

## Tutorial 2 and 3—Global Analysis

1. For  $i = 1, \dots, n$  let  $(M_i, \mathcal{A}_i)$  be a smooth manifolds. Suppose  $M := M_1 \times \dots \times M_n$  is endowed with the product topology. Then show that

$$\mathcal{A} := \{(U_1 \times \dots \times U_n, u_1 \times \dots \times u_n) : (U_i, u_i) \in \mathcal{A}_i\}$$

defines a smooth atlas on  $M$  and that the projections  $\text{pr}_i : M \rightarrow M_i$  are smooth. Moreover show that, if  $N$  is a smooth manifold and  $f_i : N \rightarrow M_i$  smooth functions, then there exists a unique smooth function  $f : N \rightarrow M$  such that  $\text{pr}_i \circ f = f_i$  and that this property characterizes the smooth manifold structure on  $M$  uniquely.

2. Suppose  $(M_i, \mathcal{A}_i)$  are smooth manifolds for  $i \in I$ , where  $I$  is countable. Consider the disjoint union

$$M := \sqcup_{i \in I} M_i = \cup_{i \in I} \{(x, i) : x \in M_i\}$$

endowed with the disjoint union topology and denote by  $\text{inj}_i : M_i \hookrightarrow M$  the canonical injections ( $\text{inj}_i(x) = (x, i)$ ). Show that  $\mathcal{A} := \cup_{i \in I} \mathcal{A}_i$  defines a smooth atlas on  $M$  and that the injections  $\text{inj}_i$  are smooth. Moreover, show that for any smooth manifold  $N$  and smooth functions  $f_i : M_i \rightarrow N$ , there exists a unique smooth function  $f : M \rightarrow N$  such that  $f \circ \text{inj}_i = f_i$  and show that this property characterizes the smooth manifold structure on  $M$  uniquely.

3. Suppose  $U \subset \mathbb{R}^m$  is open and  $f : U \rightarrow \mathbb{R}^n$  a smooth map such that  $D_x f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is of rank  $r$  for all  $x \in U$ .

Show that for any  $x_0 \in U$  there exists a diffeomorphism  $\phi$  between an open neighbourhood of  $x_0$  and an open neighbourhood of  $0 \in \mathbb{R}^m$  and a diffeomorphism  $\psi$  between an open neighbourhood of  $y_0 = f(x_0)$  and an open neighbourhood of  $0$  in  $\mathbb{R}^n$  such that the locally defined map

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^r \times \mathbb{R}^{m-r} \rightarrow \mathbb{R}^r \times \mathbb{R}^{n-r}$$

has the form  $(x_1, \dots, x_r, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$ .

**Hint:** The idea is that  $f$  locally around  $x_0$  looks like  $D_{x_0} f$ , which is a linear map  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  of rank  $r$ , which up to a basis change has the form  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$ .

- (a) Set  $E_2 := \ker(D_{x_0}f) \subset \mathbb{R}^m$  and  $E_1 := E_2^\perp$ , and  $F_1 := \text{Im}(D_{x_0}f) \subset \mathbb{R}^n$  and  $F_2 := F_1^\perp$ . Decompose

$$\mathbb{R}^m = E_1 \oplus E_2 \quad \text{and} \quad \mathbb{R}^n = F_1 \oplus F_2,$$

and consider  $f$  as a map  $f = (f_1, f_2) : E_1 \oplus E_2 \rightarrow F_1 \oplus F_2$  defined on  $U \subset E_1 \oplus E_2 = \mathbb{R}^m$ .

- (b) Show that  $\phi : E_1 \oplus E_2 \rightarrow F_1 \oplus F_2$  given by

$$\phi(x^1, x^2) = (f_1(x^1, x^2) - f_1(x_0^1, x_0^2), x^2 - x_0^2)$$

is a local diffeomorphism around  $x_0 = (x_0^1, x_0^2)$  whose local inverse will be the required map.

