Katharina Neusser

Global Analysis

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Motivation

Analysis in $\mathbb{R}^n$:

- It is concerned with the study of differentiable/smooth functions

$$f : U \to \mathbb{R}^m, \qquad U \subseteq \mathbb{R}^n \text{ open.}$$

- Sometimes already other domains occurred:
    - Method of Lagrange multipliers to find local extrema of functions $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the condition that $(x, y) \in g^{-1}(0)$ for $g : \mathbb{R}^2 \to \mathbb{R}$.
    - Theorems of Gauß, Green and Stokes: domains called curves and surfaces appear.

Such domains are called submanifolds (with or without boundary) in $\mathbb{R}^n$.

Plan of the course:

- Generalize the differential and integral calculus from open subsets of $\mathbb{R}^n$ to submanifolds of $\mathbb{R}^n$, which leads also naturally to the notion of abstract manifolds.
- Manifolds can be equipped with various geometric structures and as such they become objects of modern differential geometry:
    – Hypersurfaces in $\mathbb{R}^n$ inherit from the inner product in $\mathbb{R}^n$ a Riemannian metric. $\rightsquigarrow$ Riemannian submanifolds of $\mathbb{R}^n$.
    – Riemannian manifolds
    – Symplectic manifolds

- Lie Groups
  - appear as symmetry groups of geometric structures
  - appear in the study of PDEs

Contents

Chapter 1

Smooth Manifolds

1.1 Submanifolds of $\mathbb{R}^n$

We want to identify a class of ‚nice' subsets of $\mathbb{R}^n$, which will be called submanifolds of $\mathbb{R}^n$, on which we can develop a differential and integral calculus as on open subsets of $\mathbb{R}^n$.

For $m \leq n$ consider the inclusion

$$\mathbb{R}^m = \mathbb{R}^m \times \{0\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n. \tag{1.1}$$

Recalling that differentiability is a local concept, we may consider subsets of $\mathbb{R}^n$ that locally have the form of (1.1).

Definition 1.1. A subset $M \subset \mathbb{R}^n$ admits **local m-dimensional trivialisations**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$, an open subset V of $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to V$ such that

$$\phi(U \cap M) = V \cap \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n.$$

We may also consider graphs of smooth functions $g : \mathbb{R}^m \to \mathbb{R}^{n-m}$:

$$\mathrm{gr}(g) := \{(x, g(x)) : x \in \mathbb{R}^m\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n. \tag{1.2}$$

Localising (1.2) yields:

Definition 1.2. A subset $M \subset \mathbb{R}^n$ is **locally the m-dimensional graph of a smooth function**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$, an m-dimensional subspace $W \subset \mathbb{R}^n$, an open subset $V \subset W$ and a smooth function $g : V \to W^\perp$ such that

$$U \cap M = \mathrm{gr}(g) \subset W \oplus W^\perp = \mathbb{R}^n,$$

where $W^\perp = \{x \in \mathbb{R}^n : \langle x, w \rangle = 0 \ \forall w \in W\}$ is the orthogonal compliment of W in $\mathbb{R}^n$ with respect to the standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

We may also consider zero sets of smooth regular functions. A smooth function

$$f : U \to \mathbb{R}^{n-m}, \quad U \subset \mathbb{R}^n \text{ open },$$

is called **regular at** $y \in U$, if the derivative $D_y f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective. It is called **regular**, if f is regular at all points of U. Note that if f is regular at y, then it is so locally around y, since the rank of $D_y f$ is locally constant.

Definition 1.3. A subset $M \subset \mathbb{R}^n$ is **locally the m-dimensional zero set of a regular smooth function**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$ and smooth function $f : U \to \mathbb{R}^{n-m}$ that is regular at x such that

$$M \cap U = f^{-1}(0) = \{y \in U : f(x) = 0\}.$$

Yet another nice class of subsets arise as images of open subsets of $\mathbb{R}^m$ under immersions into $\mathbb{R}^n$:

Definition 1.4. A subset $M \subset \mathbb{R}^n$ admits **local m-dimensional parametrisations**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$, an open subset $V \subset \mathbb{R}^m$ and a smooth map $\psi : V \to U$ such that

- $D_y\psi : \mathbb{R}^m \to \mathbb{R}^n$ is injective for all $y \in V$, and
- ψ induces a homeomorphism onto its image: $\psi : V \cong M \cap U = \mathrm{Im}(\psi)$.

