INTRODUCTION TO ALGEBRAIC TOPOLOGY

MARTIN ČADEK

3. Simplicial and singular homology

3.1. Exact sequences. A sequence of homomorphisms of Abelian groups or modules over a ring

$$\dots \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} \dots$$

is called an *exact sequence* if

$$\operatorname{Im} f_n = \operatorname{Ker} f_{n-1}.$$

Exactness of the following sequences

$$O \longrightarrow A \xrightarrow{f} B, \quad B \xrightarrow{g} C \longrightarrow 0, \quad 0 \longrightarrow C \xrightarrow{h} D \longrightarrow 0$$

means that f is a monomorphism, g is an epimorphism and h is an isomorphism, respectively.

A *short exact sequence* is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0.$$

In this case $C \cong B/A$. We say that the short exact sequence splits if one of the following three equivalent conditions is satisfied:

(1) There is a homomorphism $p : B \to A$ such that $pi = \mathrm{id}_A$.
(2) There is a homomorphism $q : C \to B$ such that $jq = \mathrm{id}_C$.
(3) There are homomorphisms $p : B \to A$ and $q : C \to B$ such that $ip + qj = \mathrm{id}_B$.

The last condition means that $B \cong A \oplus C$ with isomorphism $(p, q) : B \to A \oplus C$.

Exercise. Prove the equivalence of (1), (2) and (3).

3.2. Chain complexes. The *chain complex* (C, ∂) is a sequence of Abelian groups (or modules over a ring) and their homomorphisms indexed by integers

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

such that

$$\partial_{n-1}\partial_n = 0.$$

This conditions means that $\operatorname{Im} \partial_n \subseteq \operatorname{Ker} \partial_{n-1}$. The homomorphism ∂_n is called a boundary operator. A *chain homomorphism* of chain complexes (C, ∂^C) and (D, ∂^D) is a sequence of homomorphisms of Abelian groups (or modules over a ring) $f_n : C_n \to D_n$ which commute with the boundary operators

$$\partial^D_n f_n = f_{n-1}\partial^C_n.$$

3.3. Homology of chain complexes. The n-th *homology group* of the chain complex (C, ∂) is the group

$$H_n(C) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

The elements of $\operatorname{Ker} \partial_n = Z_n$ are called *cycles* of dimension n and the elements of $\operatorname{Im} \partial_{n+1} = B_n$ are called *boundaries* (of dimension n). If a chain complex is exact, then its homology groups are trivial.

The component f_n of the chain homomorphism $f : (C, \partial^C) \to (D, \partial^D)$ maps cycles into cycles and boundaries into boundaries. It enables us to define

$$H_n(f) : H_n(C) \to H_n(D)$$

by the prescription $H_n(f)[c] = [f_n(c)]$ where $[c] \in H_n(C_*)$ and $[f_n(c)] \in H_n(D^*)$ are classes represented by the elements $c \in Z_n(C)$ and $f_n(c) \in Z_n(D)$, respectively.

3.4. Long exact sequence in homology. A sequence of chain homomorphisms

$$\ldots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \ldots$$

is exact if for every $n \in \mathbb{Z}$

$$\ldots \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow \ldots$$

is an exact sequence of Abelian groups.

Theorem. *Let* $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ *be a short exact sequence of chain complexes. Then there is a* connecting homomorphism $\partial_* : H_n(C) \to H_{n-1}(A)$ *such that the sequence*

$$\ldots \xrightarrow{\partial_*} H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(j)} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

is exact.

