Lecture 3 – Singular homology

Repetition 1) chain complex (C_*, ∂_*)

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow 0$$

$$\partial_n \circ \partial_{n+1} = 0 \qquad \operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$$

Homology groups of C_* are

$$H_n(C_*, \partial_*) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

2) chain homomorphism $f : (C_*, \partial_*) \to (D_*, \partial_*)$

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

commutes

In homology f induces $H_n(f) = f_* : H_n(C_*) \to H_n(D)$

$c \in C_n, \partial c = 0 \qquad f_*([c]) = [f(c)] \in H_n(D)$

Short exact sequence of chain complexes

$$0 \longrightarrow A_* \xrightarrow{i_*} B_* \xrightarrow{j_*} C_* \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] & B_n \arrow[r, "j_n"] & C_n \arrow[r] & 0
\end{tikzcd}$$

exact sequences of Ab. groups

Thm Short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0$$

determines a long exact sequence of homology groups

$$\to H_n(A_*) \xrightarrow{i_*} H_n(B_*) \xrightarrow{j_*} H_n(C_*) \xrightarrow{\partial_*} H_{n-1}(A_*) \to \dots$$

where ∂_* is so called connecting homomorphism.

Definition of ∂_*

$c \in C_n$, $\partial c = 0$, $\exists b \in B_n$, $j(\partial b) = 0$, there is just one $a \in A_{n-1}$, that $i(a) = \partial b$.

Define $\partial_*[c] = [a]$.

Definition is correct - tutorial and the sequence is exact - homework.

This long exact sequence is natural - it means that

$$\begin{array}{ccccccccc} 0 & \to & A_* & \to & B_* & \to & C_* & \to & 0 \\ & & \downarrow k & & \downarrow l & & \downarrow m & & \\ 0 & \to & \bar{A}_* & \to & \bar{B}_* & \to & \bar{C}_* & \to & 0 \end{array}$$

s.e.s.

induces l.e.s.

$$\begin{array}{ccccccc} H_n(A) & \to & H_n(B) & \to & H_n(C) & \xrightarrow{\partial_*} & H_{n-1}(A) \\ \downarrow k_* & & \downarrow l_* & & \downarrow m_* & & \downarrow k_* \\ H_n(\bar{A}) & \to & H_n(\bar{B}) & \to & H_n(\bar{C}) & \xrightarrow{\partial_*} & H_{n-1}(\bar{A}) \end{array}$$

Chain homotopy Two chain maps $f_*, g_* : C_* \to D_*$ are homotopic if there are s_* such that

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & C_n \arrow[r, "\partial_n^C"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "s_{n-1}"'] \\
D_{n+1} \arrow[r, "\partial_{n+1}^D"'] & D_n \arrow[r] & D_{n-1}
\end{tikzcd}$$

$$f_n - g_n = \partial^D_{n+1} \circ s_n + s_{n-1} \circ \partial_n^C$$

Thm If f and g are chain homotopic, then

$$H_n(f) = H_n(g).$$

Proof: $c \in C_n \quad \partial c = 0$

$$H_n(f)[c] - H_n(g)[c] = [f_n(c)] - [g_n(c)] =$$

$$= [f_n(c) - g_n(c)] = [\partial \circ s_n(c) + s_{n-1} \underbrace{\partial c}_{0}]$$

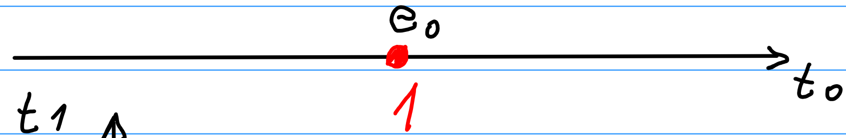
$$= [\partial \circ s_n(c)] = 0 \in H_n(D).$$

