INTRODUCTION TO ALGEBRAIC TOPOLOGY

MARTIN ČADEK

13. Homotopy excision and Hurewicz theorem

One of the reasons why the computation of homotopy groups is so difficult is the fact that we have no general excision theorem at our disposal. Nevertheless, there is a restricted version of such a theorem. It has many consequences, one of them is the Freudenthal suspension theorem which enables us to compute $\pi_n(S^n)$. At the end of this section we define the Hurewicz homomomorphism which under certain conditions compares homotopy and homology groups.

13.1. Homotopy excision theorem. Excision theorem for homology groups has the following restricted analogue for homotopy groups.

Theorem (Blakers-Massey theorem)**.** *Let A and B be subcomplexes of CW-complex $X = A \cup B$. Suppose that $C = A \cap B$ is connected, (A, C) is m-connected and (B, C) is n-connected. Then the inclusion*

$$j : (A, C) \hookrightarrow (X, B)$$

is $(m+n)$-equivalence, i. e. $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < m+n$ and an epimorphism for $i = m + n$.

Proof. We distinguish several cases.

1. Suppose that $A = C \cup \bigcup_\alpha e^{m+1}_\alpha$ and $B = C \cup e^{n+1}$. First we prove that $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is surjective for $i \leq m + n$.

Consider $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$. Using simplicial approximation lemma 12.4 we can suppose that there are simplices $\Delta^{m+1}_\alpha \subset e^{m+1}_\alpha$ and $\Delta^{n+1} \subset e^{n+1}$ such that their inverse images $f^{-1}(\Delta^{m+1}_\alpha)$, $f^{-1}(\Delta^{n+1})$ are unions of convex polyhedra on each of which f is a linear surjection $\mathbb{R}^i$ onto $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$, respectively. We will need the following statement.

Lemma. *If $i \leq m + n$ then there exist points $p_\alpha \in \Delta^{m+1}_\alpha$, $q \in \Delta^{n+1}$ and a continuous function $\varphi : I^{i-1} \to [0, 1)$ such that*

(a) *$f^{-1}(p_\alpha)$ lies above the graph of φ,*
(b) *$f^{-1}(q)$ lies below the graph of φ,*
(c) *$\varphi = 0$ on ∂I^{i-1}.*

Let us postpone the proof of the lemma for a moment. The subspace $M = \{(s, t) \in I^{i-1} \times I;\ t \geq \varphi(s)\}$ is a deformation retract of I^i with deformation retraction $h : I^i \times I \to I^i$, $h(x, 0) = x$, $h(x, 1) \in M$. Then

$$H = fh : I^i \times I \to X$$

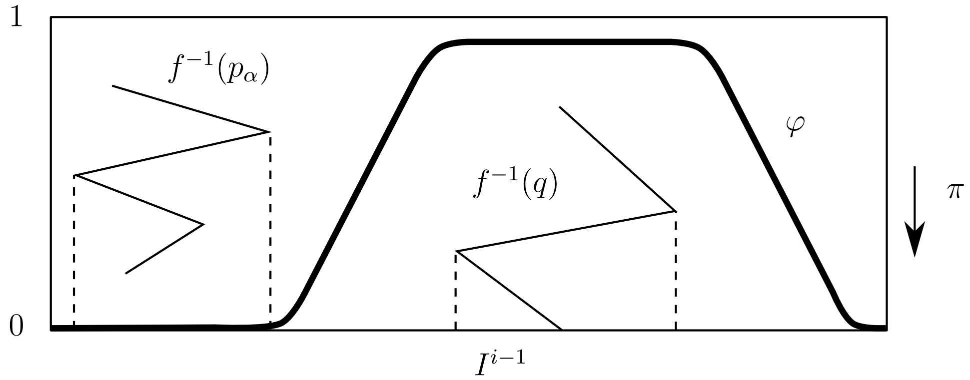

FIGURE 13.1. The graph of φ

provides a homotopy between f and

$$g : (I^i, \partial I^i, J^{i-1}) \to (X - \{q\}, X - \{q\} - \bigcup\{p_\alpha\}, x_0).$$

Obviously, g is homotopic to $\tilde{g} : (I^i, \partial I^i, J^{i-1}) \to (A, C, x_0)$. Hence $j_*[\tilde{g}] = [f]$.

