicial approximation) Let K, L be two simplicial complexes and let $f\colon |K| \to |L|$ be a continuous mapping. Then there exists a subdivision K' of the triangulation K such that f is a homotopy mapping $g\colon |K'| \to |L|$ that is simplicial with respect to the triangulations K', L.*

Before the actual proof of the simplicial approximation theorem, let's prove the dimensional invariance theorem.

Proof of the dimension invariance theorem. As said, it suffices to show that every mapping $f: S^n \to S^m$, $n < m$, is trivial. Thanks to the homeomorphisms $S^n \cong \partial\Delta^{n+1}$, $S^m \cong \partial\Delta^{m+1}$, it suffices that every mapping $f'\colon \partial\Delta^{n+1} \to \partial\Delta^{m+1}$, $n < m$, is irrelevant. By the simplicial approximation theorem, f' is homotopy to the simplicial mapping $g'\colon |K| \to |L|$. Since g' is affine on every simplex, its image is the union of simplexes of dimension at most n and, in particular, g' is not surjective (since L has a larger dimension). The backward transition to spheres is f a homotopy mapping of g which is not surjective and hence

$$g\colon S^n \to S^m \smallsetminus \{P\} \hookrightarrow S^m.$$

Since $S^m \smallsetminus \{P\} \cong \mathbb{R}^m$ is contractible, the first mapping is homotopy to constant and the same is therefore also true for the composition g. $\square$

We now return to the proof of the simplicial approximation theorem. Let us denote the vertex A of the triangulation K by its *open star*

$$\operatorname{st}(A) = \bigcup_{[A,A_1,\ldots,A_k]\in K} \operatorname{int}_c[A, A_1, \ldots, A_k],$$

i.e. union of the interiors of all simplexes containing A.

hw 20 Prove that $\operatorname{st}(A) \subseteq |K|$ is an open neighborhood of A.

Since $\bigcap_{i=0}^{k} \operatorname{st}(A_i)$ is the union of the interiors of those simplexes that contain all the vertices $A_0, \ldots, A_k$, this intersection is nonempty if and only if $[A_0, \ldots, A_k] \in K$. This will come in handy in the proof.

Proof of the simplicial approximation theorem. The simplistic mapping g is uniquely determined by its values at the vertices of the triangulation K', which will be a multiple barycentric subdivision, $K' = \operatorname{sd}^N K$. The homotopy between f and g will be linear – so we need that for each $x \in |K|$ $f(x)$, $g(x)$ lie in the same simplex L.

Let $N \gg 0$ be such that the open star of each vertex $A \in K' = \operatorname{sd}^N K$ is mapped by f to the open star of some vertex $B \in L$. This is possible because $\{\operatorname{st}(B) \mid B \in L\}$ is an open covering of $|L|$, so $\mathcal{U} = \{f^{-1}(\operatorname{st}(B)) \mid B \in L\}$ open coverage $|K|$, and $\operatorname{st}(A) \subseteq B_{\mu(K')}(A)$. So it is enough to choose N so that the fineness of $\mu(K')$ is smaller than the Lebesgue number of the open coverage $\mathcal{U}$.

[4] Formally, this "problem" can be circumvented by also considering formal "degenerate" k-simplexes $[A_0, \ldots, A_k]$ for which we do not require the vertices to be different (but we still want the set $\{A_0, \ldots, A_k\}$ to be affinely independent). These considerations lead to the so-called simplicial sets.

Now we can define $g(A)$. Choose a vertex $B \in L$ arbitrarily such that $f(\mathrm{st}(A)) \subseteq \mathrm{st}(B)$ and set $g(A) = B$. Let $[A_0, \ldots, A_k] \in K'$. Then

$$f(\mathrm{int}_c[A_0, \ldots, A_k]) \subseteq \bigcap_{i=0}^{k} \mathrm{st}(g(A_i))$$

and the intersection on the right is therefore nonempty; but this means that $[g(A_0), \ldots, g(A_k)] \in L$ and g is really simplicial. Let x lie inside $[A_0, \ldots, A_k]$. According to the previous one, then $f(x)$ lies in the interior of some simplex s containing $g(A_0), \ldots, g(A_k)$. Since $g(x)$ lies inside the simplex $[g(A_0), \ldots, g(A_k)]$, which is the wall of s, the segment connecting $f(x)$, $g(x)$ lies in $s \subseteq |L|$ and the linear homotopy between f and g really has values in $|L|$. □

end of 11. lecture

12. Compactly generated Hausdorff spaces

Definition 12.1. A topological space X is called *compactly generated Hausdorff (CGH)* if it is Hausdorff and for a subset $A \subseteq X$: if for every compact $C \subseteq X$ the intersection $C \cap A$ open in C, then A is open.

