Lecture 2 - CW-complexes

Constructive definition of a CW-complex X

(1) Discrete space X^0 - its points are called 0-dim. cells

(2) Let us have X^{n-1} constructed. We will construct X^n:

Take attaching maps $f_\alpha : \partial D^n_\alpha = S^{n-1} \longrightarrow X^{n-1}$
and define

$$X^n = \bigcup_\alpha \left(D^n_\alpha \cup_{f_\alpha} X^{n-1} \right) \quad n\text{-skeleton}$$

It is the pullback in the diagram

$$\begin{tikzcd} \bigcup \partial D^n_\alpha \arrow[r, "\cup f_\alpha"] \arrow[d, hook] & X^{n-1} \arrow[d] \\ \bigcup D^n_\alpha \arrow[r] & X^n \end{tikzcd}$$

The interiors of D^n_α are called n-cells

(3) $X = \bigcup_{n=0}^{\infty} X^n$ with inductive topology
$A \subseteq X$ closed iff $A \cap X^n$ closed in X^n for all n

Examples (1) Sphere S^n consists of 0-cell $e^0 =$ point and an n-cell $e^n =$ interior of D^n

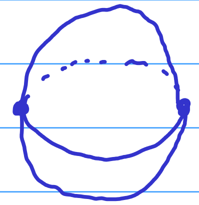
$X^0 =$ point $= X^1 = X^2 = \ldots = X^{n-1}$

attaching map $f\ \partial D^n \longrightarrow X^{n-1} =$ point

$$S^n = X^n = D^n \cup_f \text{point}$$

The sphere S^n can also have different CW-structures

S^2



2 0-cells
2 1-cells
2 2-cells

(2) Real projective space $\mathbb{R}P^n$

- lines in $\mathbb{R}^{n+1}$ going throw the origin
  (= 1-dim vector subspaces)

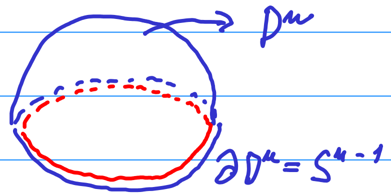
$$\mathbb{R}P^n = \mathbb{R}^{n+1}\setminus\{0\} \Big/_{\substack{v \simeq \lambda v \\ \lambda \neq 0}} = S^n \Big/_{v \simeq -v}$$

$$= D^n \Big/_{\substack{v \simeq -v \text{ for } \|v\| = 1 \\ v \in S^{n-1}}} = D^n \cup_f \partial D^n \Big/_{v \simeq -v}$$

$$= D^n \cup_{f_n} \mathbb{R}P^{n-1}$$

where the attaching map f_n is $\partial D^n = S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$

$$f_n(v) = [v] = \{v, -v\}$$



So $\mathbb{R}P^n = D^n \cup_f \mathbb{R}P^{n-1} = D^n \cup_{f_n} \mathbb{R}P^{n-1} = D^n \cup_{f_n} (D^{n-1} \cup_{f_{n-1}} \mathbb{R}P^{n-2})$

$$= \dots\dots = e^n \cup e^{n-1} \cup \dots \cup e^1 \cup e^0$$

point
$\|$
$\mathbb{R}P^0$

(3) Complex projective space $\mathbb{C}P^n$ -> tutorial

CW-complex   CW a shortcut for

C ... closure finite property (the closure of every cells consists only from a finite number of cells)

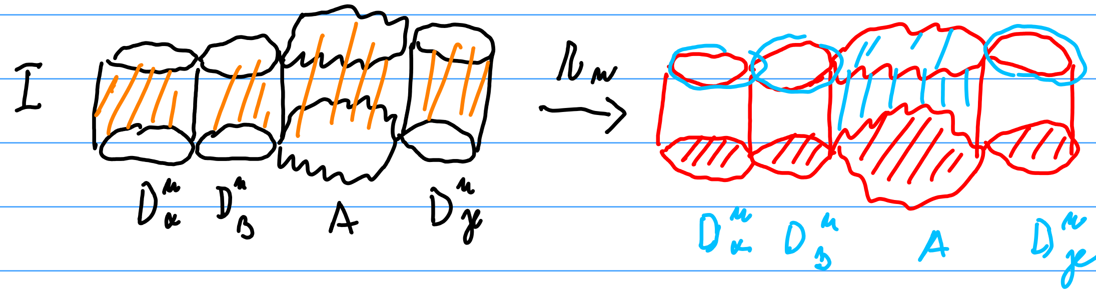
W ... weak topology $A \subseteq X$ closed iff $A \cap \overline{e^n}$ closed in $\overline{e^n}$ for every cell e^n

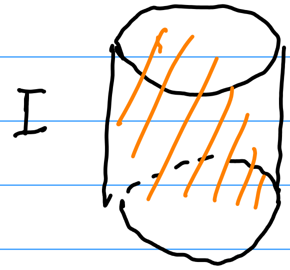
Theorem Let A be a subcomplex of a CW-complex X (i.e. cells of A are cells of X, attaching maps in A are also attaching maps in X).
Then the pair (X, A) has HEP.