Proof. Define the connecting homomorphism ∂_*. Let $[c] \in H_n(C)$ where $c \in C_n$ is a cycle. Since $j : B_n \to C_n$ is an epimorphism, there is $b \in B_n$ such that $j(b) = c$. Further, $j(\partial b) = \partial j(b) = \partial c = 0$. From exactness there is $a \in A_{n-1}$ such that $i(a) = \partial b$. Since $i(\partial a) = \partial i(a) = \partial \partial b = 0$ and i is a monomorphism, $\partial a = 0$ and a is a cycle in A_{n-1}. Put

$$\partial_*[c] = [a].$$

Now we have to show that the definition is correct, i. e. independent of the choice of c and b, and to prove exactness. For this purpose it is advantageous to use an appropriate diagram. It is not difficult and we leave it as an exercise to the reader. □

3.5. Chain homotopy. Let $f, g : C \to D$ be two chain homomorphisms. We say that they are *chain homotopic* if there are homomorphisms $s_n : C_n \to D_{n+1}$ such that

$$\partial^D_{n+1} s_n + s_{n-1} \partial^C_n = f_n - g_n \quad \text{for all } n.$$

The relation to be chain homotopic is an equivalence. The sequence of maps s_n is called a *chain homotopy*.

Theorem. *If two chain homomorphism $f, g : C \to D$ are chain homotopic, then*

$$H_n(f) = H_n(g).$$

Exercise. Prove the previous theorem from the definitions.

3.6. Five Lemma. Consider the diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle f_1}\downarrow\cong & & {\scriptstyle f_2}\downarrow\cong & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow\cong & & {\scriptstyle f_5}\downarrow\cong \\ \bar{A} & \longrightarrow & \bar{B} & \longrightarrow & \bar{C} & \longrightarrow & \bar{D} & \longrightarrow & \bar{E} \end{array}$$

If the horizontal sequences are exact and f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is also an isomorphism.

Exercise. Prove 5-lemma.

3.7. Simplicial homology. We describe two basic ways how to define homology groups for topological spaces – simplicial homology which is closer to geometric intuition and singular homology which is more general. For the definition of simplicial homology we need the notion of Δ-complex, which is a special case of CW-complex.

Let $v_0, v_1, \dots, v_n$ be points in $\mathbb{R}^m$ such that $v_1 - v_0, v_2 - v_0, v_n - v_0$ are linearly independent. The *n-simplex* $[v_0, v_1, \dots, v_n]$ with the vertices $v_0, v_1, \dots, v_n$ is the subspace of $\mathbb{R}^m$

$$\{\sum_{i=0}^{n} t_i v_i;\ \sum_{i=1}^{n} t_i = 1,\ t_i \geq 0\}$$

with a given ordering of vertices. A *face* of this simplex is any simplex determined by a proper subset of vertices in the given ordering.

Let Δ_α, $\alpha \in J$ be a collection of simplices. Subdivide all their faces of dimension i into sets F^i_β. A Δ*-complex* is a quotient space of disjoint union $\coprod_{\alpha \in J} \Delta_\alpha$ obtained by identifying simplices from every F^i_β into one single simplex via affine maps which preserve the ordering of vertices. Thus every Δ-complex is determined only by combinatorial data.

A special case of Δ-complex is a *finite simplicial complex.* It is a union of simplices the vertices of which lie in a given finite set of points $\{v_0, v_1, \dots, v_n\}$ in $\mathbb{R}^m$ such that $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent.

Example. Torus, real projective space of dimension 2 and Klein bottle are Δ-complexes as one can see from the following pictures.

In all the cases we have two sets F^2 whose elements are triangles, three sets F^1 every with two segments and one set F^0 containing all six vertices of both triangles.

These surfaces are also homeomorhic to finite simplicial complexes, but their structure as simplicial complexes is more complicated than their structure as Δ-complexes.

To every Δ-complex X we can assign the chain complex (C, ∂) where $C_n(X)$ is a free Abelian group generated by n-simplices of X (i. e. the rank of $C_n(X)$ is the number