The fact that $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is monomorphism for $i \leq m + n - 1$ can be proved by the same way as above replacing f by homotopy $h : I^i \times I \to (X, B)$. (Notice that $i + 1 \leq m + n$ now.)

Proof of the lemma. Choose arbitrary $q \in \Delta^{n+1}$. Then $f^{-1}(q)$ is a union of convex simplices of dimension $\leq i - n - 1$. Denote $\pi : I^i \to I^{i-1}$ the projection given by omitting the last coordinate. $\pi^{-1}(\pi(f^{-1}(q)))$ is the union of convex simplices of dimension $\leq i - n$. On the set $\pi^{-1}(\pi(f^{-1}(q))) \cap f^{-1}(\Delta_\alpha^{m+1})$ is f linear, hence

$$f(\pi^{-1}(\pi(f^{-1}(q)))) \cap \Delta_\alpha^{m+1}$$

is the union of simplices of dimension at most $i - n < m + 1$ for $i \leq m + n$. Consequently, there is $p_\alpha \in \Delta_\alpha^{m+1}$ such that

$$f^{-1}(p_\alpha) \cap \pi^{-1}(\pi f^{-1}(q)) = \emptyset.$$

Since $\operatorname{Im} f$ meets only finite number of cells e_α^{m+1}, the set $\bigcup \pi(f^{-1}(p_\alpha))$ is compact and disjoint from $\pi(f^{-1}(q))$. Hence there is continuous function φ, $\varphi = 0$ on $\bigcup \pi(f^{-1}(p_\alpha))$ and $\varphi = 1 - \varepsilon$ on $\pi(f^{-1}(q))$ with required properties.

2. Suppose that A is obtained from C by attaching cells e_α^{m+1} and B is obtained by attaching cells $e_\beta^{n_\beta}$ of dimensions $\geq n + 1$. Consider a map $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$. f meets only finite number of cells $e_\beta^{n_\beta}$. According to the case 1 we can show that f is homotopic to

$$\begin{aligned}
f_1 :&(I^i, \partial I^i) \to (X - e^{n_1}, B - e^{n_1}),\\
f_2 :&(I^i, \partial I^i) \to (X - e^{n_1} - e^{n_2}, B - e^{n_1} - e^{n_2}),\\
&\dots\\
f_r :&(I^i, \partial I^i) \to (A, C).
\end{aligned}$$

3. Suppose that A is obtained from C by attaching cells of dimensions $\geq m+1$ and B is obtained by attaching cells of dimensions $\geq n+1$. We may assume that the dimensions of new cells in A is $\leq m+n+1$ since higher dimensional ones have no effect on π_i for $i \leq m+n$ by cellular approximation theorem 12.5. Let A_k be a CW-subcomplex of A obtained from C by attaching cells of dimension $\leq k$, similarly let X_k be a CW-subcomplex of X obtained from B by attaching cells of dimension $\leq k$. Using the long exact sequences for triples (A_k, A_{k-1}, C) and (X_k, X_{k-1}, B), we get the diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(A_k, A_{k-1}) & \longrightarrow & \pi_i(A_{k-1}, C) & \longrightarrow & \pi_i(A_k, C) & \longrightarrow & \pi_i(A_k, A_{k-1}) & \longrightarrow & \pi_{i-1}(A_{k-1}, C) \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
\pi_{i+1}(X_k, X_{k-1}) & \longrightarrow & \pi_i(X_{k-1}, B) & \longrightarrow & \pi_i(X_k, B) & \longrightarrow & \pi_i(X_k, X_{k-1}) & \longrightarrow & \pi_{i-1}(X_{k-1}, B)
\end{array}$$

Applying the previous step for $X_k = A_k \cup X_{k-1}$ and $A_{k-1} = A_k \cap X_{k-1}$ we obtain the indicated isomorphisms. Now the induction with respect to k and 5-lemma completes the proof that $\pi_i(A_{m+n+1}, C) \to \pi_i(X_{m+n+1}, B)$ is an isomorphism for $i < m+n$ and an epimorphism for $i = m+n$.

