Exercise 1. *Show that $\mathbb{C}P^n$ is a CW-complex.*

Complex projective space

$$\mathbb{C}P^n = \{ \text{1-dim } \mathbb{C}\text{-vector subspaces in } \mathbb{C}^{n+1} \}$$

$$= \mathbb{C}^{n+1} \setminus \{0\} / \sim \qquad v \sim \lambda v \text{ for } \lambda \in \mathbb{C} \setminus \{0\}$$

$\mathbb{C}P^0$ is a point. $\mathbb{C}^1$

Now we want to get $\mathbb{C}P^n$ from $\mathbb{C}P^{n-1}$ by attaching some cells.

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\} / v \sim \lambda v,\ \lambda \in \mathbb{C} \setminus 0 =$$

$$= S^{2n+1} / v \sim \lambda v \quad |\lambda| = 1$$

$$= \underbrace{w_1, w_2, \dots w_n}_{w}, w_{n+1} \qquad \|w\| = 1 \qquad w_{n+1} \neq 0 \qquad \mathbb{C}$$

$$= \{ (w, \sqrt{1 - \|w\|^2}) \in \mathbb{C}^{n+1},\ w \in D^{2n} \} / w \sim \lambda w \quad \begin{matrix} \|w\| = 1 \\ |\lambda| = 1 \end{matrix}$$

$$= (D^{2n} \cup \underbrace{S^{2n-1}) / w \sim \lambda w \text{ for } w \in S^{2n-1}}_{} \quad |\lambda| = 1$$

$$= D^{2n} \cup_{f_n} \mathbb{C}P^{n-1}$$

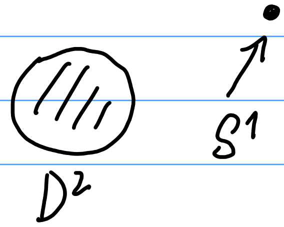
$$f_n : \partial D^{2n} = S^{2n-1} \longrightarrow \underbrace{S^{2n-1} / w \sim \lambda w}_{\mathbb{C}P^{n-1}} \quad |\lambda| = 1$$

$$\mathbb{C}P^n = D^{2n} \cup_{f_n} \mathbb{C}P^{n-1} = D^{2n} \cup_{f_n} \left(D^{2n-2} \cup_{f_{n-1}} \mathbb{C}P^{n-2} \right)$$

$$= \cdots\cdots = \qquad \mathbb{C}P^0$$

$$= e^{2n} \cup e^{2n-2} \cup \cdots \cup e^2 \cup e^0$$

$$\mathbb{C}P^1 = e^2 \cup e^0 = S^2$$

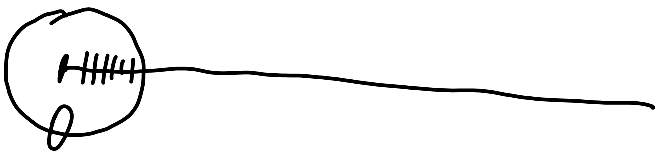


Exercise 2. *Prove that $A := \{\frac{1}{n}, n \in \mathrm{N}\} \cup \{0\}$ as a subspace of $\mathbb{R}$ is not a CW-complex. Then show that $X := I \times \{0\} \cup A \times I$ is not a CW-complex either.*

$A = \{\frac{1}{n} \in \mathbb{R},\ n \in \mathbb{N}^+\} \cup \{0\}$ topology comes from $\mathbb{R}$

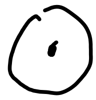
$C \subseteq A$ closed iff $C = A \cap O$, O closed in $\mathbb{R}$

$0 \in A$ does not lie in a cell of dim ≥ 1



any neighborhood of 0 is disconnected

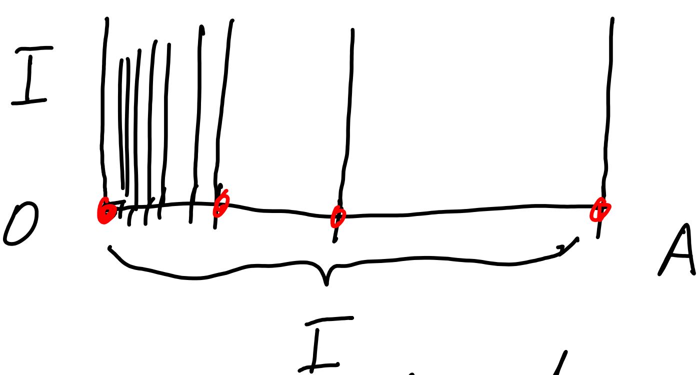
If 0 lies in a cell of dim ≥ 1, it is not the case



0 has to be zero cell in A

A^0 has discrete top., but $\{0\}$ is not open in A

$X = I \times \{0\} \cup A \times I$



$I = [0,1]$

X is not a CW-c

(1) X does not contain any cell of dim ≥ 2 after removing any point the remaining neighb. is disconnected.