In what follows, we will call the set A for which $C \cap A \subseteq C$ is open *compactly open*. A *compactly closed* set is defined analogously. Thus, X is compactly generated if every compactly open set is open.

Example 12.2. Every locally compact Hausdorff space X is compactly generated: let U be compactly open and $x \in U$. There exists a compact neighborhood $C \ni x$ and by the definition of compact openness, $C \cap U \subseteq C$ is open, that is, by the intersection of $C \cap V$ with the open set $V \subseteq X$. Since both C, V are neighborhoods of x, $C \cap U = C \cap V$ is also a neighborhood of x and even more $U \supseteq C \cap U$.

Let X be a Hausdorff space. Let kX denote the set X together with the topology given by the system of compact open subsets.

cv **Exercise 12.3.** The mapping $f\colon kX \to Y$ is continuous if and only if $f\colon X \to Y$ is continuous on every compact subset.

Lemma 12.4. *The space kX is a compactly generated Hausdorff space.*

Proof. Since there are more open sets in kX, it is apparently a Hausdorff space. We show that it has the same compact subspaces. The identical mapping $kX \to X$ is continuous and therefore every compact subspace of kX is also compact in X. Conversely, let $C \subseteq X$ be compact. According to the exercise, the composition $C \to X \to kX$ is continuous (since $\mathrm{id}\colon kX \to kX$ is continuous), so its image is a compact set. The two possible topologies on C as a subspace of X and as a subspace of kX are identical because both identities on C are continuous. Therefore, the compactly open sets X and kX are the same and therefore kX is compactly generated (a compactly open subset of kX is compactly open in X, thus open in kX). □

Definition 12.5. Let Y, Z be topological spaces. On a set of continuous views

$$Z^Y = \{f\colon Y \to Z \mid f \text{ continuous}\}$$

let's define a *compact-open* topology using a subbase

$$M(C,U) = \{f \in Z^Y \mid f(C) \subseteq U\},$$

where $C \subseteq Y$ is compact and $U \subseteq Z$ is open.

$*$ **Example 12.6.** Let Y be a compact Hausdorff space and Z a metric space. Let us define the *Uniform Convergence Metric* on Z^Y using a prescription

$$\operatorname{dist}(f,g) = \max\{\operatorname{dist}(f(y), g(y)) \mid y \in Y\}.$$

This metric specifies exactly the compact-open topology on Z^Y.

More generally, for a locally compact Hausdorff space Y, the compact-open topology on Z^Y is given by uniform convergence on compact subsets.

To represent $f\colon X \times Y \to Z$, let us define $f^\flat\colon X \to Z^Y$, $f^\flat(x)(y) = f(x,y)$. Conversely, for $g\colon X \to Z^Y$, let us define $g^\sharp\colon X \times Y \to Z$, $g^\sharp(x,y) = g(x)(y)$.

We define $X \times_k Y = k(X \times Y)$. The importance of this construction lies in the following sentence.

Theorem 12.7 (about adjunction)**.** *Let X, Y be compactly generated Hausdorff spaces and Z an arbitrary space. Then the mapping $f\colon X \times_k Y \to Z$ is continuous if and only if the mapping $f^\flat\colon X \to Z^Y$ is continuous.*

Proof. The continuity of $f^\flat$ just needs to be verified on each compact subset of $C \subseteq X$ and can be expressed as follows. Let $M(D,U)$ be a subbasic set. Then

$$\{x \in C \mid \forall y \in D\colon f(x,y) \in U\}$$

is open in C. This follows from the continuity of $f\colon X \times Y \to Z$ on $C \times D$ and from the compactness of D using the "tube lemma".

The continuity of f is equivalent to the continuity of $f\colon X \times Y \to Z$ on every compact set $C \times D \subseteq X \times Y$. Let $U \subseteq Z$ be open and let $f(x,y) \in U$. Since $f(x,-)\colon Y \to Z$ is continuous and $D \subseteq Y$ is locally compact, there exists a compact neighborhood $y \in D' \subseteq D$ such that $f(x,D') \subseteq U$. This means that $x \in (f^\flat)^{-1}(M(D',U))$ and from the continuity $f^\flat$ there is a neighborhood $x \in C' \subseteq C$ such that $f^\flat(C') \subseteq M(D',U)$, i.e. $f(C' \times D') \subseteq U$. Thus, f is continuous on $C \times D$. $\square$

Remark. The continuity $f^\flat$ is equivalent to the continuity $f^\flat\colon X \to k(Z^Y)$. This means that the category of compactly generated Hausdorff spaces is Cartesian closed (since $X \times_k Y$ is a product in this category and $k(Z^Y)$ is an object of functions).

