INTRODUCTION TO ALGEBRAIC TOPOLOGY

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8. Vector bundles and Thom isomorphism

In this section we introduce the notion of vector bundle and define its important algebraic invariants Thom and Euler classes. The Thom class is involved in so called Thom isomorphism. Using this isomorphism we derive the Gysin exact sequence which is an important tool for computing cup product structure in cohomology.

8.1. Fibre bundles. A *fibre bundle* structure on a space E, with fiber F, consists of a projection map $p : E \to B$ such that each point of B has a neighbourhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes. Here pr_1 is the projection on the first factor. h is called a *local trivialization*, the space E is called the *total space* of the bundle and B is the *base space*.

A *subbundle* (E', B, p') of a fibre bundle (E, B, p) has the total space $E' \subseteq E$, the fibre $F' \subseteq F$, $p' = p/E'$ and local trivializations in E' are restrictions of local trivializations of E.

A *vector bundle* is a fibre bundle (E, B, p) whose fiber is a vector space (real or complex). Moreover, we suppose that for each $b \in B$ the fiber $p^{-1}(b)$ over b is a vector space and all local trivializations restricted to $p^{-1}(b)$ are linear isomorphisms. The dimension of a vector bundle is the dimension of its fiber. For $p^{-1}(U)$ where $U \subseteq B$ we will use notation E_U. Further, E^0_U will stand for E_U without zeroes in vector spaces $E_x = p^{-1}(x)$ for $x \in U$.

8.2. Orientation of a vector space. Let V be a real vector space of dimension n. The orientation of V is the choice of a generator in $H^n(V, V - \{0\}; \mathbb{Z}) = \mathbb{Z}$. If R is a commutative ring with a unit, the R-orientation of V is the choice of a generator in $H^n(V, V - \{0\}; R) = R$. For $R = \mathbb{Z}$ we have two possible orientations, for $R = \mathbb{Z}_2$ only one.

8.3. Orientation of a vector bundle. Consider a vector bundle (E, B, p) with fiber $\mathbb{R}^n$. The *R-orientation* of the vector bundle E is a choice of orientation in each vector space $p^{-1}(b)$, $b \in B$, i. e. a choice of generators $t_b \in H^n(E_b, E_b^0; R) = R$ such that for each $b \in B$ there is a neighbourhood U and an element

$$t_U \in H^n(E_U, E_U^0; R)$$

with the property

$$i_x^*(t_U) = t_x$$

for each $x \in U$ and the inclusion $i_x : E_x \hookrightarrow E_U$.

If a vector bundle has an R-orientation, we say that it is R-orientable. An R-oriented vector bundle is a vector bundle with a chosen R-orientation. Talking on orientation we will mean $\mathbb{Z}$-orientation.

Example. Every vector bundle (E, B, p) is $\mathbb{Z}_2$-orientable. After we have some knowledge of fundamental group, we will be able to prove that vector bundles with $\pi_1(B) = 0$ are orientable.

8.4. Thom class and Thom isomorphism. The Thom class of a vector bundle (E, B, p) of dimension n is an element $t \in H^n(E, E^0; R)$ such that $i_b^*(t)$ is a generator in $H^n(E_b, E_b^0; R) = R$ for each $b \in B$ where $i_b : E_b \hookrightarrow E$ is an inclusion.

It is clear that any Thom class determines a unique orientation. The reverse statement is also true.

Theorem (Thom Isomorphism Theorem)**.** *Let (E, B, p) be an R-oriented vector bundle of real dimension n. Then there is just one Thom class $t \in H^n(E, E^0; R)$ which determines the given R-orientation. Moreover, the homomorphism*

$$\tau : H^k(B; R) \to H^{k+n}(E, E^0; R), \quad \tau(a) = p^*(a) \cup t$$

is an isomorphism (so called Thom isomorphism).

Remark. Notice that Thom Isomorphism Theorem is a generalization of the Künneth Formula 7.5 for $(Y, A) = (\mathbb{R}^n, \mathbb{R}^n - \{0\})$. We use it in the proof.

Proof. (1) First suppose that $E = B \times \mathbb{R}^n$. Then according to Theorem 7.5

$$\begin{aligned} H^*(E, E^0; R) &= H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\}); R) = H^*(B; R) \otimes H^*(R^n, \mathbb{R}^n - \{0\}); R) \\ &\cong H^*(B; R)[e]/\langle e^2 \rangle \end{aligned}$$

where $e \in H^n(R^n, \mathbb{R}^n - \{0\}); R)$ is the generator given by the orientation of E. Now, there is just one Thom class $t = 1 \times e$ and

$$\tau(a) = p^*(a) \cup t = a \times e$$

is an isomorphism.