4. Consider a general case. Then according to Corrolary 12.6 there is a CW-pair (A', C) homotopy equivalent to (A, C) and a CW-pair (B', C) homotopy equivalent to (B, C) such that $A' - C$ contains only cells of dimension $\geq m+1$ and $B' - C$ contains only cells of dimension $\geq n+1$. Then $X' = A' \cup B'$ is homotopy equivalent to $X = A \cup B$. According to the previous case $j' : (A', C) \to (X', B')$ is an $(m+n)$-equivalence, consequently $j : (A, C) \to (X, B)$ is an $(m+n)$-equivalence as well. ☐

Corollary. *If a CW-pair (X, A) is r-connected and A is s-connected with $r, s \geq 0$, then the homomorphism*

$$\pi_i(X, A) \to \pi_i(X/A)$$

induced by the quotient map $X \to X/A$ is an isomorphism for $i \leq r+s$ and an epimorphism for $i \leq r+s+1$.

Proof. Consider the diagram:

$$\begin{array}{ccccccc}
\pi_i(X, A) & \longrightarrow & \pi_i(X \cup CA, CA) & \longrightarrow & \pi_i(X \cup CA/CA) & \xrightarrow{\cong} & \pi_i(X/A) \\
& & \updownarrow \cong & \nearrow \cong & & & \\
& & \pi_i(X \cup CA) & & & &
\end{array}$$

The first homomorphism is $(r+s+1)$-equivalence by the homotopy excision theorem for $(s+1)$-connected pair (CA, A) and r-connected pair (X, A). The vertical isomorphism comes from the long exact sequence for the pair $(X \cup CA, CA)$ and the remaining isomorphisms are induced by a homotopy equivalence and the identity $X \cup CA/CA = X/A$. ☐

13.2. Freudenthal suspension theorem. We have defined the suspension of a space in 1.5 and the reduced suspension of a space with distinquished point in 1.6. In

4.3 we have introduced the suspension of a map. In a similar way we can define the reduced suspension of a map which preserves distinquished points. This notion defines so called suspension homomorphism $\pi_i(X) \to \pi_{i+1}(X)$, $[f] \mapsto [\Sigma f]$ for every space X.

Theorem (Freudenthal suspension theorem)**.** *Let X be $(n-1)$-connected CW-complex, $n \geq 1$. Then the suspension homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i \leq 2n-2$ and an epimorphism for $i \leq 2n-1$.*

Proof. The suspension ΣX is a union of two reduced cones $\widetilde{C}_+X$ and $\widetilde{C}_-X$ with intersection X. Now, we get

$$\pi_i(X) \cong \pi_{i+1}(\widetilde{C}_+X, X) \to \pi_{i+1}(\Sigma X, \widetilde{C}_-X) \cong \pi_{i+1}(\Sigma X)$$

where the first and the last isomorphisms come from the long exact sequences for pairs $(\widetilde{C}_+X, X)$ and $(\Sigma X, \widetilde{C}_-X)$, respectively, and the middle homomorphism comes from homotopy excision theorem for n-connected pairs $(\widetilde{C}_+X, X)$ and $(\widetilde{C}_-X, X)$. What remains is to show that the induced map on the level of homotopy groups is the same as suspension homomorphism which is left to the reader. □

13.3. Stable homotopy groups. The Freudenthal suspension theorem enables us to define *stable homotopy groups.* Consider a based space X and an integer j. The n-times iterated reduced suspension $\Sigma^n X$ is at least $(n-1)$-connected. If $n \geq j+2$, then $i = j+n \leq 2n-2$, so the assumptions of the Freudenthal suspension theorem are satisfied and we get

$$\pi_{j+(j+2)}(\Sigma^{j+2}X) \cong \pi_{j+(j+3)}(\Sigma^{j+3}X) \cong \pi_{j+(j+4)}(\Sigma^{j+4}X) \cong \dots$$

Hence we define the j-th stable homotopy group of the space X as

$$\pi_j^s(X) = \lim_{n\to\infty} \pi_{j+n}(\Sigma^n X).$$

We will write π_j^s for the j-th stable homotopy group of S^0.