Let $\sim$ be an equivalence relation on a compactly generated Hausdorff space X such that $X/{\sim}$ is again Hausdorff. Then the Hausdorff is compactly generated. This follows from the fact that the projection $X \to X/{\sim}$ induces a continuous mapping $X = kX \to k(X/{\sim})$. But since $X/{\sim}$ is the largest topology for which this mapping is continuous, and $k(X/{\sim})$ has more open sets, $k(X/{\sim}) must be = X/{\sim}$, i.e. $X/{\sim}$ is compactly generated Hausdorff.

Corollary 12.8. *Let $\sim$ be an equivalence relation on a compactly generated Hausdorff space X such that $X/\sim$ is also Hausdorff. Then there is a homeomorphism*

$$(X \times_k Y)/\sim \cong (X/\sim) \times_k Y.$$

Proof. The continuous mapping $(X \times_k Y)/\sim \to (X/\sim) \times_k Y$ is equivalently given as $X \times_k Y \to (X/\sim) \times_k Y$ respecting the relation. So just take $p \times \mathrm{id}$, where $p\colon X \to X/\sim$ is the canonical projection. It then follows from the continuity that $(X \times_k Y)/\sim$ is also a Hausdorff space.

In the opposite direction, the continuous mapping $(X/\sim) \times_k Y \to (X \times_k Y)/\sim$ is equivalently given as $X/\sim \to ((X \times_k Y)/\sim)^Y$, thus as the relation-respecting mapping $X \to ((X \times_k Y)/\sim)^Y$, and thus as the mapping $X \times_k Y \to (X \times_k Y)/\sim$ respecting the session. Just take the canonical projection. □

An important special case is when Y is locally compact.

Proposition 12.9. *Let X be a compactly generated Hausdorff, Y a locally compact Hausdorff. Then $X \times_k Y = X \times Y$.*

Proof. Let $A \subseteq X \times Y$ be compactly open and $(x_0, y_0) \in A$. Then also $(\{x_0\} \times Y) \cap A$ is compactly open in $\{x_0\} \times Y \cong Y$ and due to the compact generation of Y is open. Thus, there exists a compact neighborhood $y_0 \in D \subseteq Y$ with property $\{x_0\} \times D \subseteq A$. Consider a set

$$U = \{x \in X \mid \{x\} \times C \subseteq A\} \subseteq X.$$

We show that U is compactly open, that is, open – if $C \subseteq X$ is compact, then $(C \times D) \cap A$ is open in $C \times D$; $C \cap U$ is then open according to the "tube lemma". Therefore $(x_0, y_0) \in U \times D \subseteq A$. Since (x_0, y_0) was arbitrary, A is open.

An alternative proof consists of the following: the representation $\mathrm{in}\colon X \to (X \times_k Y)^Y$ is continuous by the adjoint theorem and the evaluation $\mathrm{ev}\colon Z^Y \times Y \to Z$, $(f, y) \mapsto f(y)$, is continuous thanks to a very simple argument (if $U \ni f(y)$ is open, then from the continuity of f and the local compactness of Y there exists a compact neighborhood $C \ni y$; then $M(C, U) \times C$ is the neighborhood of (f, y) that appears in U). Therefore, the composition is also continuous

$$X \times Y \xrightarrow{\mathrm{in} \times \mathrm{id}_Y} (X \times_k Y)^Y \times Y \xrightarrow{\mathrm{ev}} X \times_k Y$$

and therefore $X \times Y$ is compactly generated. □

13. Algebras of continuous functions

Recall that the (associative, with unit) $\mathbb{C}$-*algebra* A is a vector space over $\mathbb{C}$ together with the bilinear mapping $A \times A \to A$ that makes A Circle. The mapping $\mathbb{C} \to A$, $z \mapsto z1$, is then a circle homomorphism. Conversely, every homomorphism of circuits $\iota\colon \mathbb{C} \to A$ imposes on A the structure of the vector space over $\mathbb{C}$ by $za = \iota(z) \cdot a$. One simply verifies that it is a $\mathbb{C}$-algebra if and only if the image ι lies at the center of the circle A. In particular, the commutative $\mathbb{C}$-algebra is exactly a homomorphism of the circles $\mathbb{C} \to A$.

The $\mathbb{C}$-algebra homomorphism $\Phi: A \to B$ is a homomorphism of circuits that is also linear. Equivalently, it commutes the diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[dl] \arrow[dr] & \\ A \arrow[rr, "\Phi"'] & & B \end{tikzcd}$$

Let X be a compact Hausdorff space. Let's define

$$C(X) = \{f\colon X \to \mathbb{C} \text{ continuous}\}.$$

Together with addition and multiplication functions, this is a circuit. The embedding of constant functions is a homomorphism $\mathbb{C} \to C(X)$ and is therefore a $\mathbb{C}$-algebra.