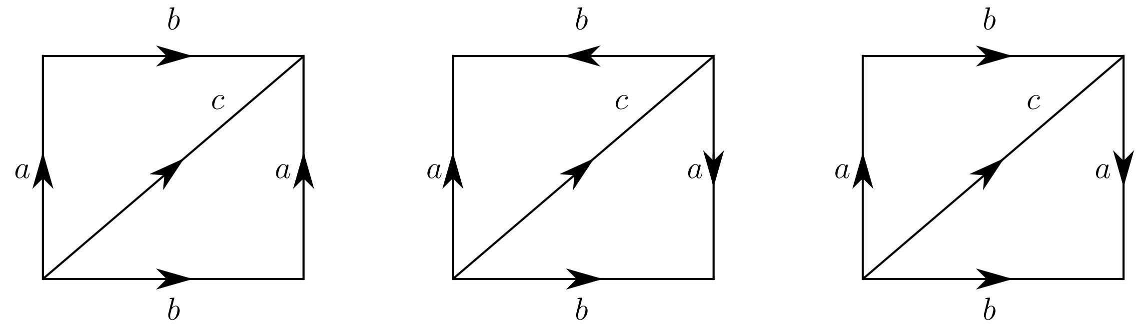

FIGURE 3.1. Torus, $\mathbb{R}P^2$ and Klein bottle as Δ-complexes

of the sets F^n and the boundary operator on generators is given by

$$\partial[v_0, v_1, \dots, v_n] = \sum_{i=0}^{n}(-1)^i[v_0, \dots, \hat{v}_i \dots, v_n].$$

Here the symbol $\hat{v}_i$ means that the vertex v_i is omitted. Prove that $\partial\partial = 0$.

The *simplicial homology groups* of Δ-complex X are the homology groups of the chain complex defined above. Later, we will show that these groups are independent of Δ-complex structure.

Exercise. Compute simplicial homology of S^2 (find a Δ-complex structure), $\mathbb{RP}^2$, torus and Klein bottle (with Δ-complex structures given in example above).

Let X and Y be two Δ-complexes and $f : X \to Y$ a map which maps every simplex of X into a simplex of Y and it is affine on all simplexes. Using appropriate sign conventions we can define the chain homomorphism $f_n : C_n(X) \to C_n(Y)$ induced by the map f. This chain map enables us to define homomorphism of simplicial homology groups induced by f.

Having a Δ-subcomplex A of a Δ-complex X (i. e. subspace of X formed by some of the simplices of X) we can define simplicial homology groups $H_n(X, A)$. The definition is the same as for singular homology in paragraph 3.9. These groups fit into the long exact sequence

$$\dots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \dots$$

See again 3.9.

3.8. Singular homology. The *standard n-simplex* is the n-simplex

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1};\ \sum_{i=0}^{n} t_i = 1;\ t_i \geq 0\}.$$

The j-th face of this standard simplex is the $(n-1)$-dimensional simplex $[e_0, \dots, \hat{e}_j, \dots, e_n]$ where e_j is the vertex with all coordinates 0 with the exception of the j-th one which is 1. Define

$$\varepsilon_n^j : \Delta^{n-1} \to \Delta^n$$

as the affine map $\varepsilon_n^j(t_0, t_1, \dots, t_{n-1}) = (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1})$ which maps

$$e_0 \to e_0,\ \dots,\ e_{j-1} \to e_{j-1},\ e_j \to e_{j+1},\ \dots,\ e_{n-1} \to e_n.$$

It is not difficult to prove

Lemma. $\varepsilon_{n+1}^k \varepsilon_n^j = \varepsilon_{n+1}^{j+1} \varepsilon_n^k$ *for* $k < j$.

A *singular n-simplex* in a space X is a continuous map $\sigma : \Delta^n \to X$. Denote the free Abelian group generated by all the singular n-simplices by $C_n(X)$ and define the boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \varepsilon_n^i$$

for $n \geq 0$. Put $C_n(X) = 0$ for $n < 0$. Using the lemma above one can show that

$$\partial_{n+1}\partial_n = 0.$$

The chain complex (C_n, ∂_n) is called the *singular chain complex* of the space X. The *singular homology groups* $H_n(X)$ of the space X are the homology groups of the chain complex $(C_n(X), \partial_n)$, i. e.

$$H_n(X) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

Next consider a map $f : X \to Y$. Define the chain homomorhism $C_n(f) : C_n(X) \to C_n(Y)$ on singular n-simplices as the composition

$$C_n(f)(\sigma) = f\sigma.$$

From definitions it is easy to show that these homomorphisms commute with boundary operators. Hence this chain homomorphism induces homomorphisms

$$f_* = H_n(f) : H_n(X) \to H_n(Y).$$

Moreover, $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$ and $H_n(fg) = H_n(f)H_n(g)$. It means that H_n is a functor from the category Top to the category Ab of Abelian groups and their homomorphisms. This functor is the composition of the functor C from Top to chain complexes and the n-th homology functor from chain complexes to abelian groups.