(2) If U is open subset of B, then the orientation of (E, B, p) induces an orientation of the vector bundle (E_U, U, p). Suppose that U and V are two open subsets in B such that the statement of Theorem is true for E_U, E_V and $E_{U\cap V}$ with induced orientations. Denote the corresponding Thom classes by t_U, t_V and $t_{U\cap V}$. The uniqueness of $t_{U\cap V}$ implies that the restrictions of both classes t_U and t_V on $H^n(E_{U\cap V}, E^0_{U\cap V}; R)$ are $t_{U\cap V}$. We will show that Theorem holds for $E_{U\cup V}$.

Consider the Mayer-Vietoris exact sequence 5.13 for $A = E_U$, $B = E_V$, $C = E_U^0$, $D = E_V^0$ together with the Mayer-Vietoris exact sequence for $A = U$, $B = V$ and $C = D = \emptyset$. Omitting coefficients these exact sequences together with Thom isomorphisms τ_U, τ_V and $\tau_{U\cap V}$ yield the following diagram where DE_U stands for the pair (E_U, E_U^0)

$$\begin{tikzcd}
{} \arrow[r, "\delta^*"] & H^{k+n}(DE_{U\cup V}) \arrow[r, "(j_U^*, j_V^*)"] & H^{k+n}(DE_U) \oplus H^{k+n}(DE_V) \arrow[r, "i_U^* - i_V^*"] & H^{k+n}(DE_{U\cap V}) \arrow[r] & {} \\
{} \arrow[r, "\delta^*"] & H^k(U \cup V) \arrow[u, dashed, "\tau_{U\cup V}"] \arrow[r, "(j_U^*, j_V^*)"] & H^k(U) \oplus H^k(V) \arrow[u, "\tau_U \oplus \tau_V"] \arrow[r, "i_U^* - i_V^*"] & H^k(U \cap V) \arrow[u, "\tau_{U\cap V}"] \arrow[r] & {}
\end{tikzcd}$$

(At the moment we do not need commutativity.) From the first row of this diagram we get that

$$H^i(E_{U\cup V}, E_{U\cup V}^0) = 0 \quad \text{for } i < n$$

and that there is just one class $t_{U\cup V} \in H^n(E_{U\cup V}, E_{U\cup V}^0)$ such that

$$(j_U^*, j_V^*)(t_{U\cup V}) = (t_U, t_V).$$

This is the Thom class for $E_{U\cup V}$ and we can define the homomorphism $\tau_{U\cup V} : H^k(U \cup V) \to H^{k+n}(E_{U\cup V}, E_{U\cup V}^0)$ by

$$\tau_{U\cup V}(a) = p_*(a) \cup t_{U\cup V}.$$

Complete the diagram by this homomorphism. When we check the commutativity of the completed diagram, it suffices to apply Five Lemma to show that $\tau_{U\cup V}$ is an isomorphism.

To prove the commutativity we have to go into the cochain level from which the Mayer-Vietoris sequences are derived in natural way. Let t'_U and t'_V be cocycles representing the Thom classes t_U and t_V. We can choose them in such a way that

$$i_U^* t'_U = i_V^* t'_V = t'_{U\cap V}$$

where $t'_{U\cap V}$ represents the Thom class $t_{U\cap V}$. Consider the diagram where the rows are the short exact sequences inducing the Mayer-Vietoris exact sequences above.

$$\begin{tikzcd}
0 \arrow[r] & C_0^*(E_U + E_V) \arrow[r, "(j_U^*, j_V^*)"] & C_0^*(E_U) \oplus C_0^*(E_V) \arrow[r, "i_U^* - i_V^*"] & C^*(E_{U\cap V}) \arrow[r] & 0 \\
0 \arrow[r] & C^*(U + V) \arrow[u, dashed, "\tau'_{U\cup V}"] \arrow[r, "(j_U^*, j_V^*)"] & C^*(U) \oplus C^*(V) \arrow[u, "\tau'_U \oplus \tau'_V"] \arrow[r, "i_U^* - i_V^*"] & C^*(U \cap V) \arrow[u, "\tau'_{U\cap V}"] \arrow[r] & 0
\end{tikzcd}$$

Here we use the following notation: $C_*(U + V)$ is the free Abelian group generated by simplices lying in U and V, $C^*(U + V) = \mathrm{Hom}_R(C_*(U + V), R)$. $C_0^*(E_U + E_V)$ are the cochains from $C^*(E_U + E_V)$ which are zeroes on simplices from $C_*(E_U^0 + E_V^0)$. $\tau'_U(a) = p^*(a) \cup t'_U$. (According to Lemma in 3.12 the cohomology of $C_0^*(E_U + E_V)$ is $H^*(E_{U\cup V}, E_{U\cup V}^0; R)$.)

There is just one cocycle $t'_{U\cup V}$ representing the Thom class $t_{U\cup V}$ such that

$$(j_U^*, j_V^*)(t'_{U\cup V}) = (t'_U, t'_V).$$

If we show that τ'_U, τ'_V, $\tau'_{U\cap V}$ and $\tau'_{U\cup V}$ are cochain homomorphisms which make the diagram commutative, then the diagram with the Mayer-Vietoris exact sequences will be also commutative. To prove the commutativity of the cochain diagram above is not difficult and left to the reader. Here we prove that τ'_U is a cochain homomorhism. (The proof for the other τ' is the same.)

