Lecture 9: Homotopy groups

Definition: n-th homotopy group of a space X with distinguished point x_0 is as a set

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$$
$$= [(I^n, \partial I^n), (X, x_0)]$$

$\pi_0(X, x_0)$ ... the set of path connected components with a distinguished element – the component containing x_0

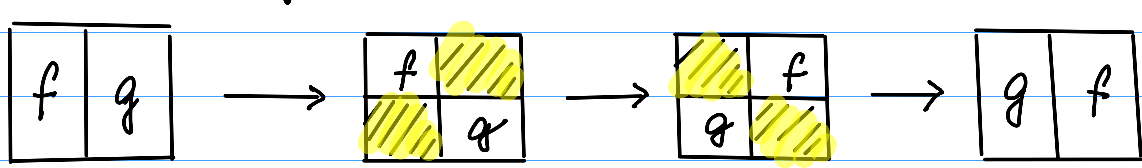
$n \geq 1$ $\pi_n(X, x_0)$ is a group with an operation induced by

$\boxed{f \mid g}$
$$(f+g)(t_1, \dots t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$

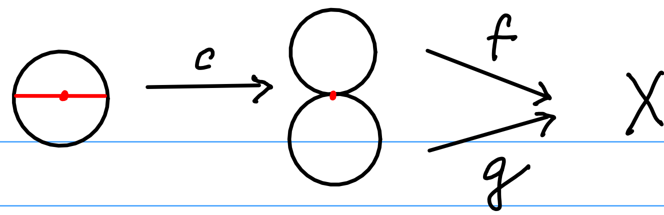
For homotopy classes:
$$[f] + [g] := [f+g]$$
$(f+g)+h \neq f+(g+h)$
$\boxed{fg \mid h} \sim \boxed{f \mid g \mid h}$

well defined, associative, with inverse given by
$$-f(t_1, t_2, \dots t_n) = f(1-t_1, t_2, \dots, t_n)$$

For $n \geq 2$ the groups are abelian – proof by the following picture:



In the interpretation of $\pi_n(X, x_0)$ as $[(S^n, s_0), (X, x_0)]$ the operation is
$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$$



$F : (X, x_0) \longrightarrow (Y, y_0)$ induces $\pi_n(X, x_0) \xrightarrow{F_*} \pi_n(Y, y_0)$

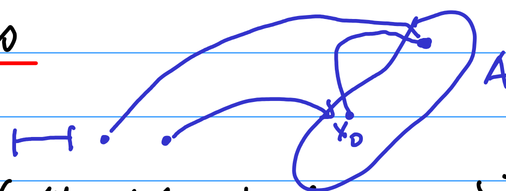
$$F_*([f]) = [F \circ f]$$

for $f : (S^n, s_0) \longrightarrow (X, x_0)$.

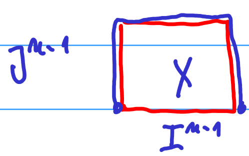
π_n is a functor from $Top^* \longrightarrow$ Groups.

Relative homotopy groups

$x_0 \in A \subseteq X$



$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$



I^n

$$= [(I^n, \partial I^{n-1}, J^{n-1}), (X, A, x_0)]$$

$\partial I^n \to A$

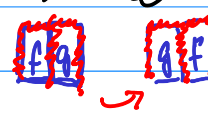
$J^{n-1} \to x_0$

where $J^{n-1} = \overline{(\partial I^{n-1} \setminus I^{n-1})}$ (closure)

Well defined for $n \geq 1$.

$n = 1$ only a set

$n \geq 2$ a group with the operation defined in the same way as for $\pi_n(X, x_0)$



$n \geq 3$ an abelian group $\pi_2(X, x_0)$

Prove that $n \geq 3$ abelian groups.

How to represent a neutral element in the homotopy groups?