Prove the lemma above and $\partial_{n+1}\partial_n = 0$.

Show directly from the definition that the singular homology groups of a point are $H_0(*) = \mathbb{Z}$ and $H_n(*) = 0$ for $n \neq 0$.

3.9. Singular homology groups of a pair. Consider a pair of topological spaces (X, A). Then the $C_n(A)$ is a subgroup of $C_n(X)$. Hence we get this short exact sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} \frac{C_n(X)}{C_n(A)} \to 0.$$

Since the boundary operators in $C_n(A)$ are restrictions of boundary operators in $C_n(X)$, we can define boundary operators

$$\partial_n : \frac{C_n(X)}{C_n(A)} \to \frac{C_{n-1}(X)}{C_{n-1}(A)}.$$

We will denote this chain complex as $(C(X,A),\partial)$ and its homology groups as $H_n(X,A)$. Notice that the factor $C_n(X)/C_n(A)$ is a free Abelian group generated by singular simplices $\sigma:\Delta^n\to X$ such that $\sigma(\Delta^n)\nsubseteq A$. We will need it later.

A map $f:(X,A)\to(Y,B)$ induces the chain homomorphism $C_n(f):C_n(X)\to C_n(Y)$ which restricts to a chain homomorphism $C_n(A)\to C_n(B)$ since $f(A)\subseteq B$. Hence we can define the chain homomorphism

$$C_n(f):C_n(X,A)\to C_n(Y,B)$$

which in homology induces the homomorphism

$$f_*=H_n(f):H_n(X,A)\to H_n(Y,B).$$

We can again conclude that H_n is a functor from the category Top^2 into the category Ab of Abelian groups. This functor extends the functor defined on the category Top since every object X and every morphism $f:X\to Y$ in Top can be considered as the object $(X,\emptyset)$ and the morphism $\hat{f}=f:(X,\emptyset)\to(Y,\emptyset)$ in the category Top^2 and

$$H_n(X,\emptyset)=H_n(X),\quad H_n(\hat{f})=H_n(f).$$

3.10. Long exact sequence for singular homology. Consider inclusions of spaces $i:A\to X$, $i':B\to Y$ and maps $j:(X,\emptyset)\to(X,A)$, $j':(Y,\emptyset)\to(Y,B)$ induced by id_X and id_Y, respectively. Let $f:(X,A)\to(Y,B)$ be a map. Then there are connecting homomorphisms ∂_*^X and ∂_*^Y such that the following diagram

$$\begin{array}{ccccccccc}
\xrightarrow{\partial_*^X} & H_n(A) & \xrightarrow{i_*} & H_n(X^1) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\partial_*^X} & H_{n-1}(A) & \xrightarrow{i_*} \\
 & \big\downarrow{\scriptstyle (f/A)_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (f/A)_*} & \\
\xrightarrow{\partial_*^Y} & H_n(B) & \xrightarrow{i'_*} & H_n(Y) & \xrightarrow{j'_*} & H_n(Y,B) & \xrightarrow{\partial_*^Y} & H_{n-1}(B) & \xrightarrow{i'_*}
\end{array}$$

commutes and its horizontal sequences are exact.

An analogous theorem holds also for simplicial homology.

Remark. Consider the functor $I:\mathsf{Top}^2\to\mathsf{Top}^2$ which assigns to every pair (X,A) the pair $(A,\emptyset)$. The commutativity of the last square in the diagram above means that ∂_* is a natural transformation of functors H_n and $H_{n-1}\circ I$ defined on Top^2.