Let $a \in C^k(U)$. Since t'_U is cocycle we get

$$\delta\tau'_U(a) = \delta(p^*(a) \cup t'_U) = \delta(p^*(a)) \cup t'_U + (-1)^k p^*(a) \cup \delta(t'_U) = p^*(\delta(a)) \cup t'_U = \tau'_U\delta(a).$$

(3) Let B be compact (particullary a finite CW-complex). Then there is a finite open covering $U_1, U_2, \ldots, U_m$ such that E_{U_i} is homeomorphic to $U_i \times \mathbb{R}^n$. So according to (1) the statement of Theorem holds for all E_{U_i}. Using (2) and induction we can show that Theorem holds for $E = \bigcup_{i=1}^m E_{U_i}$ as well.
(4) The proof for the other base spaces B needs a limit transitions in cohomology and the fact that for any B there is always a CW-complex X and a map $f : B \to X$ inducing isomorphism in cohomology. Here we omit this part. □

8.5. Euler class. Let $\xi = (E, B, p)$ be oriented vector bundle of dimension n with the Thom class $t_\xi \in H^n(E, E^0; \mathbb{Z})$. Consider the standard inclusion $j : E \to (E, E^0)$. Since $p : E \to B$ is a homotopy equivalence, there is just one class $e(\xi) \in H^n(B; \mathbb{Z})$, called the *Euler class* of ξ, such that

$$p^*(e(\xi)) = j^*(t_\xi).$$

For R-oriented vector bundles we can define the Euler class $e(\xi) \in H^n(B; R)$ in the same way. Particulary, any vector bundle $\xi = (E, B, p)$ has an Euler class with $\mathbb{Z}_2$-coefficients called the *n-th Stiefel-Whitney class* $w_n(\xi) \in H^n(B; \mathbb{Z}_2)$.

8.6. Gysin exact sequence. The following theorem gives us a useful tool for computation of the ring structure of singular cohomology of various spaces.

Theorem (Gysin exact sequence)**.** *Let $\xi = (E, B, p)$ be an R-oriented vector bundle of dimension n with the Euler class $e(\xi) \in H^n(B; R)$. Then there is a homomorphism $\Delta^* : H^*(E^0; R) \to H^*(B; R)$ of modules over $H^*(B; R)$ such that the sequence*

$$\ldots \xrightarrow{p^*} H^{k+n-1}(E^0; R) \xrightarrow{\Delta^*} H^k(B; R) \xrightarrow{\cup e(\xi)} H^{k+n}(B; R) \xrightarrow{p^*} H^{k+n}(E^0; R) \xrightarrow{\Delta^*} \ldots$$

is exact.

Proof. The definition of Δ^* and the exactness follows from the following cummutative diagram where we have used the long exact sequence for the pair (E, E^0) and the Thom isomorphism τ:

$$\begin{tikzcd}
H^{k+n-1}(E^0) \arrow[r, "\delta^*"] \arrow[dr, dashed, "\Delta^*"'] & H^{k+n}(E, E^0) \arrow[r, "j^*"] & H^{k+n}(E) \arrow[r, "i^*"] & H^{k+n}(E^0) \\
& H^k(B) \arrow[u, "\cong"', "\tau"] \arrow[r, dashed, "\cup e(\xi)"'] & H^{k+n}(B) \arrow[u, "\cong"', "p^*"] \arrow[ur, "p^*"'] &
\end{tikzcd}$$

The right action of $b \in H^*(B)$ on $H^*(E^0)$ is given by

$$x \cdot b = x \cup i^* p^*(b), \quad x \in H^*(E^0).$$

Using the definition of the connecting homomorphism and the properties of cup product one can show that

$$\Delta^*(x \cdot b) = \Delta^*(x) \cup b.$$

The details are left to the reader. □

Example. Consider the canonical one dimensional vector bundle $\gamma = (E, \mathbb{RP}^n, p)$ where

$$E = \{(l, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1};\ v \in l\},$$

the elements of $\mathbb{RP}^n$ are identified with lines in $\mathbb{R}^{n+1}$ and $p(l, v) = l$. The space E_0 is equal to $\mathbb{R}^{n+1} - \{0\}$ and homotopy equivalent to S^n.

Using the Gysin exact sequence with $\mathbb{Z}_2$-coefficients and the fact that $H^k(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 \leq k \leq n$, we get successively that the first Stiefel-Whitney class $w_1(\gamma) \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ is different from zero and that

$$H^*(\mathbb{RP}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma)]/\langle w_1(\gamma)^{n+1}\rangle.$$

Exercise. Using the Gysin exact sequence show that

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/\langle x^{n+1}\rangle$$

where $x \in H^2(\mathbb{CP}^n; \mathbb{Z})$.

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