SINGULAR HOMOLOGY

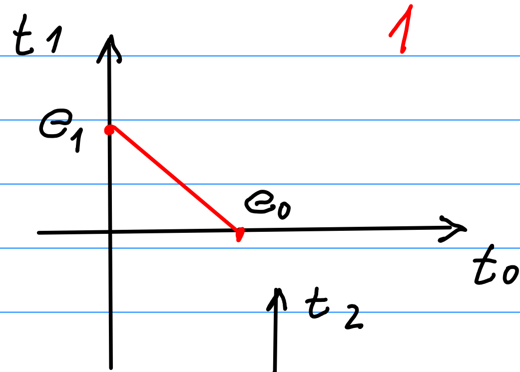
Standard n-simplex

$$\Delta^n = \{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \}$$

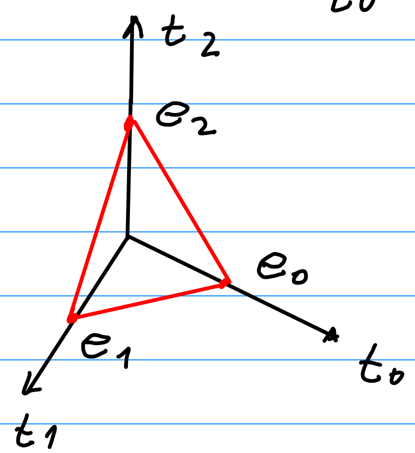
Δ^0



Δ^1



Δ^2



j-th face of Δ^n is $(n-1)$-simplex

$$(e_0, e_1, \dots, \hat{e}_j, e_{j+1}, \dots e_n)$$

omitted

Define $\quad \varepsilon_n^j : \Delta^{n-1} \longrightarrow \Delta^n$

$$\varepsilon_n^j (t_0, t_1, \dots, t_{n-1}) = (t_0, t_1, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1})$$

$$e_0 \longmapsto e_0, \dots\ e_{j-1} \longrightarrow e_{j-1},\ e_j \longrightarrow e_{j+1}, \dots\ e_{n-1} \longrightarrow e_n$$

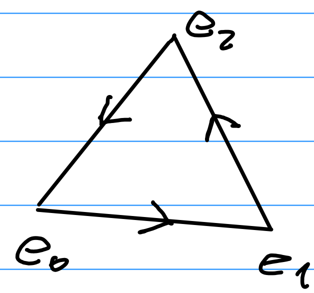
Lemma $\quad \varepsilon_{n+1}^k \circ \varepsilon_n^j = \varepsilon_{n+1}^{j+1} \circ \varepsilon_n^k$ for $k < j$.

Proof – text, tutorial

Singular n-simplex in a space X is a map
$$\sigma : \Delta^n \longrightarrow X$$

Free abelian group of all n-simplices in X is denoted
$$C_n(X)$$
and define boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$
$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \varepsilon_n^i$$

Geometrical meaning - we restrict σ on the boundary $\partial\Delta^n$ of Δ^n with suitable signs.



$$\partial(e_0 e_1 e_2) = e_0 e_1 + e_1 e_2 - e_0 e_2$$

It holds $\quad \partial_{n+1} \circ \partial_n = 0$

It follows from the lemma above - tutorial

$(C_*(X), \partial_*)$ is called the singular chain complex of the space X

Let $f : X \to Y$ be a map. It induces chain homo: $f_* \; C_*(X) \longrightarrow C_*(Y)$

$$f_*(\sigma) = f \circ \sigma \qquad \begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[rr, bend right, dashed] & X \arrow[r, "f"] & Y \end{tikzcd}$$

We get a functor $\quad Top \longrightarrow Chain$

Now we can define (singular) homology groups of a space X as

$$H_n(X) = H_n(C_*(X), \partial_*)$$

and for $f: X \to Y$

$$H_n(f) : H_n((C_*(X), \partial_*) \longrightarrow H_n((C_*(Y), \partial_*)$$

So we have $\quad Top \longrightarrow Chain \longrightarrow$ Graded Ab Groups

$$X \longmapsto C_*(X), \partial_* \longmapsto H_*(X)$$

Singular homology groups of a pair (X, A)