In $\pi_n(X, x_0)$ the answer is easy - any map homotopic to the constant map.

$$f : S^n \to X \qquad f \sim \text{const}.$$

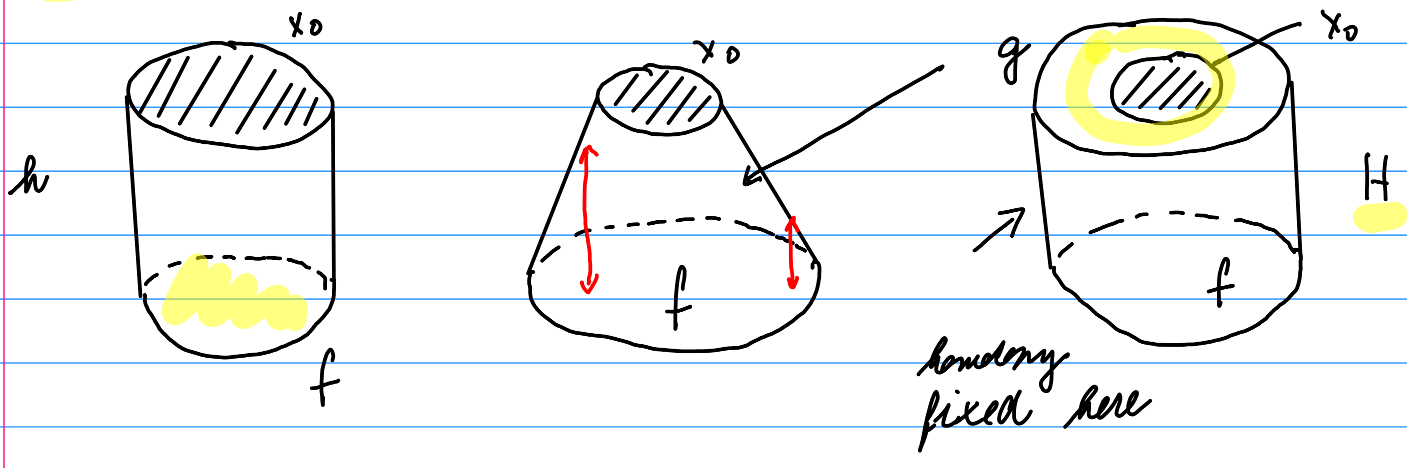
In $\pi_n(X, A, x_0)$ it is a little bit more complicated.

$f, g : (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ are homotopic rel S^{n-1}, if there is a homotopy $h : D^n \times I \longrightarrow X$ such that $\forall t \in I \; \forall x \in S^{n-1}$: $h(x,t) = f(x) = g(x)$.

Proposition A map $f : (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ represents neutral element in $\pi_n(X, A, x_0)$ if and only if it is homotopic rel S^{n-1} to a map with the image in A.

Proof: $\Leftarrow$ If $f \sim g$ rel S^{n-1} and $g(D^n) \subseteq A$, then $g = g \circ id_{D^n} \sim g \circ const = const$, so f is homotopic to a constant map a the homotopy on S^{n-1} takes values only in A. So f represents the neutral element of $\pi_n(X, A, x_0)$.

$\Rightarrow$ f homotopic to the constant map via homotopy $h : D^n \times I \longrightarrow X_0$ such that $h(S^{n-1} \times I) \subseteq A$.



$x \in S^{n-1}$: $g(x) = H(x,t) = f(x)$

Long exact sequence of homotopy groups

Theorem Let (X,A) be a pair of topological spaces with a distinguished point $x_0 \in A$. Then the sequence

$$\pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\delta} \pi_{n-1}(A,x_0)$$
$$\dots \longrightarrow \pi_0(A,x_0) \longrightarrow \pi_0(X,x_0)$$

is exact. Here $i : A \hookrightarrow X$, $j : (X,x_0) \hookrightarrow (X,A)$.

Proof: Tutorial and homework.

Remark: Boundary operator $\delta : \pi_n(X,A,x_0) \to \pi_{n-1}(X,x_0)$ is defined: $f : (D^n, S^{n-1}, s_0) \longrightarrow (X,A,x_0)$

$$\delta([f]) = [f/S^{n-1}] : (S^{n-1}, s_0) \longrightarrow (A,x_0)$$

Fibrations:

A map $p: E \to B$ has the homotopy lifting property (HLP) with respect to a pair (X,A), if the following solid diagram can be completed by a map $X \times I \to E$

$$\begin{tikzcd} X\times\{0\} \cup A\times I \arrow[rr] \arrow[d, "i"'] & & E \arrow[d, "p"] \\ X \times I \arrow[rr] \arrow[urr, dashed] & & B \end{tikzcd}$$