Theorem 13.1. *There is a natural bijection between the points of X and the maximal ideals of $C(X)$.*

Proof. We first describe the maximal ideals corresponding to the points X. Let $x \in X$. We define

$$\mathfrak{m}_x = \{f \in C(X) \mid f(x) = 0\}.$$

Since $\mathfrak{m}_x$ is the kernel of a surjective homomorphism of $\mathbb{C}$-algebras (especially circles)

$$\mathrm{ev}_x\colon C(X) \to \mathbb{C}, \quad f \mapsto f(x)$$

and $\mathbb{C}$ is a solid, is $\mathfrak{m}_x = \ker \mathrm{ev}_x$ really a maximal ideal.

hw 21 Show that the assignment $x \mapsto \mathfrak{m}_x$ is injective.

It remains to show that every maximal ideal is of the form $\mathfrak{m}_x$ for some $x \in X$. Suppose by argument that $I \subsetneq C(X)$ is a maximal ideal different from $\mathfrak{m}_x$. Then there exists $f_x \in I \smallsetminus \mathfrak{m}_x$. By multiplying by the complex combined function $\overline{f_x}$, we get the non-negative function $g_x = \overline{f_x} f_x \in I$ with the property $g_x(x) > 0$. Let us set $U_x = \{y \in X \mid g_x(y) > 0\}$. We thus obtain the open coverage $\mathcal{U} = \{U_x \mid x \in X\}$. Thanks to its compactness

$$X = U_{x_1} \cup \dots \cup U_{x_n}$$

and the function $g = g_{x_1} + \dots + g_{x_n}$ is positive on all X. Therefore g^{-1} exists and I contains $1 = g^{-1}g$ and cannot be maximal. □

Remark. The previous theorem does not hold without the compactness condition of X. Ideal

$$I = \{f\colon \mathbb{R}^n \to \mathbb{C} \mid \exists C \text{ is compact: } f = 0 \text{ on } \mathbb{R}^n \smallsetminus C\}$$

does not lie in any maximal ideal $\mathfrak{m}_x$. Thus, it must lie in some maximal ideal different from $\mathfrak{m}_x$ and, in particular, such maximal ideals of $\mathfrak{m}$ exist. Note also that the dimension $C(X)/\mathfrak{m}$
** will always be infinite. (Assume this dimension is finite and set $f(x) = |x|$. Then $[1, f, f^2, \dots]$ has infinite dimension – $\sum a_n f^n(x) = 0$ only for $f(x)$ with a root of $\sum a_n z^n$, there are finitely many of them and therefore the equality cannot hold for all x. Therefore, there must be some $h = \sum a_n f^n \ in \mathfrak{m}$ and the zero set $Z(h)$ is compact; we proceed as in the proof of the theorem.)

Thus, X can be reconstructed as a set from the algebra $C(X)$; let us now show how the topology can be reconstructed. If $I \subseteq C(X)$ is an arbitrary ideal, is a set

$$Z(I) = \bigcap_{f \in I} f^{-1}(0) = \{x \in X \mid \forall f \in I \colon f(x) = 0\}$$

closed (this is the set of common zeros of the ideal I).

hw 22 Show that every closed set arises in this way from some ideal.

Theorem 13.2. *There is a natural bijection between continuous mappings $\varphi \colon X \to Y$ and $\mathbb{C}$-algebra homomorphisms $\Phi \colon C(Y) \to C(X)$.*

Proof. Let $\varphi : X \to Y$ be a continuous mapping. Let us define the homomorphism $\mathbb{C}$-algebra $\varphi^* : C(Y) \to C(X)$ by the prescription $\varphi^*(f) = f \circ \varphi$. We now show that every homomorphism $\mathbb{C}$-algebra $\Phi \colon C(Y) \to C(X)$ is of this form. Let $x \in X$ and consider the ideals $\mathfrak{m}_x \subseteq C(X)$, $\Phi^{-1}(\mathfrak{m}_x) \subseteq C(Y)$. The induced mapping $C(Y)/\Phi^{-1}(\mathfrak{m}_x) \to C(X)/\mathfrak{m}_x$ is clearly injective. Both quotients contain $\mathbb{C}$ as a subbody, and this is fixed by this representation. Since $C(X)/\mathfrak{m}_x$ is equal to $\mathbb{C}$, this is an isomorphism. Therefore, $\Phi^{-1}(\mathfrak{m}_x)$ is also maximal and $\Phi^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{\varphi(x)}$ for some $\varphi(x)\ inY$. This determines the representation of $\varphi \colon X \to Y$.