Theorem 1.5. *Assume $M \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$. Then the following are equivalent:*

(a) M admits local m-dimensional trivialisations.

(b) M is locally the m-dimensional zero set of a regular smooth function.

(c) M is locally the m-dimensional graph of a smooth function.

(d) M admits local m-dimensional parametrisations.

The proof is based on the Inverse Function Theorem, which we recall now:

Theorem 1.6 (Inverse Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open subset, $F : U \to \mathbb{R}^n$ a smooth map, and $x \in U$. If the derivative $D_xF : \mathbb{R}^n \to \mathbb{R}^n$ of F at x is an isomorphism, then there exist open neighbourhoods V of x and W of $F(x)$ such that $F(V) = W$ and*

$$F|_V : V \to W$$

is a diffeomorphism.

Proof. See Analysis/Calculus class. □

An immediate corollary is:

Corollary 1.7 (Implicit Function Theorem)**.** *Assume $m \leq n$. Suppose*

$$f : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$$

is a smooth function with $f(0,0) = 0$ and

$$\partial_2 f(0,0) := D_{(0,0)}F|_{\mathbb{R}^{n-m}} : \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$$

is an isomorphism. Then there exists locally a unique solution $g(x)$ of $f(x, g(x)) = 0$ and $x \mapsto g(x)$ is smooth.

Proof. Consider $F : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ given by $F(x,y) = (x, f(x,y))$. Note that F is smooth, $F(0,0) = (0,0)$ and

$$D_{(0,0)}F = \begin{pmatrix} \mathrm{Id}_m & 0 \\ * & \partial_2 f(0,0) \end{pmatrix}$$

is invertible. By Theorem 1.6, F^{-1} exists locally around $(0,0)$ and is smooth. By construction of F, the local inverse F^{-1} is of the form $F^{-1}(u,v) = (u, G(u,v))$ with G smooth. Hence,

$$\begin{aligned} f(x,y) = 0 &\iff F(x,y) = (x,0) \\ &\iff (x,y) = F^{-1}(x,0) = (x, G(x,0)) \\ &\iff y = G(x,0) =: g(x). \end{aligned}$$

□

Proof of Theorem 1.5. We start with showing

(a) $\Longrightarrow$ (b) Assume $x \in M$, $U, V \subset \mathbb{R}^n$ open and $\phi : U \to V$ a diffeommorphism as in Definition 1.1. Set $f := \pi \circ \phi : U \to \mathbb{R}^{n-m}$, where $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ is the natural projection. By construction, $f^{-1}(0) = U \cap M$ and f is smooth. Moreover,

$$D_y f = D_{\phi(y)}\pi \circ D_y\phi = \pi \circ D_y\phi : \mathbb{R}^n \cong \mathbb{R}^n \to \mathbb{R}^{n-m}$$

is surjective for all $y \in U$.

(b) $\Longrightarrow$ (c) Assume $x \in M$ and $f : U \to \mathbb{R}^{n-m}$ as in Definition 1.3. Then $D_x f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective and $\ker(D_x f) =: W \subset \mathbb{R}^n$ an m-dimensional subspace. Identify $\mathbb{R}^n = W \oplus W^\perp$ and write $x = w + w^\perp$. Then $D_x f|_{W^\perp} : W^\perp \to \mathbb{R}^{n-k}$ is an isomorphism. Hence, by Corollary 1.7, there exists open neighbourhoods $V \subset W$ and $V' \subset W^\perp$ of w respectively $w^\perp$ and a smooth function $g : V \to V' \subset W^\perp$ such that

$$M \cap (V \times V') = f^{-1}(0) \cap (V \times V') = \{(v, g(v)); v \in V\}.$$

