Exercise 1. *Prove the following equalities (assuming some conditions):*

$$\bar{H}_*(X \vee Y) = \bar{H}_*(X) \oplus \bar{H}_*(Y)$$
$$\bar{H}_*(\bigvee_{i=1}^{n} X_i) = \bigoplus_{i=1}^{n} \bar{H}_*(X_i)$$
$$\bar{H}_*(\bigvee_{i=1}^{\infty} X_i) = \bigoplus_{i=1}^{\infty} \bar{H}_*(X_i)$$

Solution. Denote z the distinguished point of $X \vee Y$. For the pair $(X \vee Y, X)$ we have the following long exact sequence

$$\cdots \to \bar{H}_{i+1}(X \vee Y, X) \xrightarrow{0} \bar{H}_i(X) \to \bar{H}_i(X \vee Y) \to \bar{H}_i(X \vee Y, X) \xrightarrow{0} \bar{H}_{i-1}(X) \to \cdots .$$

Thus we have short exact sequence, which splits, because we have continuous (cts) map $\mathrm{id} \vee \mathrm{const}_z \colon X \vee Y \to X$ (which maps Y to z). Thus we have $\bar{H}_*(X \vee Y) \cong \bar{H}_*(X) \oplus \bar{H}_*(X \vee Y, X)$. Now it remains to prove $\bar{H}_*(X \vee Y, X) \cong \bar{H}_i(Y)$. If $X \vee Y$ is a CW-complex and X its subcomplex, it is known that $\bar{H}_i(X \vee Y, X) \cong \bar{H}_i(X \vee Y / X) = \bar{H}_i(Y)$. More generally, let U be some (sufficiently small) neighborhood of z in X. From excision theorem we have:

$$\bar{H}_i(X \vee Y, X) \cong \bar{H}_i\big(X \vee Y \smallsetminus (X \smallsetminus U), X \smallsetminus (X \smallsetminus U)\big) = \bar{H}_i(U \vee Y, U).$$

Because U should be[1] contractible, $\bar{H}_i(U \vee Y, U) \cong \bar{H}_i(Y, z) = \bar{H}_i(Y)$.

The second equality we get from the first by induction.

Let us prove the third equality. Denote $Y_n = X_1 \vee X_2 \vee \cdots \vee X_n$ and $Y = \bigvee_{n=1}^{\infty} Y_n$ and denote z the distinguished point of Y and Y_n for every n. We have the following diagram (where each arrow is an inclusion):

$$\begin{array}{ccccc} C_*(Y_1, z) & \longrightarrow & C_*(Y_2, z) & \longrightarrow & \cdots \\ & \searrow & \downarrow & \swarrow & \\ & & C_*(Y, z) & & \end{array}$$

Since Δ^k is compact, every continuous (cts) map $\Delta^k \to Y$ has image in some Y_n, thus it is easy to prove $C_*(Y, z) = \operatorname{colim} C_*(Y_n, z)$, thus

$$\bar{H}_*(Y) = \operatorname{colim} \bar{H}_*(Y_n) = \operatorname{colim} \bigoplus_{i=1}^{n} X_i = \bigoplus_{i=1}^{\infty} X_i. \qquad \square$$

[1] It is true at least for X locally contractible. It is not true generally.

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for each sufficiently large i. Every finitely generated abelian group can be written as $\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-times} \oplus \operatorname{Tor}$, where Tor *denote torsion part of the group. The number k is called the* rank *of the group.*

Euler characteristic χ of X is defined by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X)$$

Example. *We know* $H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & otherwise. \end{cases}$ *Thus* $\chi(S^n) = 1 - (-1)^n$.

Exercise 2. *Let (C_*, ∂) be a chain complex with homology $H_*(C_*)$. Prove that $\chi(X) = \chi(C_*)$, where*

$$\chi(C_*) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} C_i.$$

Solution. We have two short exact sequences:

$$0 \to Z_i \hookrightarrow C_i \xrightarrow{\partial} B_{i-1} \to 0$$
$$0 \to B_i \hookrightarrow Z_i \to Z_i/B_i = H_i \to 0,$$

where C_i, cycles Z_i and boundaries B_i are free abelian groups, thus $\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1}$ and $\operatorname{rank} H_i = \operatorname{rank} Z_i - \operatorname{rank} B_i$. Thus we have

$$\begin{aligned} \chi(C_*) &= \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} Z_i + \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} B_{i-1} \\ &= \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} Z_i - \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} B_i = \chi(X). \end{aligned}$$ □