end of 12. lecture

It remains to show that φ is continuous and that $\Phi = \varphi^*$. Let $x_0 \in X$, $y_0 = \varphi(x_0)$; then $\mathfrak{m}_{y_0} = \Phi^{-1}(\mathfrak{m}_{x_0})$, so $\Phi(\mathfrak{m}_{y_0}) \subseteq \mathfrak{m}_{x_0}$. Let's do the math

$$\Phi(f) = \Phi(\underbrace{f(y_0)}_{\text{const}} + (f f(y_0))) = f(y_0) + \Phi(\underbrace{f f(y_0)}_{\in \mathfrak{m}_{y_0}}) \in f(y_0) + \mathfrak{m}_{x_0},$$

i.e. $\Phi(f)(x_0) = \mathrm{ev}_{x_0} \Phi(f) = \mathrm{ev}_{x_0} f(y_0) = f(y_0) = f(\varphi(x_0)) = (\ varphi^* f)(x_0)$. Since $x_0 \in X$ was arbitrary, we have $\Phi(f) = \varphi^* f$, i.e. $\Phi = \varphi^*$.

For continuity, just note that $Z(I) = \{y \in Y \mid I \subseteq \mathfrak{m}_y\}$. Then $Z(\Phi(I))$ is the set of those $x \in X$ for which

$$\Phi(I) \subseteq \mathfrak{m}_x \;\Leftrightarrow\; I \subseteq \Phi^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{\varphi(x)} \;\Leftrightarrow\; \varphi(x) \in Z(I),$$

thus $Z(\Phi(I)) = \varphi^{-1}(Z(I))$ and, in particular, $\varphi^{-1}(Z(I))$ is closed. Since every closed set is of the form $Z(I)$, φ is continuous. □

14. Topological groups, Pontryagin duality

Definition 14.1. The topological group G is a Hausdorff topological space and at the same time a group in such a way that group operations

$$\mu \colon G \times G \to G, \quad \mu(x, y) = xy; \quad \nu \colon G \to G, \quad \nu(x) = x^{-1}$$

are continuous.

Let us define a left translation $\lambda_y : x \mapsto yx$ and a right translation $\rho_y : x \mapsto xy$. Both are homeomorphisms since $(\lambda_y)^{-1} = \lambda_{y^{-1}}$ and $(\rho_y)^{-1} = \rho_{y^{-1}}$. Similarly, the inversion ν and conjugation homeomorphisms are $x \mapsto yxy^{-1}$.

Lemma 14.2. *Every open subgroup is closed at the same time. In particular, a subgroup containing some unit neighborhood e contains the entire unit component.*

Proof. The complement of a closed subgroup $H \subseteq G$ is the union $G \setminus H = \bigcup_{x \notin H} xH$, where $xH = \lambda_x(H)$ is open – λ_x is a homeomorphism and H is open; thus $G \setminus H$ is also open and H is indeed closed.

A unit component G_e is a continuous closed subgroup – the images $G_e \cdot G_e = \mu(G_e \times G_e)$, $G_e^{-1} = \nu(G_e)$ are also continuous and contain e , therefore must lie in G_e. If $H \subseteq G$ is an arbitrary subgroup containing some neighborhood $U \ni e$, then it is clearly open – for every $x \in H$ also contains some neighborhood $xU \ni x$. Therefore, the intersection of $G_e \cap H$ is an open subgroup of the continuous group G_e and must therefore be equal to G_e. $\square$

Lemma 14.3. *A subgroup closure is a subgroup. The closure of a normal subgroup is a normal subgroup.*

Proof. If $H \subseteq G$ is a subgroup, then $H \times H \subseteq \mu^{-1}(H)$ holds. It is not difficult to verify[5] that $\overline{H} \times \overline{H} = \overline{H \times H}$ and therefore also $\overline{H} \times \overline{H} \subseteq \mu^{-1}(\overline{H})$, i.e. $\overline{H} \cdot \overline{H} \subseteq \overline{H}$. Together with $\overline{H} \subseteq \nu^{-1}(\overline{H})$, i.e. $\overline{H}^{-1} \subseteq \overline{H}$, this means that $\overline{H}$ is a group. Normality flows in a similar way using conjugations. $\square$

Example 14.4.

hw 23 1. Prove that the Hausdorffness of the topological group follows from the weaker requirement T_1, in fact from the closure of $\{e\}$. (Hint: if U, V are two neighborhoods of e and x, y are two points of G, then $xU \cap yV = \emptyset$, if and only if $x^{-1}y \notin U \cdot V^{-1}$.)