- (c) Show that  $g := f \circ \phi^{-1} : F_1 \oplus F_2 \rightarrow F_1 \oplus F_2$  has the form

$$g(y^1, y^2) = (g_1((y^1, y^2)), g_2((y^1, y^2))) = (y^1 + y_0^1, g_2(y^1, 0)).$$

Now  $\psi$  is easily seen to be...?

4. Suppose  $M$  and  $N$  are manifolds of dimension  $m$  respectively  $n$  and let  $f : M \rightarrow N$  be a smooth map of constant rank  $r$ . Deduce from (1) that for any fixed  $y \in f(M)$  the preimage  $f^{-1}(y) \subset M$  is a submanifold of dimension  $m - r$  in  $M$ .
5. Consider the Grassmannian of  $r$ -planes in  $\mathbb{R}^n$ :

$$\text{Gr}(r, n) := \{E \subset \mathbb{R}^n : E \text{ is a } r\text{-dimensional subspace of } \mathbb{R}^n\}.$$

Denote by  $\text{St}_r(\mathbb{R}^n)$  the set of  $r$ -tuples of linearly independent vectors in  $\mathbb{R}^n$ . Identifying an element  $X \in \text{St}_r(\mathbb{R}^n)$  with a  $n \times r$  matrix

$$X = (x^1, \dots, x^r) \quad x^i \in \mathbb{R}^n,$$

shows that  $\text{St}_r(\mathbb{R}^n)$  equals the subset of rank  $r$  matrices in the vector space  $\mathbf{M}_{n \times r}(\mathbb{R})$ , which we know from Tutorial 1 is an open subset. Write

$$\pi : \text{St}_r(\mathbb{R}^n) \rightarrow \text{Gr}(r, n)$$

for the natural projection given by  $\pi(X) = \text{span}(x^1, \dots, x^r)$  and equip  $\text{Gr}(r, n)$  with the quotient topology with respect to  $\pi$ .

- (a) Fix  $E \in \text{Gr}(r, n)$  and let  $F \subset \mathbb{R}^n$  be a subspace of dimension  $n - r$  such that  $\mathbb{R}^n = E \oplus F$ . Show that

$$U_{(E,F)} = \{W \in \text{Gr}(r, n) : W \cap F = \{0\}\} \subset \text{Gr}(r, n)$$

is an open neighbourhood of  $E$ .

- (b) Show that any element  $W \in U_{(E,F)}$  determines a unique linear map

$$\widetilde{W} : E \rightarrow F$$

such that its graph equals  $W$ , i.e.  $W = \{(x, \widetilde{W}x) : x \in E\}$ .

(c) Show that the map  $u_{E,F} : U_{(E,F)} \rightarrow \text{Hom}(E, F)$  given by  $u_{E,F}(W) = \widetilde{W}$  is a homeomorphism.

(d) Show that

$$\mathcal{A} := \{(U_{(E,F)}, u_{(E,F)}) : E, F \subset \mathbb{R}^n \text{ complimentary subspaces of dimension } r \text{ resp. } n-r\}$$

is a smooth atlas for  $\text{Gr}(r, n)$ .

6. For a topological space  $M$  denote by  $C^0(M)$  the vector space of continuous real-valued functions  $f : M \rightarrow \mathbb{R}$ . Any continuous map  $F : M \rightarrow N$  between topological spaces  $M$  and  $N$  induces a map  $F^* : C^0(N) \rightarrow C^0(M)$  given by  $F^*(f) := f \circ F : M \rightarrow \mathbb{R}$ .

(a) Show that  $F^*$  is linear.

(b) If  $M$  and  $N$  are (smooth) manifolds, show that  $F : M \rightarrow N$  is smooth  $\iff F^*(C^\infty(N)) \subset C^\infty(M)$ .

(c) If  $F$  is a homeomorphism between (smooth) manifolds, show that  $F$  is a diffeomorphism  $\iff F^*$  is an isomorphism.