Proof. We have the following commutative diagram of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(A) & \xrightarrow{C(i)} & C(X) & \xrightarrow{C(j)} & C(X,A) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle C(f/A)} & & \big\downarrow{\scriptstyle C(f)} & & \big\downarrow{\scriptstyle C(f)} & & \\
0 & \longrightarrow & C(B) & \xrightarrow{C(i')} & C(Y) & \xrightarrow{C(j')} & C(Y,B) & \longrightarrow & 0
\end{array}$$

with exact horizontal rows. Then Theorem 3.4 and the construction of connecting homomorphism ∂_* imply the required statement. $\square$

Remark. It is useful to realize how $\partial_*:H_n(X,A)\to H_{n-1}(A)$ is defined. Every element of $H_n(X,A)$ is represented by a chain $x\in C_n(X)$ with a boundary $\partial x\in$

$C_{n-1}(A)$. This is a cycle in $C_n(A)$ and from the definition in 3.4 we have

$$\partial_*[x] = [\partial x].$$

3.11. Homotopy invariance. If two maps $f, g : (X, A) \to (Y, B)$ are homotopic, then they induce the same homomorphisms

$$f_* = g_* : H_n(X, A) \to H_n(Y, B).$$

Proof. We need to prove that the homotopy between f and g induces a chain homotopy between $C_*(f)$ and $C_*(g)$. For the proof see [Hatcher], Theorem 2.10 and Proposition 2.19 or [Spanier], Chapter 4, Section 4. □

Corollary. *If X and Y are homotopy equivalent spaces, then*

$$H_n(X) \cong H_n(Y).$$

3.12. Excision Theorem. There are two equivalent versions of this theorem.

Theorem (Excision Theorem, 1st version)**.** *Consider spaces $C \subseteq A \subseteq X$ and suppose that $\bar{C} \subseteq \operatorname{int} A$. Then the inclusion*

$$i : (X - C, A - C) \hookrightarrow (X, A)$$

induces the isomorphism

$$i_* : H_n(X - C, A - C) \xrightarrow{\cong} H_n(X, A).$$

Theorem (Excision Theorem, 2nd version)**.** *Consider two subspaces A and B of a space X. Suppose that $X = \operatorname{int} A \cup \operatorname{int} B$. Then the inclusion*

$$i : (B, A \cap B) \hookrightarrow (X, A)$$

induces the isomorphism

$$i_* : H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A).$$

The second version of Excision Theorem holds also for simplicial homology if we suppose that A and B are Δ-subcomplexes of a Δ-complex X and $X = A \cup B$. In this case the proof is easy since the inclusion

$$C_n(i) : C_n(B, A \cap B) \to C_n(A \cup B, A)$$

is an isomorphism, namely the both chain complexes are generated by the same n-simplices.

Exercise. Show that the theorems above are equivalent.

The proof of Excision Theorem for singular homology can be found in [Hatcher], pages 119 – 124, or in [Spanier], Chapter 4, Sections 4 and 6. The main step (a little bit technical for beginners) is to prove the following lemma which we will need later.

Lemma. *Let $\mathcal{U} = \{U_\alpha;\ \alpha \in J\}$ be a collection of subsets of X such that $X = \bigcup_{\alpha\in J} \operatorname{int} U_\alpha$. Denote the free chain complex generated by singular simplices σ with $\sigma(\Delta^n) \in U_\alpha$ for some α as $C_n^{\mathcal{U}}(X)$. Then*

$$C_n^{\mathcal{U}}(X)) \hookrightarrow C_n(X)$$

induces isomorphism in homology.

Proof of Excision Theorem. Consider $\mathcal{U} = \{A, B\}$. Then the inclusion

$$C_n(i) : C_n(B, A \cap B) \to \frac{C_n^{\mathcal{U}}(X)}{C_n(A)}$$

is an isomorphism and, moreover, according to the previous lemma, the homology of the second chain complex is $H_n(X, A)$. □

3.13. Homology of disjoint union. Let $X = \coprod_{\alpha\in J} X_\alpha$ be a disjoint union. Then

$$H_n(X) = \bigoplus_{\alpha\in J} H_n(X_\alpha).$$

The proof follows from the definition and connectivity of $\sigma(\Delta^n)$ in X for every singular n-simplex σ.