(c) $\Longrightarrow$ (d) Assume $x \in M$, $U, V \subset W$, and $g : V \to W^\perp$ as in Definition 1.2. Now consider the map $\psi : V \to W \oplus W^\perp = \mathbb{R}^n$ given by $\psi(v) = (v, g(v))$. It is smooth and $\psi(V) = M \cap U$. Moreover, since $\pi_W \circ \psi = \mathrm{Id}$, where $\pi_W : W \oplus W^\perp \to W$ is the natural projection, ψ is a homeomorphism onto its image. Also, for $D_v\psi : W \to W \oplus W^\perp$ one has

$$D_v\psi(w) = (w, D_v g\, w) = (0,0) \iff w = 0.$$

(d) $\Longrightarrow$ (a) Assume $x \in M$, $V \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ open and $\psi : V \to U$ as in Definition 1.4. Without loss of generality we may assume $0 \in V$ and $\psi(0) = x$. Then $W := \mathrm{Im}(D_0\psi) \subset \mathbb{R}^n$ is an m-dimensional subspace and we identify $\mathbb{R}^n = W \oplus W^\perp$. Now define

$$\Phi : V \times W^\perp \to \mathbb{R}^n$$
$$\Phi(v, w) := \psi(v) + w.$$

Note that $\Phi(0,0) = x$ and with respect to the identification $\mathbb{R}^n = W \oplus W^\perp$ the derivative of Φ at $(0,0)$ has the form

$$D_{(0,0)}\Phi = \begin{pmatrix} D_0\psi & 0 \\ 0 & \mathrm{Id}_{W^\perp} \end{pmatrix}.$$

Hence, $D_{(0,0)}\Phi : W \oplus W^\perp \to \mathbb{R}^n$ is an isomorphism and, by Theorem 1.6, there exist open subsets $V_1 \subset V$, $V_2 \subset W^\perp$ and $S \subset \mathbb{R}^n$ with $x \in S$ such that $\Phi : V_1 \times V_2 \to S$ is a diffeomorphism. Since $\psi : V \to U \cap M$ is a homeomorphism, there exists an open subset $\widetilde{S} \subset \mathbb{R}^n$ with $\psi(V_1) = \widetilde{S} \cap M$. Set $\widetilde{U} := U \cap S \cap \widetilde{S} \subset \mathbb{R}^n$, which is an open neighbourhood of x by construction, and define

$$\phi := (\Phi^{-1})|_{\widetilde{U}} : \widetilde{U} \to \phi(\widetilde{U}) := \widetilde{V}.$$

Then ϕ is a diffeomorphism between the open subsets $\widetilde{U} \subset \mathbb{R}^n$ and $\widetilde{V} \subset V_1 \times V_2 \subset V \times W^\perp \subset \mathbb{R}^n$. Moreover, if $y \in M \cap \widetilde{U}$, then in particular $y \in M \cap \widetilde{S}$, which implies that there exists $v_1 \in V_1$ such that $\psi(v_1) = y$. Since $y \in S$, this shows $\phi(y) = (v_1, 0)$. Conversely, if $(v_1, 0) \in \widetilde{V} \cap W$, then $\Phi(v_1, 0) = \psi(v_1) \in \widetilde{U} \cap M$ by definition of ψ. Hence, $\phi(\widetilde{U} \cap M) = \widetilde{V} \cap W$.

$\square$

Definition 1.8. Assume $1 \leq m \leq n$ are integers. A subset $M \subset \mathbb{R}^n$ is called a **(smooth) submanifold of $\mathbb{R}^n$ of dimension** m, if M satisfies any of the equivalent conditions in Theorem 1.5.

Note that as a subset of $\mathbb{R}^n$ a submanifold $M \subset \mathbb{R}^n$ inherits a topology from $\mathbb{R}^n$, namely the subspace topology:

$$U \subset M \text{ is open} \iff U = \widetilde{U} \cap M \text{ for some open subset } \widetilde{U} \subset \mathbb{R}^n.$$

Remark 1.9.

- If one replaces smooth/C^∞ everywhere by C^r for $1 \leq r < \infty$ or by C^ω, one obtains the notion of C^r-submanifolds respectively real analytic submanifolds of $\mathbb{R}^n$.
- Similarly, if one replaces $\mathbb{R}$ by $\mathbb{C}$ and smooth by holomorphic, one obtains complex submanifolds of $\mathbb{C}^n$.
- Replacing C^∞ in Definition 1.1 by C^0 leads to topological submanifolds of $\mathbb{R}^n$. In this case, not all the definitions 1.1–1.4 are equivalent!