13.4. Computations. In this paragraph we compute n-th homotopy groups of $(n-1)$-connected CW-complexes.

Theorem A. *$\pi_n(S^n) \cong \mathbb{Z}$ generated by the identity map for all $n \geq 1$. Moreover, this isomorphism is given by the degree map $\pi_n(S^n) \to \mathbb{Z}$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\pi_1(S^1) & \xrightarrow{\text{epi}} & \pi_2(S^2) & \xrightarrow{\cong} & \pi_3(S^3) & \xrightarrow{\cong} & \dots \\
\text{deg}\downarrow\cong & & \text{deg}\downarrow & & \text{deg}\downarrow & & \\
\mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & \xrightarrow{=} & \dots
\end{array}$$

where the horizontal homomorphisms are suspension homomorphisms and the left vertical isomorphism is known from Section 11 and determined by degree. The statement follows now from the fact that $\deg f = \deg \Sigma f$. □

Exercise. Prove that $\pi_n(\prod_{\alpha\in A} X_\alpha) = \prod_{\alpha\in A} \pi_n(X_\alpha)$.

Theorem B. $\pi_n(\bigvee_{\alpha\in A} S^n_\alpha) = \bigoplus_{\alpha\in A}\mathbb{Z}$ *for* $n \geq 2$.

Proof. Suppose first that A is finite. Then CW-complex $\bigvee_{\alpha\in A} S^n_\alpha$ is a subcomplex of CW-complex $\prod_{\alpha\in A} S^n_\alpha$. The pair

$$\Big(\prod_{\alpha\in A} S^n_\alpha, \bigvee_{\alpha\in A} S^n_\alpha\Big)$$

is $(2n-1)$-connected since $\prod_{\alpha\in A} S^n_\alpha$ is obtained from $\bigvee_{\alpha\in A} S^n_\alpha$ by attaching cells of dimension $\geq 2n$. Hence

$$\pi_n\Big(\bigvee_{\alpha\in A} S^n_\alpha\Big) = \pi_n\Big(\prod_{\alpha\in A} S^n_\alpha\Big) = \prod_{\alpha\in A}\pi_n(S^n_\alpha) = \bigoplus_{\alpha\in A}\pi_n(S^n_\alpha) = \bigoplus_{\alpha\in A}\mathbb{Z}.$$

If A is infinite, consider homomorphism $\phi : \bigoplus_{\alpha\in A}\pi_n(S^n_\alpha) \to \pi_n(\bigvee_{\alpha\in A} S^n_\alpha)$ induced by inclusions $\pi_n(S^n_\alpha) \to \bigvee_{\alpha\in A} S^n_\alpha$. ϕ is surjective since any $f : S^n \to \bigvee_{\alpha\in A} S^n_\alpha$ has a compact image and meets only finitely many S^n_α's. Similarly, if $h : S^n \times I \to \bigvee_{\alpha\in A} S^n_\alpha$ is homotopy between f and the constant map, it meets only finitely many S^n_α's, so $\phi^{-1}([f])$ is zero. □

Theorem C. *Suppose* $n \geq 2$. *If* X *is obtained from* $\bigvee_{\alpha\in A} S^n_\alpha$ *by attaching cells* e^{n+1}_β *via base point preserving maps* $\varphi_\beta : S^n \to \bigvee_{\alpha\in A} S^n_\alpha$, *then*

$$\pi_i(X) = \begin{cases} 0 & \text{if } i < n, \\ \bigoplus_{\alpha\in A}\pi_n(S^n_\alpha)/N & \text{if } i = n. \end{cases}$$

where N *is a subgroup of* $\bigoplus_{\alpha\in A}\pi_n(S^n_\alpha)$ *generated by* $[\varphi_\beta]$.

Proof. The first equality is clear from the cellular approximation theorem. Consider the long exact sequence for the pair $(X, X^n = \bigvee_{\alpha\in A} S^n_\alpha)$

$$\pi_{n+1}(X, X^n) \xrightarrow{\partial} \pi_n(X^n) \to \pi_n(X) \to 0.$$

The pair (X, X^n) is n-connected, X^n is $(n-1)$-connected, hence by Corollary 13.1

$$\pi_{n+1}(X, X^n) \to \pi_{n+1}(X/X^n) = \pi_{n+1}\Big(\bigvee_{\beta\in B} S^{n+1}_\beta\Big) = \bigoplus_{\beta\in B}\mathbb{Z}$$