$A \subseteq X$ space X with a subspace A

We define $\quad C_*(X, A) = \dfrac{C_*(X)}{C_*(A)}$

where $\partial_n : C_n(X,A) \to C_{n-1}(X,A)$ is induced from

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X).$$

We have the following short exact sequence of chain complexes:

$$0 \longrightarrow C_*(A) \hookrightarrow C_*(X) \longrightarrow \underbrace{\frac{C_n(X)}{C_n(A)}}_{C_n(X,A)} \longrightarrow 0$$

We define $H_n(X,A) = H_n(C_*(X,A), \partial_*)$

clearly $H_n(X,\emptyset) = H_n(X)$

The short exact sequence above induces the long exact sequence of homology groups

$$\to H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \to$$

This sequence is natural and moreover ∂_* is the natural transformation of functors

$$H_n : Top^2 \longrightarrow Ab\ Groups$$

$$Top^2 \longrightarrow Ab\ Groups$$
$$(X,A) \longmapsto A \longmapsto H_{n-1}(A).$$

Homotopy invariance of singular homology

Thm If $f, g : (X,A) \to (Y,B)$ are homotopic, then
$$H_n(f) = H_n(g).$$

Corollary: If X and Y are homotopy equivalent spaces, then $H_n(X) \cong H_n(Y)$.

$f : X \to Y$, $\quad g : Y \to X \qquad f \circ g \sim id_Y$, $\quad g \circ f \sim id_X$

Then $\quad H_n(fg) = H_n(id_Y) \qquad H_n(gf) = H_n(id_X)$

$\quad H_n(f)\,H_n(g) = id_{H_n(Y)} \qquad H_n(g)\,H_n(f) = id_{H_n(X)}$

Excision Theorem

1st version Consider $C \subset A \subset X$, $\bar{C} \subseteq \operatorname{int} A$.
Then the inclusion
$$i : (X \setminus C, A \setminus C) \hookrightarrow (X, A)$$
induces the isomorphism of homology groups
$$i_* : H_n(X \setminus C, A \setminus C) \longrightarrow H_n(X, A)$$

2nd version Consider two subspaces A, B of a space X.
Let $X = \operatorname{int} A \cup \operatorname{int} B$. Then the inclusion
$$i : (B, A \cap B) \hookrightarrow (X, A)$$
induces the isomorphism in homology
$$i_* \; H_n(B, A \cap B) \longrightarrow H_n(X, A)$$

Proof of equivalence - tutorial

Singular homology of a point

Let $*$ be a one point space. Then
$$H_n(*) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

For every n there is just one map
$$\Delta^n \longrightarrow *$$
That is why
$$C_n(*) = \mathbb{Z} \quad \text{for} \quad n \geq 0$$
The boundary operator is

σ_n ... generator of $C_n(*)$ i.e. constant map $\Delta^n \to *$

$$\partial \sigma_n = \sum (-1)^i \sigma_n \circ \varepsilon_n^i = \sigma_{n-1} - \sigma_{n-1} + \sigma_{n-1} - \dots$$

$$= \begin{cases} 0 & \text{for } n \text{ odd} \\ \sigma_{n-1} & \text{for } n \text{ even} \end{cases}$$

$$\begin{array}{ccccccccccc} C_{2k}(*) & \xrightarrow{0} & C_{2k-1}(*) & \xrightarrow{id} & C_{2k-2}(*) & \dots & C_1 & \xrightarrow{0} & C_0 & \xrightarrow{0} & \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{0} & 0 \end{array}$$

Hence

$$H_{2k}(*) = \frac{\ker 0}{\operatorname{im} id} = \mathbb{Z}/\mathbb{Z} = 0 \qquad 2k > 0$$

$$H_{2k-1}(*) = \frac{\ker id}{\operatorname{im} 0} = 0/0 = 0 \qquad 2k-1 > 0$$

$$H_0(*) = \frac{C_0(*)}{0} = \mathbb{Z}.$$

Next property - boundary of disjoint union

$$H_n\Big(\bigsqcup_{\alpha \in A} X_\alpha\Big) = \bigoplus_{\alpha \in A} H_n(X_\alpha)$$