$D^k \times \{0\}$

$\downarrow$

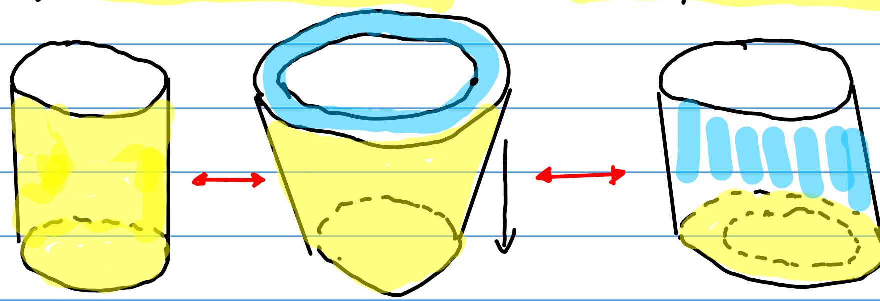
$D^k \times I$

p is called a fibration (Serre fibration, weak fibration) if it has HLP with respect to all pairs $(D^k, \emptyset)$.

Theorem: If $p: E \to B$ is a fibration, then it has HLP with respect to all pairs (X, A) of CW-complexes.

Proof: (1) $p: E \to B$ has HLP with respect to all pairs (D^k, S^{k-1}), since the pair

$$(D^k \times I,\ D^k \times \{0\} \cup S^{k-1} \times I) \cong (D^k \times I,\ D^k \times \{0\})$$



homeomorphisms

(2) Induction from $(k-1)$-skeleton to k-skeleton using the following diagram X, A X^k k-skeleton of X and A together

$$\begin{tikzcd}
\bigsqcup_\alpha S^{k-1}_\alpha \times I \cup \bigsqcup_\alpha D^k_\alpha \times 0 \arrow[r] \arrow[d] & X^{k-1} \times I \cup X^k \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\bigsqcup_\alpha D^k_\alpha \times I \arrow[r] \arrow[urr, red] & X^k \times I \arrow[r] \arrow[ur, green] \arrow[ul, phantom, "\ulcorner"] & B
\end{tikzcd}$$

$$\begin{tikzcd}
X^k \times 0 \cup A \times I \arrow[r] \arrow[d, hook] & E \arrow[d] \\
X^k \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

By induction

$$\begin{tikzcd}
X^k \times 0 \cup X^{k-1} \times I \arrow[r] \arrow[d, hook] & E \arrow[d] \\
X^k \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

Fibre bundle $(p: E, B, F)$ is a map $p: E \to B$ such that every $b \in B$ has a neighbourhood U and a homeomorphism $p^{-1}(U) \longrightarrow U \times F$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$ commutes.

Lemma: In every fibre bundle $(p \mid E, B, F)$ the projection $p: E \to B$ is a fibration.

Proof: See tutorial.

Trivial f. bundle $E = B \times F$
$$\downarrow p_1$$
$$B$$

Examples of fibre bundles:

(1) Projection $p: S^n \to \mathbb{R}P^n$, fibre S^0

(2) Projection $p: S^{2n+1} \to \mathbb{C}P^n$, fibre S^1

(3) The special case is so called Hopf fibration
$$S^1 \to S^3 \to S^2 \cong \mathbb{C}P^1$$

(4) Quaternionic projective spaces
$$p: S^{4n+3} \to \mathbb{H}P^n \qquad \mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$$
with the fibre S^3.
Especially: $S^3 \to S^7 \to \mathbb{H}P^1 = S^4$
(called also Hopf fibration) "$e_0 \cup e_4$"

(5) Cayley numbers (octonions) give
$$S^7 \to S^{15} \to S^8$$

(6) Let H be a Lie subgroup of G. Then the projection
$$p: G \to G/H$$
is a fibre bundle with the fibre H.

(7) $V_{n,k}$ Stiefel manifolds (k-tuples of orthonormal vectors in $\mathbb{R}^n$) For $n \geq k > l \geq 1$ we get the projection
$$p \quad V_{n,k} \to V_{n,l}$$
with the fibre $V_{n-l,\, k-l}$.

(8) $G_{n,k}$ Grassmann manifolds ... k-dim vector subspaces of $\mathbb{R}^n$

The projection $p: V_{n,k} \longrightarrow G_{n,k}$ is a fibration with the fibre $O(k)$.