Some trivial examples and natural constructions:

Example 1.1 (Open subsets)**.** Any open subset $U \subset \mathbb{R}^n$ is an n-dimensional submanifold of $\mathbb{R}^n$ and all n-dimensional submanifolds of $\mathbb{R}^n$ are of this form. More generally, any open subset of a submanifold in $\mathbb{R}^n$ is again a submanifold (of the same dimension). Note also that of course any open subset of $\mathbb{R}^n$ can be seen as an n-dimensional submanifold of $\mathbb{R}^d$ via the standard inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^d$ for $n \leq d$.

Example 1.2 (Products)**.** If $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds of dimensions m respectively k of $\mathbb{R}^n$ respectively $\mathbb{R}^\ell$, then

$$M \times K \subset \mathbb{R}^n \times \mathbb{R}^\ell = \mathbb{R}^{n+\ell}$$

is an $m+k$ dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^\ell$.

Some non-trivial examples:

Example 1.3. Consider $\mathbb{R}^{m+1}$ equipped with its standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$. Then the m**-dimensional (unit) sphere**

$$S^m := \{x \in \mathbb{R}^m : ||x|| = 1\} \subset \mathbb{R}^{m+1}$$

is the prototypical example of an m-dimensional submanifold of $\mathbb{R}^{m+1}$. For $m = 1$, one gets the unit circle S^1 in $\mathbb{R}^2$. To see this, note that S^m can be described globally as the zero set of the smooth function $f : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \langle x, x \rangle - 1$, i.e. $f^{-1}(0) = S^m$. Since for any $x \in \mathbb{R}^{m+1} \setminus \{0\}$ and $v \in \mathbb{R}^{m+1}$ one has

$$\begin{aligned} D_x f\, v &= \frac{d}{dt}|_{t=0} \langle x + tv, x + tv \rangle - 1 = \frac{d}{dt}|_{t=0} \langle x, x \rangle + 2t \langle x, v \rangle + t^2 \langle v, v \rangle \\ &= 2\langle x, v \rangle, \end{aligned}$$

the derivative $D_x f : \mathbb{R}^{m+1} \to \mathbb{R}$ is surjective by non-degeneracy of $\langle \cdot, \cdot \rangle$. Hence, f is regular.

Example 1.4. For fixed positive real integers $a_1, ..., a_{m+1} \in \mathbb{R}_{>0}$ consider the function

$$\begin{aligned} &f : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R} \\ &f(x_1, ..., x_{m+1}) := \sum_{i=1}^{k} \frac{x_i^2}{a_i^2} - \sum_{i=k+1}^{m+1} \frac{x_i^2}{a_i^2} - 1. \end{aligned}$$

It is smooth and regular. Hence, $f^{-1}(0) := M$ is an m-dimensional submanifold of $\mathbb{R}^{m+1}$. Depending on k, these submanifolds are m**-dimensional ellipsoids or hyperboloids**.

Example 1.5. Consider $\mathbb{C}^m \cong \mathbb{R}^{2m}$ as real vector space. Then

$$T^m := \{z \in \mathbb{C}^m : |z_1| = ... = |z_m| = 1\} \subset \mathbb{R}^{2m}$$

is an m-dimensional submanifold of $\mathbb{R}^{2m}$, since $f^{-1}(0) = T^m$, where $f : \mathbb{C}^m \setminus \{0\} \to \mathbb{R}^m$ is the smooth regular function given by

$$f(z_1, ..., z_m) = (|z_1| - 1, ..., |z_m| - 1).$$

Of course, also

$$T^m \cong \underbrace{S^1 \times ... \times S^1}_{m-times} \subset \underbrace{\mathbb{R}^2 \times ... \times \mathbb{R}^2}_{m-times} = \mathbb{R}^{2m},$$

so T^m is an m-dimensional submanifold of $\mathbb{R}^{2m}$ by Examples 1.3 and 1.2. It is called the **m-dimensional torus**.