** 2. Every topological group is a regular topological space.

Proposition 14.5. *The quotient G/H of a topological group G by a closed normal subgroup $H \subseteq G$ is a topological group. (Here G/H is equipped with a quotient topology.)*

Proof. Thanks to the previous example, it suffices to show that multiplication and inversion on G/H are continuous, and that G/H is T_1. Let $p : G \to G/H$ denote the canonical projection. Any point G/H is closed because its pattern is the class $xH = \lambda_x(H)$.

First, let us note that the projection p is open – for any open $U \subseteq G$, $p(U) \subseteq G/H$ is also open – namely $p^{-1}(p(U)) = \bigcup_{y \in U} yH = U \cdot H = \bigcup_{x \in H} Ux$.

The continuity of multiplication follows from the following diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, two heads, "p \times p"'] & G \arrow[d, two heads, "p"] \\ G/H \times G/H \arrow[r, dashed, "\mu'"'] & G/H \end{tikzcd}$$

If $W \subseteq G/H$ is open, then $(\mu')^{-1}(W) = (p \times p)(\mu^{-1}(p^{-1}(W)))$ open due to the openness of the view $p \times p$. Inversion continuity is similar but simpler. $\square$

[5] It is true that $\overline{A \times B}$ is the set of mass points $A \times B$, i.e. those (x, y) whose every neighborhood intersects $A\ timesB$. Obviously, it suffices to restrict oneself to an arbitrary basis of the neighborhood, e.g. to the neighborhood of the form $U \times V$. Then the intersection condition of $A \times B$ is exactly $A \cap U \neq \emptyset$ & $B \cap V \neq \emptyset$. This is equivalent to $x \in \overline{A}$ & $y \in \overline{B}$.

The quotient of the group G under the (closed) *nonnormal* subgroup H is just a set, in our case a Hausdorff topological space. We call it *homogeneous space*. Homogeneous spaces are characterized by the following theorem in the case of the compact group G. There is also an extension of this theorem to locally compact groups, but it is technically more demanding.

Proposition 14.6. *Let G be a compact topological group having a continuous action on the Hausdorff space X. Then display*

$$G/G_x \to G(x), \quad gG_x \mapsto gx$$

is a homeomorphism of the quotient G/G_x according to the stabilizer x to the orbit $G(x)$ passing through x.

Proof. The continuity of the mapping $G/G_x \to G(x)$ follows from the universal property of the quotient. Since it is also a bijection and G/G_x is compact and $G(x)$ is Hausdorff, it is a homeomorphism. □

Example 14.7.

cv 1. Show that $\mathrm{GL}_+(n)$, $\mathrm{SO}(n)$ are continuous. (This can also be shown via $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$ and due to the connection S^n – it is convenient that the projection $\mathrm{SO}(n+1) \to S^n$ is open.)

cv 2. $\mathrm{O}(n+1)/\mathrm{O}(n) \cong S^n$.

cv 3. $\mathrm{O}(n)/(\{E\} \times \mathrm{O}(nk)) \cong V_k(\mathbb{R}^n)$.

cv 4. $\mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(nk)) \stackrel{\text{def}}{=} G_k(\mathbb{R}^n)$.

Let G be a locally compact abelian group and define $\Gamma = \widehat{G} = \mathsf{hom}(G, \mathbb{T}) \subseteq \mathbb{T}^G$, i.e., the space of continuous homomorphisms $G \to \mathbb{T}$ into complex units $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Again, this is a locally compact abelian group – its elements are called *characters*. Let us now consider the second dual $\widehat{\Gamma}$. There is a natural display

$$E\colon G \to \widehat{\Gamma}, \quad x \mapsto (\mathrm{ev}_x\colon \chi \mapsto \chi(x)).$$

The essence of Pontryagin duality is that E is an isomorphism of topological groups.

Example 14.8. Valid $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{T}} = \mathbb{Z}$, $\widehat{\mathbb{Z}} = \mathbb{T}$.

Then on G there exists a measure μ defined on the set of Borel subsets $E \subseteq G$, i.e. the smallest σ-algebra containing closed sets, with the following properties

1. is regular, $\mu(E) = \sup\{\mu(C) \mid C \subseteq E \text{ compact}\} = \inf\{\mu(U) \mid U \supseteq E \text{ open}\}$,
2. is translation invariant, $\mu(xE) = \mu(E)$,
3. is not identically zero.

Such a measure is called the Haar measure, it exists and is unique up to a multiple. In the case $G = \mathbb{R}$, the Lebesgue measure is the Haar measure. For a general G , the construction of the Haar measure is performed as follows: an appropriate continuous linear form $C_c(G) \to \mathbb{C}$ is constructed, where $C_c(G)$ are compact carrier functions; according to Riesz's representation theorem, this linear form corresponds to a single measure, while the properties of the measure are derived from the properties of this functional.