Long exact sequence of a fibration

Consider a fibration $p: E \to B$, take $b_0 \in B$ a base point, put $p^{-1}(b_0) = F$ and choose $x_0 \in F \subseteq E$.

Lemma: For all $n \geq 1$ the map

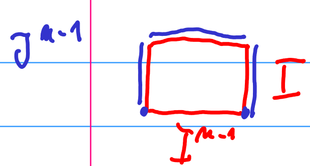
$$p_* \quad \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0)$$

is an isomorphism.

Proof: (1) p_* is an epimorphism. Let $f: (I^n, \partial I^n) \to (B, b_0)$

$$\begin{tikzcd}
J^{n-1} = I^{n-1}\times\{1\} \cup \partial I^{n-1}\times I \arrow[r, "\text{const. } x_0"] \arrow[d] & E \arrow[d, "p"] \\
I^{n-1}\times I \arrow[r, "f"'] \arrow[ur, dashed, "F"] & B
\end{tikzcd}$$

x_0



Let F be a lift in the diagram above.
Then $pF(\partial I^n) \subseteq \{b_0\}$, hence $F(\partial I^n) \subseteq F$ and $F(J^{n-1}) = x_0$. F represents an element in $\pi_n(E, F, x_0)$ such that $p_*[F] = [f]$.

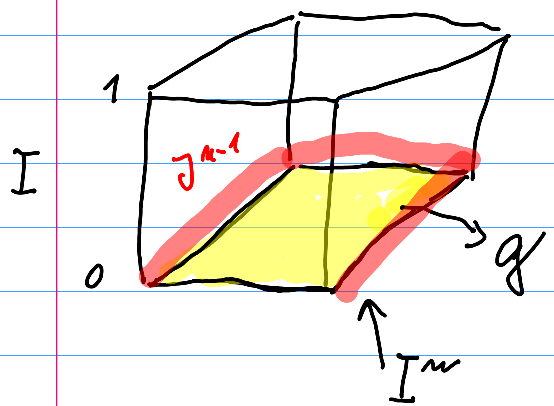
(2) p_* is a mono. Let $g: (I^n, \partial I^n, J^{n-1}) \longrightarrow (E, F, x_0)$ and $p_*[g] = 0$. Consider the homotopy

$\in \pi_n(B, b_0)$ $\qquad$ $[g] \in \pi_n(E, F, x_0)$.

$G : (I^n \times I, \partial I^n \times I) \longrightarrow (B, b_0)$ $G : p_* g \sim$ const

between $p \circ g$ and const.

$$\begin{tikzcd} J^{n-1} \times I \cup I^n \times \{0\} \cup I^n \times \{1\} \arrow[r, "\text{const} \cup g \cup \text{const}"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "G"'] \arrow[ur, "H"] & B \end{tikzcd}$$



H is a homotopy between g (lower face) and const (upper face) such that

$H(\partial I^n \times I) \subseteq F$.

$H(J^{n-1} \times I) = x_0$

Theorem: If $p : E \to B$ is a fibration, $p^{-1}(b_0) = F$, $x_0 \in F \subseteq E$ and B is path connected, then we have the following exact sequence:

$$\pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \dots$$

$$\dots \longrightarrow \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

Proof: Insert $p_* : \pi_n(E, F, x_0) \cong \pi_n(B, b_0)$ into the long exact sequence of the pair (E, F).

$$\begin{tikzcd} \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] \arrow[dr, dashed, "p_*"'] & \pi_n(E, F) \arrow[r, "\partial"] \arrow[d, "p_*", "\cong"'] & \pi_{n-1}(F) \\ & & \pi_n(B) \arrow[ur, dashed, "\partial"'] & \end{tikzcd}$$

Computation of ∂. Take $[f] \in \pi_n(B)$. Make a lift $\bar{f}$ of f into E and restrict $\bar{f}/S^{n-1}$. $\rightsquigarrow \pi_{n-1}(F)$

The sequence of the pair finishes with

$$\pi_0(F) \longrightarrow \pi_0(E, x_0).$$

If B is path connected it is a bijection between sets of path connected components, so we can add $\pi_0(B)$ to the end.