Example 1.6. Consider the vector space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Via a choice of basis of $\mathbb{R}^n$,

$$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2},$$

where $M_{n\times n}(\mathbb{R})$ denotes the vector space of real $n \times n$ matrices. Since the determinant $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is continuous (polynomial in the eneries of the matrix), the subset

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) : \det(A) \neq 0\} \subset M_{n\times n}(\mathbb{R})$$

is open and as such an n^2-dimensional submanifold of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Note that $\mathrm{GL}(n, \mathbb{R})$ is also a group with respect to matrix multiplication. It is called the **general linear group**.

In fact, $\det : \mathrm{GL}(n, \mathbb{R}) :\to \mathbb{R}$ is smooth and also regular, since for any $A \in \mathrm{GL}(n, \mathbb{R})$ one has

$$\begin{aligned}(D_A \det)(A) &= \frac{d}{dt}|_{t=0} \det(A + tA)\\ &= \frac{d}{dt}|_{t=0} \det((1 + t)A) = \frac{d}{dt}|_{t=0}(1 + t)^n \det(A) = n \det(A) \neq 0,\end{aligned}$$

which shows that $D_A \det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ is surjective for all $A \in \mathrm{GL}(n, \mathbb{R})$. Hence, also $f := \det -1 : \mathrm{GL}(n, \mathbb{R}) :\to \mathbb{R}$ is a smooth regular function. Therefore,

$$\mathrm{SL}(n, \mathbb{R}) := f^{-1}(0) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det A = 1\} \subset M_{n\times n}(\mathbb{R})$$

is an (n^2-1)-dimensional submanifold of $M_{n\times n}(\mathbb{R})$. It is also a group with respect to matrix multiplication, called the **special linear group**.

Now consider the map

$$\begin{aligned} f &: \mathrm{GL}(n,\mathbb{R}) \to M_{n\times n}(\mathbb{R}) \\ f(A) &:= AA^t - \mathrm{Id} \end{aligned}$$

and set

$$\mathrm{O}(n) := f^{-1}(0) = \{A \in \mathrm{GL}(n,\mathbb{R}) : AA^t = \mathrm{Id}\}.$$

Note that $f(A)^t = f(A)$. Hence, f has values in the subspace $M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ of symmetric $n\times n$-matrices. The function

$$f : \mathrm{GL}(n,\mathbb{R}) \to M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$

is obviously smooth. To see that it is also regular, note that $(A,B) \mapsto AB^t$ is bilinear as a map $M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$. Therefore, for any $A \in \mathrm{GL}(n,\mathbb{R})$ and $B \in M_{n\times n}(\mathbb{R})$, one has $D_A fB = AB^t + BA^t$. So, if $A \in \mathrm{O}(n)$ and $S \in M^{\mathrm{sym}}_{n\times n}(\mathbb{R})$ is arbitrary, then for $B := \frac{1}{2}SA$ one has

$$D_A fB = \frac{1}{2}(\underbrace{AA^t}_{=\mathrm{Id}} S^t + S\underbrace{AA^t}_{=\mathrm{Id}}) = \frac{1}{2}(S^t + S) = S,$$

which shows that $D_A f : M_{n\times n}(\mathbb{R}) \to M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$ is surjective for any $A \in \mathrm{O}(n)$. Therefore, the set $\mathrm{O}(n)$ of orthogonal $n \times n$-matrices is a submanifold of $\mathbb{R}^{n^2}$ of dimension $\frac{n(n-1)}{2}$. It is also closed under matrix multiplication and hence a group, called the **orthogonal group**.

For submanifolds of $\mathbb{R}^n$, we have an obvious notion of defining smooth maps between them:

Definition 1.10. Suppose $M \subset \mathbb{R}^n$ is an m-dimensional submanifold.

- A map $f : M \to \mathbb{R}^\ell$ is **smooth**, if for every point $x \in M$ there exists an open neighbourhood $\widetilde{U}$ of x in $\mathbb{R}^n$ and a smooth function $\tilde{f} : \widetilde{U} \to \mathbb{R}^\ell$ such that $\tilde{f}|_{M\cap\widetilde{U}} = f|_{M\cap\widetilde{U}}$.
- For a k-dimensional submanifold $K \subset \mathbb{R}^\ell$ a map $f : M \to K$ is **smooth**, if it is is smooth as a map $M \to \mathbb{R}^\ell$.