The Fourier transform is then defined

$$L^1(G) \to C(\Gamma), \quad \widehat{f}(\chi) = \int_G f(x)\overline{\chi}(x)\mathrm{d}\mu,$$

where $L^1(G)$ is the space of absolutely integrable functions. The inverse Fourier transform is given by

$$L^1(\Gamma) \to C(G), \quad \check{g}(x) = \int_\Gamma g(\chi)\chi(x)\mathrm{d}\nu,$$

where ν is a certain "dual" measure on Γ.

These transformations are mutually inverse on the space of functions that are absolutely integrable even with their square and specify an isometry

$$L^2(G) \xrightarrow{\cong} L^2(\Gamma)$$

(the so-called Plancherel theorem). From these considerations, Pontryagin's duality follows quite simply.

end of 13. lecture

** 15. Paracompact Spaces

Definition 15.1. Let $\mathcal{U}$ be the coverage of the space X. We say that the coverage of $\mathcal{V}$ is a *refinement* of the coverage of $\mathcal{U}$ if every element of $V \in \mathcal{V}$ lies in some $U \in \mathcal{U}$.

We say that a coverage of $\mathcal{V}$ is *locally finite* if every $x \in X$ has a neighborhood of $N \ni x$ that intersects only finitely many $V \in \mathcal{V}$.

We say that a Hausdorff topological space X is *paracompact* if every open covering of it has a locally finite open refinement.

Lemma 15.2. *The union of a locally finite system of closed sets is closed.*

Proof. Let $\mathcal{F}$ be a locally finite system of closed sets and $x \notin \mathcal{F}$. Then some of its neighborhoods $N \ni x$ intersect only finitely many elements of $\mathcal{F}$ and thus $N \cap \bigcup \mathcal{F}$ is closed in N and does not contain x, i.e. $N \smallsetminus \bigcup\ mcF$ is a neighborhood of x and $X \smallsetminus \bigcup \mathcal{F}$ is open. □

Lemma 15.3. *A closed subspace of a paracompact space is paracompact.*

Proof. similar to compact. □

Proposition 15.4. *Every paracompact space is normal.*

Proof. We can prove regularity, then normality can be proved in the same way. Let $x \notin F$ where $F \subseteq X$ is closed. For each $y \in F$, we choose an open neighborhood $U_y \ni y$ such that $x \notin \overline{U_y}$; we thus obtain an open coverage $\{U_y \mid y \in Y\}$ of the set F. Since this is paracompact by the previous lemma, there exists a locally finite open refinement $\mathcal{V}$ of it. Then $\bigcup_{V \in \mathcal{V}} \overline{V}$ is a closed (due to local coequality) neighborhood (containing $\bigcup \mathcal{V}$) of the set F that does not contain x. □

Definition 15.5. The *Carrier* of a continuous function $f : X \to \mathbb{R}$ is the set $\operatorname{supp} f = \overline{f^{-1}(\mathbb{R} \smallsetminus \{0\})}$.

Let $\mathcal{U}$ be an open covering of X. We say that the system of functions $f_\lambda \colon X \to I$, $\lambda \in \Lambda$, is a *unit decomposition* subordinate to $\mathcal{U}$ if $\{\operatorname{supp} f_\lambda \; mid \lambda \in \Lambda\}$ locally finite refinement of $\mathcal{U}$ and holds $\sum_{\lambda\in\Lambda} f_\lambda = 1$.

The sum in the definition makes sense because the system of carriers is locally finite, i.e. around every point this sum is finite. For the same reason, such a sum is always a continuous function.

Theorem 15.6. *Let $\mathcal{U}$ be an open covering of the paracompact space X. Then there exists a unit decomposition subordinate to $\mathcal{U}$.*

Proof. We can assume that $\mathcal{U}$ is a locally finite coverage (eventual transition to refinement). Let $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ and choose a good ordering on the index set Λ. We will construct the decomposition of the unit by transfinite induction. Obviously, it suffices to construct a system of functions f_λ such that $\operatorname{supp} f_\lambda \subseteq U_\lambda$ and $f = \sum_{\lambda\in\Lambda} f_\lambda > 0$ – such a system then it is enough to normalize, i.e. replace each f_λ with the quotient f_λ/f.

For the already constructed functions f_λ, let us denote $V_\lambda = f_\lambda^{-1}(0,1]$. By induction, we will assume that for $i < \lambda$ $\overline{V_i}$ $subseteq U_i$ and $\bigcup_{i<\lambda} V_i \cup \bigcup_{i\geq\lambda} U_i = X$. For this reason,

$$F_\lambda = \Big(\bigcap_{i<\lambda}(X \smallsetminus V_i) \cap \bigcap_{i>\lambda}(X \smallsetminus U_i)\Big) \subseteq U_\lambda$$

and we choose $f_\lambda \colon X \to I$ arbitrarily such that it is 1 on F_λ and has a carrier inside U_λ – this is possible due to the normality of X. It is straightforward to verify that $\sum_{\lambda\in\Lambda} f_\lambda > 0$ as we wish. □

It holds that every locally compact Hausdorff space with a countable topology basis is paracompact (the proof is not difficult). Also, every metric space is paracompact, but the proof of this claim is quite demanding.