is an isomorphism. Hence

$$\pi_n(X) = \pi_n(X^n)/\operatorname{Im}\partial = \pi_n\Big(\bigvee_{\alpha\in A} S^n_\alpha\Big)/N$$

since $\operatorname{Im}\partial$ is generated by $[\varphi_\beta]$. □

13.5. Hurewicz homomorphism. The *Hurewicz map* $h : \pi_n(X, A, x_0) \to H_n(X, A)$ assigns to every element in $\pi_n(X, A, x_0)$ represented by $f : (D^n, \partial D^n, s_0) \to (X, A, x_0)$ the element $f_*(\iota) \in H_n(X, A)$ where $\iota \in H_n(D^n, \partial D^n) = H_n(\Delta^n, \partial\Delta^n)$ is the generator induced by the identity map $\Delta^n \to \Delta^n$. In the same way we can define the Hurewicz map $h : \pi_n(X) \to H_n(X)$.

Proposition 13.6. *The Hurewicz map is a homomorphism.*

Proof. Let $c : D^n \to D^n \vee D^n$ be the map collapsing equatorial D^{n-1} into a point, $q_1, q_2 : D^n \vee D^n \to D^n$ quotient maps and $i_1, i_2 : D^n \to D^n \vee D^n$ inclusions. We have the diagram

$$\begin{array}{ccccc} H_n(D^n, \partial D^n) & \xrightarrow{c_*} & H_n(D^n \vee D^n, \partial D^n \vee \partial D^n) & \xrightarrow{f \vee g} & H_n(X, A) \\ & & {\scriptstyle i_{1*}+i_{2*}}\uparrow \ \downarrow{\scriptstyle q_{1*}\oplus q_{2*}} & & \\ & & H_n(D^n, \partial D^n) \oplus H_n(D^n, \partial D^n) & & \end{array}$$

Since $i_{1*} + i_{2*}$ is an inverse to $q_{1*} \oplus q_{2*}$, we get

$$\begin{aligned} h([f] + [g]) &= (f + g)_*(\iota) = (f \vee g)_* c_*(\iota) \\ &= \big((f \vee g)_*(i_{1*} + i_{2*})\big)\big((q_{1*} \oplus q_{2*})c_*\big)(\iota) = (f_* + g_*)(\iota \oplus \iota) \\ &= f_*(\iota) + g_*(\iota) = h([f]) + h([g]). \end{aligned}$$

□

We leave the reader to prove the following properties of the Hurewicz homomorphism directly from the definition:

Proposition 13.7. *The Hurewicz homomorphism is natural, i. e. the diagram*

$$\begin{array}{ccc} \pi_n(X, A) & \xrightarrow{f_*} & \pi_n(Y, B) \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ H_n(X, A) & \xrightarrow[f_*]{} & H_n(Y, B) \end{array}$$

commutes for any $f : (X, A) \to (Y, B)$.

The Hurewicz homomorphisms make commutative also the following diagram with long exact sequences of a pair (X, A)*:*

$$\begin{array}{ccccccc} \pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) \\ \downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_{(X,A)}} & & \downarrow{\scriptstyle h_A} \\ H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) \end{array}$$

13.8. Hurewicz theorem. The previous calculations of $\pi_n(\bigvee_{\alpha \in A} S^n_\alpha)$ enable us to compare homotopy and homology groups of $(n-1)$-connected CW-complexes via the Hurewicz homomorphism.

Theorem A (Absolute version of the Hurewicz theorem)**.** *Let* $n \geq 2$. *If* X *is a* $(n-1)$-*connected, then* $\tilde{H}_i(X) = 0$ *for* $i < n$ *and* $h : \pi_n(X) \to H_n(X)$ *is an isomorphism.*

For the case $n = 1$ see Theorem 11.5.

Proof. We will carry out the proof only for CW-complexes X. For general method which enables us to enlarge the result to all spaces see [Hatcher], Proposition 4.21.

First, realize that $h : \pi_n(S^n) \to H_n(S^n)$ is an isomorphism. It follows from the characterization of $\pi_n(S^n)$ by degree in Theorem 13.4.