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for every sufficiently large i. Let $f \colon X \to X$ be a continuous map. Map f induces homomorphism on the chain complex $f_* \colon C_*(X) \to C_*(X)$ and on the homologiy groups $H_* f \colon H_*(X) \to H_*(X)$, where $H_* f(\operatorname{Tor} H_*(X)) \subseteq \operatorname{Tor} H_*(X)$. Thus it induces homomorphism

$$H_* f \colon H_*(X)/\operatorname{Tor} H_*(X) \to H_*(X)/\operatorname{Tor} H_*(X).$$

Since $H_*(X)/\operatorname{Tor} H_*(X) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\operatorname{rank} H_*(X)}$, map $H_* f$ can be written as a matrix, thus we can compute its trace. So we can define the Lefschetz number of a map f:

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f.$$

Similarly to the case of the Euler characteristic, it can be proved that[2]

$$\sum_{i=0}^{\infty}(-1)^i \operatorname{tr} H_i f = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr} f_i.$$

Theorem. *If $L(f) \neq 0$, then f has a fixed point.*

Exercise 3. *Use the theorem above to show, that every cts map f on D^n and $\mathbb{R}P^n$ where n is even has a fixed point.*

Solution. We know that that $H_i(D^n) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$ Because $H_0 f\colon H_0(D^n) \cong \mathbb{Z} \to \mathbb{Z} \cong H_0(D^n)$ can be only the identity, we have $L(f) = 1$, thus f has a fixed point.

Since $H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/2, & i < n, i \text{ odd;} \\ 0, & \text{otherwise,} \end{cases}$ and $\mathbb{Z}/2$ is torsion, we have $L(f) = 1$ as in the previous case. □

Exercise 4. *Let M be a smooth compact manifold. Prove, that there is a nonzero vector field on M if and only if $\chi(M) = 0$.*

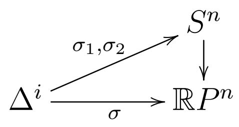
Solution. We will prove only implication $\Rightarrow$. Let v be a nonzero vector field on M. Define a map $X\colon [0,1] \times M \to M$ which satisfies $\dot{X}(t,x) = v(X(t,x))$ for every $x \in M$ and $X(0,x) = x$. There exists t_0 such that $X(t_0,x) \neq x$. Denote $f(x) = X(t_0,x)$, thus f has no fixed point, thus $L(f) = 0$. Because f is homotopic to id and $\operatorname{tr} H_i \mathrm{id} = \operatorname{rank} H_i(M)$, we get from homotopy invariance $0 = L(f) = L(\mathrm{id}) = \chi(M)$. □

Exercise 5. *Use $\mathbb{Z}/2$ coefficients to show, that every cts map $f\colon S^n \to S^n$ satisfying $f(-x) = -f(x)$ has an odd degree.*

Solution. The map f induces a map $g\colon \mathbb{R}P^n \to \mathbb{R}P^n$, since $f(\{x,-x\}) \subseteq \{f(x), -f(x)\}$. We have the short exact sequence[3]

$$\begin{array}{ccccccccc} & & \sigma & \longmapsto & \sigma_1 + \sigma_2 & \longmapsto & 2\sigma = 0 & & \\ 0 & \to & C_*(\mathbb{R}P^n, \mathbb{Z}/2) & \to & C_*(S^n, \mathbb{Z}/2) & \to & C_*(\mathbb{R}P^n, \mathbb{Z}/2) & \to & 0, \end{array}$$

where $\sigma\colon \Delta^i \to \mathbb{R}P^n$ is an arbitrary element of $C_*(\mathbb{R}P^n)$, σ_1, σ_2 are its preimages of a projection:



[2] $f_i\colon C_i(X) \to C_i(X)$

[3] $2\sigma = 0$ because of the $\mathbb{Z}/2$ coefficient.

From the short exact sequence we get the long exact sequence

$$\begin{array}{ccccccccc}
H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & H_i(S^n;\mathbb{Z}/2) & \longrightarrow & H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} & & \\
H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & H_i(S^n;\mathbb{Z}/2) & \longrightarrow & H_i(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

Because $H_0(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2$ and g_0 on $H_0(\mathbb{R}P^n;\mathbb{Z}/2)$ is an isomorphism, we can show by induction, that $H_i(\mathbb{R}P^n,\mathbb{Z}/2) = \mathbb{Z}/2$ and g_i is an isomorphism for every $i \le n-1$. An induction step is shown on the following diagram (three isomorphisms imply the fourth):

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow[\cong]{} & \mathbb{Z}/2
\end{array}$$

For $i = n$ we have the following situation (the vertical isomorphisms were proved by induction):

$$\begin{array}{ccccccccc}
\mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle ?} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
\mathbb{Z}/2 & \xrightarrow[\cong]{} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow[\cong]{} & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

Thus f_* (the arrow marked by ?) has to be an isomorphism for H_n, thus it maps $[1]_2$ to $[1]_2$, hence f has degree $1 \mod 2$. □