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16. Uniform Spaces

Uniform space is another abstraction of metric space. In the topological space, we can compare the proximity of points to the given point x – they are similarly close if they belong to some neighborhood of x. However, we cannot compare the proximity of arbitrary pairs of points as in metric space. The basic “topological” concept in this direction is not a continuous representation, but a uniformly continuous representation. The formalization of this concept is the so-called uniform spaces.

Definition 16.1. Let X be a set. *Uniformity* on X is a system of sets $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ satisfying

1. $\forall U, V \in \mathcal{U} \colon U \cap V \in \mathcal{U}$,
2. $\forall U \in \mathcal{U}, \forall V \in \mathcal{P}(X \times X) \colon V \in \mathcal{U}$,
3. $\forall U \in \mathcal{U} \colon \Delta_X \subseteq U$,
4. $\forall U \in \mathcal{U} \colon U^{-1} \in \mathcal{U}$,
5. $\forall U \in \mathcal{U} \colon \exists V \in \mathcal{U} \colon V \circ V \subseteq U$.

We call the set X together with the uniformity *uniform space.*

Since every $U \in \mathcal{U}$ is a relation on X, we will write xUy instead of $(x, y) \in U$. The first three conditions say that $\mathcal{U}$ is in some sense a neighborhood system of Δ_X. It is more precisely expressed in the following construction. Let X be a uniform space and $x \in X$. We say that N is a neighborhood of the point x if there exists $U \in \mathcal{U}$ such that N is equal to $xU = \{y \in X \mid xUy\}$.

According to property 5. let us choose a sequence $U_n \in \mathcal{U}$ such that $U_0 = U$, $U_n \circ U_n \subseteq U_{n-1}$. Then

$$V = U_1 \cup U_1U_2 \cup U_1U_2U_3 \cup \cdots$$

is an element of $\mathcal{U}$ (for example, because it contains $U_1 \in \mathcal{U}$) and for $y \in xV$ $xU_1 \cdots U_ny$ holds for some n and then for $z \in yU_{n+1}$ holds $xU_1 \cdots U_nyU_{n+1}z$, i.e. xVz. Therefore, xV includes the point y and its neighborhood yU_{n+1} and xV is therefore open, while $xV \subseteq xU$.

Let us now deal with the uniformity relation $\mathcal{U}$ to the induced topology on X in more detail. We first show that each $U \in \mathcal{U}$ is a neighborhood of $\Delta_X \subseteq X \times X$ in the product topology. Let $(x, x) \in \Delta_X$. Let us choose V using properties 4. and 5. such that $V^{-1} \circ V \subseteq U$. Then $yV^{-1}xVz \Rightarrow yUz$ holds for $(y, z) \in xV \times xV$, i.e. $(y, z) \in U$.

Example 16.2. A typical example of a uniform space is the topological group G. Let $N \ni e$ be the unit neighborhood. Let us define the appropriate neighborhood of Δ_G as $U_N = \{(x, y) \mid y^{-1}x \in N\}$. Then let us set $\mathcal{U} = \{U_N \mid N \ni e \text{ unit neighborhood}\}$.

Example 16.3. Let X be a compact Hausdorff space. Let us define uniformity on X using the system of *all* neighborhoods of Δ_X. The only axiom that is not obvious is 5. First, let's realize that due to normality, closed neighborhoods form the basis of all Δ_X neighborhoods. So let U be an arbitrary open neighborhood of Δ_X. It is easy to show that

$$\bigcap\{V \circ V \mid V \text{ is the neighborhood of } \Delta_X\} = \Delta_X,$$

for each $x \neq y$ it is enough to choose a closed neighborhood $N \supseteq \Delta_X$ not containing (x, y); then $V \circ V \not\ni (x, y)$ holds for $V = N \setminus (xN \times \{y\})$.

In other words, the union of U and all complements of sets of the form $V \circ V$ is the whole $X \times X$. By compactness, $X \times X$ is the union of U and finally many such complements, or

$$(V_1 \circ V_1) \cap \cdots \cap (V_n \circ V_n) \subseteq U.$$

Let's put $V = V_1 \cap \cdots \cap V_n$, then $V \circ V \subseteq U$.

uniformly continuous mapping, topological groups, compact T_2, uniformizability is the same as $\mathrm{T}_{3\frac{1}{2}}$.