Proof We use the criterion
(X,A) has HEP $\iff$ there is a retraction
$$r: I \times X \longrightarrow \{0\} \times X \cup I \times A$$
i.e for $i: \{0\} \times X \cup I \times A \hookrightarrow I \times X$
it holds
$$r \circ i = id$$

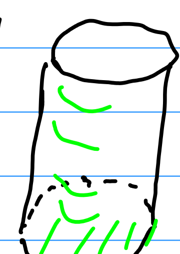
We will do the proof only for finite dimensional complex X, i.e. $X = X^n$ for some n.

(1) We will find a retraction
$$I \times X^n \longrightarrow \{0\} \times X^n \cup I \times (X^{n-1} \cup A)$$





We already know $\xrightarrow{q}$ that there is



D^n a retraction $\{0\} \times D^n \cup I \times \partial D^n$

Formally:

$$\begin{tikzcd}
\coprod_\alpha \left(I \times \partial D^n_\alpha \cup \{0\} \times D^n_\alpha\right) \arrow[rr, "\cup_\alpha (id_I \times f_\alpha) \cup id_I \times id_A"] \arrow[d, hook] & & I \times X^{n-1} \cup I \times A \cup \{0\} \times X^n \arrow[d, hook] \\
\coprod_\alpha \left(I \times D^n_\alpha\right) \arrow[u, "q_\alpha"] \arrow[rr, "\text{onto}"'] & & I \times X^n \arrow[u, dashed, "r_n"]
\end{tikzcd}$$

We can define $r_n : I \times X^n \to I \times (X^{n-1} \cup A) \cup \{0\} \times X^n$ using q_α

(2) Now we will analogously proceed in lower dimensions

$$I \times X \xrightarrow{r_n} \{0\} \times X^n \cup I \times (A \cup X^{n-1})$$

$$\Big\downarrow id \cup r_{n-1}$$

$$\{0\} \times X^n \cup \{0\} \times X^{n-1} \cup I \times (A \cup X^{n-2})$$

$$\Big\downarrow id \cup id \cup r_{n-2}$$

$$\{0\} \times X^n \cup \{0\} \times X^{n-1} \cup \{0\} \times X^{n-2} \cup I \times (A \cup X^{n-3})$$

$$\Big\downarrow id \; r_{n-1}$$

$$\vdots$$

$$\Big\downarrow id \; r_0$$

$$\{0\} \times X^n \cup \{0\} \times X^{n-1} \cup \dots \{0\} \times X^0 \cup I \times A$$

$$\|$$

$$\{0\} \times X^n \cup I \times A$$

This composition is a retraction we search for!

Some algebra as a preparation for homology groups

A_n abelian groups. The sequence of homomorphisms

$$\longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow$$

is exact if for all n

$$\operatorname{im} f_{n+1} = \ker f_n$$

Special cases of exact sequences

$0 \longrightarrow A \xrightarrow{f} B$ $\quad$ $\operatorname{im} 0 = 0 = \ker f$

f is a monomorphism

$A \xrightarrow{g} B \longrightarrow 0$ $\quad$ $\operatorname{im} g = \ker 0 = B$

g is an epimorphism

$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ $\quad$ f is an isomorphism

Short exact sequence is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

i mono, j epi and $\operatorname{im} i = \ker j$ which means

$$\begin{array}{l} B/\operatorname{im} i = B/\ker j \cong \operatorname{im} j = C \\ \quad \rotatebox{90}{$\cong$} \\ B/A \end{array}$$

Two basic examples

(1) $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$

$a \longmapsto 2a$

$b \longmapsto b \bmod 2$

(2) $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$

$a \longmapsto (a, 0)$

$(a, b) \longmapsto b$

The second short exact sequence splits.

The short exact sequence splits means

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] \arrow[l, bend left, dashed, "p"] & C \arrow[r] \arrow[l, bend left, dashed, "q"] & 0 \end{tikzcd}$$

(1) $\exists p : B \to A \quad p \circ i = id_A$

(2) $\exists q : C \to B \quad j \circ q = id_C$

(3) $\exists p : B \to A \; \exists q : C \to B \quad i \circ p + q \circ j = id_B$

These three statement are equal!
We will prove it on tutorial.

Chain complex $C_* = (C_n, \partial_n : C_n \to C_{n-1})_{n \geq 0}$
C_n abelian group, $\partial_n : C_n \to C_{n-1}$ homo

$$\partial_n \circ \partial_{n+1} = 0$$

$$\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots C_0 \to 0$$

Homomorphism of chain complexes $f_* \; C_* \to D_*$
$f_n : C_n \to D_n$ homo of ab. groups

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \end{tikzcd}$$

commutes

In chain complex $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$

That is why we can define homology groups of a chain complex C_* as

$$H_n(C_*, \partial) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

For f_* a homo of $C_* \longrightarrow D_*$ we define

$$H_n(f_*) : H_n(C_*) \longrightarrow H_n(D_*)$$

as

$$H_n(f_*)[c] = [f_n(c)]$$

$c \in \ker \partial_n^C$ and $f_n(c) \in \ker \partial_n^D$.

The definition is correct !