According to Corollary 12.6 every $(n-1)$-connected CW-complex X is homotopy equivalent to a CW-complex obtained by attaching cells of dimension $\geq n$ to a point. Moreover cells of dimension $\geq n+2$ do not play any role in computing π_i and H_i for $i \leq n$. Hence we may suppose that

$$X = \bigvee_{\alpha \in A} S^n_\alpha \cup_{\varphi_\beta} \bigcup_{\beta \in B} e^{n+1}_\beta = X^{n+1}$$

where φ_β are base point preserving maps. Then $\tilde{H}_i(X) = 0$ for $i < n$.

Using the long exact sequences for the pair (X, X^n) and the Hurewicz homomorphisms between them we get

$$\begin{array}{ccccccc}
\pi_{n+1}(X, X^n) & \xrightarrow{\partial} & \pi_n(X^n) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\
\downarrow h & & \downarrow h & & \downarrow h & & \\
H_{n+1}(X, X^n) & \xrightarrow{\partial} & H_n(X^n) & \longrightarrow & H_n(X) & \longrightarrow & 0
\end{array}$$

Since $\pi_{n+1}(X, X^n)$ is isomorphic to $\pi_{n+1}(X/X^n) = \bigoplus \pi_{n+1}(S^{n+1}_\beta)$ and $\pi_n(X^n) = \bigoplus \pi_n(S^n_\alpha)$, the first and the second Hurewicz homomorphisms are isomorphisms. According to the 5-lemma so is $h : \pi_n(X) \to H_n(X)$. □

Let $[\gamma] \in \pi_1(A, x_0)$, $[f] \in \pi_n(X, A, x_0)$. Then $\gamma \cdot f$ and f are homotopic (although the homotopy does not keep the base point x_0 fixed), and consequently,

$$(\gamma \cdot f)_*(\iota) = f_*(\iota)$$

for $\iota \in H_n(D^n, \partial D^n)$. Hence $h([\gamma] \cdot [f]) = h([f])$.

Let $\pi'_n(X, A, x_0)$ be the factor of $\pi_n(X, A, x_0)$ by the normal subgroup generated by $[\gamma] \cdot [f] - [f]$. Let $h' : \pi'_n(X, A, x_0) \to H_n(X, A)$ be the map induced by the Hurewicz homomorphism h.

Theorem B (Relative version of the Hurewicz theorem)**.** *Let $n \geq 2$. If a pair (X,A) of the path connected spaces is $(n-1)$-connected, then $H_i(X, A) = 0$ for $i < n$ and $h' : \pi'_n(X, A, x_0) \to H_n(X, A)$ is an isomorphism.*

Proof. We will prove the theorem for a pair (X, A) of CW-complexes where A is supposed to be simply connected. In this case $\pi'_n(X, A, x_0) = \pi_n(X, A, x_0)$ and $h' = h$. For general proof see [Hatcher], Theorem 4.37, pages 371–373.

Since (X, A) is $(n-1)$-connected and A is 1-connected, Corollary 13.1 implies that the quotient map $\pi_n(X, A) \to \pi_n(X/A)$ is an isomorphism and X/A is $(n-1)$-connected. The absolute version of the Hurewicz theorem and the commutativity of the diagram

$$\begin{array}{ccc}
\pi_n(X, A) & \xrightarrow{\cong} & \pi_n(X/A) \\
h \downarrow & & \cong \downarrow h \\
H_n(X, A) & \xrightarrow{\cong} & H_n(X/A)
\end{array}$$

imply immediately the required statement. □

13.9. Homology version of Whitehead theorem. Since computations in homology are much easier that in homotopy, the following homology version of the Whitehead theorem gives a very useful method how to prove that two spaces are homotopy equivalent.

Theorem (Whitehead theorem)**.** *A map $f : X \to Y$ between two simply connected CW-complexes is homotopy equivalence if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n.*

Proof. Replacing Y by the mapping cylinder M_f we can consider f to be an inclusion $X \hookrightarrow Y$. Since X and Y are simply connected, we have $\pi_1(Y, X) = 0$. Using the relative version of the Hurewicz theorem and the induction with respect to n, we get successively that

$$\pi_n(Y, X) = H_n(Y, X) = 0.$$

The long exact sequence of homotopy groups for the pair (Y, X) yields that $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n. Applying now the Whitehead theorem 12.3 we get that f is a homotopy equivalence. □

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