3.14. Reduced homology groups. For every space $X \neq \emptyset$ we define the *augmented chain complex* $(\tilde{C}(X), \tilde{\partial})$ as follows

$$\tilde{C}_n(X) = \begin{cases} C_n(X) & \text{for } n \neq -1, \\ \mathbb{Z} & \text{for } n = -1. \end{cases}$$

with $\tilde{\partial}_n = \partial_n$ for $n \neq 0$ and $\partial_0(\sum_{i=1}^j n_i\sigma_i) = \sum_{i=1}^j n_i$. The *reduced homology groups* $\tilde{H}_n(X)$ are the homology groups of the augmented chain complex. From the definition it is clear that

$$\tilde{H}_n(X) = H_n(X) \quad \text{for } n \neq 0$$

and

$$\tilde{H}_n(*) = 0 \quad \text{for all } n.$$

For pairs of spaces we define $\tilde{H}_n(X, A) = H_n(X, A)$ for all n. Then theorems on long exact sequence, homotopy invariance and excision hold for reduced homology groups as well.

Considering a space X with distinguished point $*$ and applying the long exact sequence for the pair $(X, *)$, we get that for all n

$$\tilde{H}_n(X) = \tilde{H}_n(X, *) = H_n(X, *).$$

Using this equality and the long exact sequence for unreduced homology we get that

$$H_0(X) \cong H_0(X, *) \oplus H_0(*) \cong \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Lemma. *Let (X, A) be a pair of CW-complexes, $X \neq \emptyset$. Then*

$$\tilde{H}_n(X/A) = H_n(X, A)$$

and we have the long exact sequence

$$\dots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \dots$$

Proof. According to example in Section 2

$$(X, A) \to (X \cup CA, CA) \to (X \cup CA/CA, *) = (X/A, *)$$

is the composition of an excision and a homotopy equivalence. Hence $\tilde{H}_n(X/A) = H_n(X, A)$. The rest folows from the long exact sequence of the pair (X, A). □

Exercise. Prove that $\tilde{H}_n(\bigvee X_\alpha) \cong \oplus \tilde{H}_n(X_\alpha)$.

$\tilde{H}_n$ can be considered as a functor from Top_* to Abelian groups.

3.15. The long exact sequence of a triple. Three spaces (X, B, A) with the property $A \subseteq B \subseteq X$ are called a *triple*. Denote $i : (B, A) \to (X, A)$ and $j : (X, A) \to (X, B)$ maps induced by the inclusion $B \hookrightarrow X$ and id_X, respectively. Analogously as for pairs one can derive the following long exact sequence:

$$\dots \xrightarrow{\partial_*} H_n(B, A) \xrightarrow{i_*} H_n(X, A) \xrightarrow{j_*} H_n(X, B) \xrightarrow{\partial_*} H_{n-1}(B, A) \xrightarrow{i_*} \dots$$

3.16. Singular homology groups of spheres. Consider the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda^{n-1} = \partial\Delta^n - \Delta^{n-1})$:

$$\dots \to H_i(\Delta^n, \Lambda^{n-1}) \to H_i(\Delta^n, \partial\Delta^n) \xrightarrow{\partial *} H_{i-1}(\partial\Delta^n, \Lambda^{n-1}) \to H_{i-1}(\Delta^n, \Lambda^{n-1}) \to \dots$$

The pair $(\Delta^n, \Lambda^{n-1})$ is homotopy equivalent to $(*, *)$ and hence its homology groups are zeroes. Next using Excision Theorem and homotopy invariance we get that $H_i(\Delta^n, \Lambda^{n-1}) \cong H_i(\Delta^{n-1}, \partial\Delta^{n-1})$. Consequently, we get an isomorphism

$$H_i(\Delta^n, \partial\Delta^n) \cong H_{i-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

Using induction and computing $H_i(\Delta^1, \partial\Delta^1) = H_i([0, 1], \{0, 1\}) \cong H_{i-1}(\{0, 1\}, \{0\})$ we get that

$$H_i(\Delta^n, \partial\Delta^n) = \begin{cases} \mathbb{Z} & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

Doing the induction carefully we can find that the generator of the group $H_n(\Delta^n, \partial\Delta^n) = \mathbb{Z}$ is determined by the singular n-simplex id_{Δ^n}.

The pair (D^n, S^{n-1}) is homeomorphic to $(\Delta^n, \partial\Delta^n)$. Hence it has the same homology groups. Using the long exact sequence for this pair we obtain

$$\tilde{H}_{i-1}(S_{n-1}) = H_i(D^n, S^{n-1}) = \begin{cases} 0 & \text{for } i \neq n, \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

3.17. Mayer-Vietoris exact sequence. Denote inclusions $A \cap B \hookrightarrow A$, $A \cap B \hookrightarrow B$, $A \hookrightarrow X$, $B \hookrightarrow X$ by i_A, i_B, j_A, j_B, respectively. Let $C \hookrightarrow A \cap B$ and suppose that $X = \operatorname{int} A \cup \operatorname{int} B$. Then the following sequence

$$\dots \xrightarrow{\partial_*} H_n(A \cap B, C) \xrightarrow{(i_{A*}, i_{B*})} H_n(A, C) \oplus H_n(B, C)$$
$$\xrightarrow{j_{A*} - j_{B*}} H_n(X, C) \xrightarrow{\partial_*} H_{n-1}(A \cap B, C) \to \dots$$

is exact.

Proof. The covering $\mathcal{U} = \{A, B\}$ satisfies conditions of Lemma 3.12. The sequence of chain complexes

$$0 \longrightarrow \frac{C(A \cap B)}{C(C)} \xrightarrow{i} \frac{C(A)}{C(C)} \oplus \frac{C(B)}{C(c)} \xrightarrow{j} \frac{C^{\mathcal{U}}(X)}{C(C)} \longrightarrow 0$$

where $i(x) = (x, x)$ and $j(x, y) = x - y$ is exact. Consequently, it induces a long exact sequence. Using Lemma 3.12 we get that $H_n(C^{\mathcal{U}}(X), C(C)) = H_n(X, C)$, which completes the proof. □

3.18. Equality of simplicial and singular homology. Let (X, A) be a pair of Δ-complexes. Then the natural inclusion of simplicial and singular chain complexes $C^{\Delta}(X, A) \hookrightarrow C(X, A)$ induces the isomorphism of simplicial and singular homology groups

$$H_n^{\Delta}(X, A) \cong H_n(X, A).$$

Outline of the proof. Consider the long exact sequences for the pair (X^k, X^{k-1}) of skeletons of X. We get

$$\begin{array}{ccccccccc}
H_{n+1}^{\Delta}(X^k, X^{k-1}) & \longrightarrow & H_n^{\Delta}(X^{k-1}) & \longrightarrow & H_n^{\Delta}(X^k) & \longrightarrow & H_n^{\Delta}(X^k, X^{k-1}) & \longrightarrow & H_{n-1}^{\Delta}(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

Using induction by k we have $H_i^{\Delta}(X^{k-1}) = H_i(X^{k-1})$ for all i. Further, $C_i^{\Delta}(X^k, X^{k-1})$ is according to definition zero if $i \neq k$ and free Abelian of rank equal the number of i-simplices Δ_α^i if $i = k$. The homology groups $H_i^{\Delta}(X^k, X^{k-1})$ have the same description.

Since

$$\coprod_\alpha \Delta_\alpha^k / \coprod_\alpha \partial \Delta_\alpha^k = X^k / X^{k-1}$$

we get the isomorphism

$$H_i^{\Delta}(X^k / X^{k-1}) \to H_i(\coprod_\alpha \Delta_\alpha^k / \coprod_\alpha \partial \Delta_\alpha^k) = H_i(X^k / X^{k-1}).$$

Applying 5-lemma (see 3.6) in the diagram above, we get that $H_n^{\Delta}(X^k) \to H_n(X^k)$ is an isomorphism.

If X is finite Δ-complex, we are ready. If it is not, we have to prove that $H_n^{\Delta}(X) = H_n(X)$. See [Hatcher], page 130. □

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