(2) $(a,0) \in X$ after removing one gets at least 3 components in any neighbourh.
$0 < a < 1$ $a \in A$

$(a,0)$ $0<a<1$ is 0-cell

X^0 has to be closed

$(0,0) \in X^0$, contradiction with the

fact that X^0 has discrete topology.

Exercise 3. *Prove that every compact set A in a CW-complex X can have a nonempty intersection with only finitely many cells.*

$J = \{ \beta \mid e^\beta \cap A \neq \emptyset \}$

B is the set of just one point from every intersection $e^\beta \cap A$ $\quad \beta \in J$

We will prove

B is closed $\quad$ since $B \subseteq A$ compact $\Rightarrow$ B compact

B is discrete, $\quad$ B discrete $\wedge$ compact $\Rightarrow$ B is finite

By induction $B \cap X^n$ is closed.

$B \cap X^0$ discrete $\Rightarrow$ closed

Suppose $B \cap X^{n-1}$ closed

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ D^n \arrow[r, "F"'] & X^n \end{tikzcd}$$

$B \cap X^n$ closed

$\Leftrightarrow$ $F^{-1}(B \cap X^n)$ is closed in D^n

$F^{-1}(B \cap X^n) \cap S^{n-1}$ closed

$B \cap X^{n-1}$ is closed ✓

$$F^{-1}(B \cap X^n) = \underbrace{\underbrace{F^{-1}(B \cap X^n) \cap S^{n-1}}_{\text{closed}} \cup \text{one interior point}}_{\text{closed}}$$

$B \cap X^n$ is closed

B is discrete $\iff$ every $b \in B$ is open $\{b\}$

$\iff$ $B \setminus b$ is closed

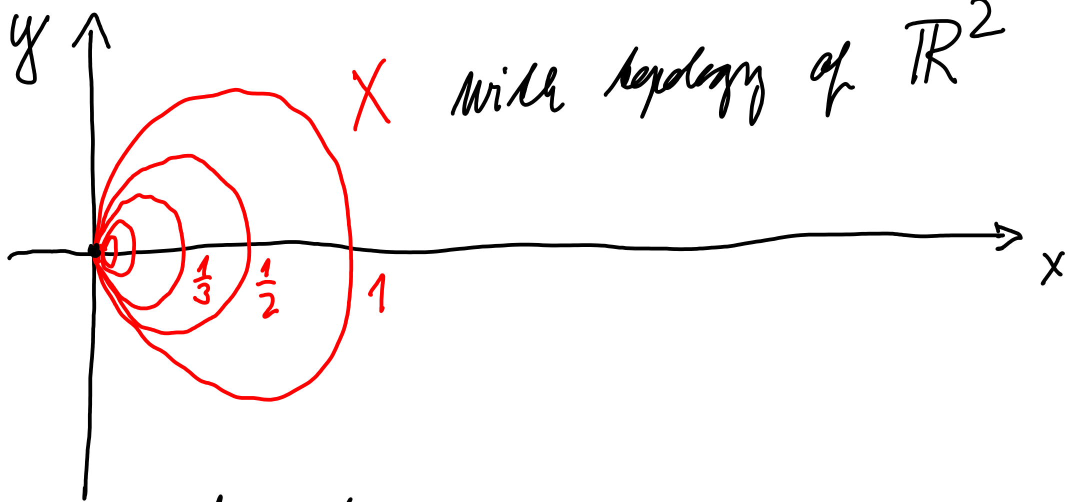
this can be proved in the same way as that B is closed

B discrete

Exercise 4. *Show that the Hawaiian earring given by*

$$X = \{(x, y) \in \mathbb{R}^2, (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2} \text{ for some } n\}$$

is not a CW-complex.



X with topology of $\mathbb{R}^2$

By contradiction

Suppose X is a CW-C

(0,0) must be a 0-cell, after removing you have ∞ many components in any neighb.

X has to have ∞-many 0-cells or 1-cells

X is compact, according to ex. 4 it can have only finitely many cells.

Exercise 5. *Show that for a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *of abelian groups (or more generally modules over a commutative ring) the following are equivalent:*

(1) *There exists* $p : B \to A$ *such that* $pf = \mathrm{id}_A$.

(2) *There exists* $q : C \to B$ *such that* $gq = \mathrm{id}_C$.

(3) *There exist* $p : B \to A$ *and* $q : C \to B$ *such that* $fp + qg = \mathrm{id}_B$.

Another equivalent condition is $B \cong A \oplus C$, *with* (p, g) *and* $f + q$ *being the respective inverse isomorphisms.*

Prove (1) implies (2) and (3) and (3) implies (1) and (2). The rest is for homework.

$$0 \to \underline{A} \underset{p}{\overset{f}{\rightleftarrows}} \underline{B} \underset{q}{\overset{g}{\rightleftarrows}} \underline{C} \to 0$$

f is a mono $\qquad \ker g = \operatorname{im} f$

g is an epi

reduction

$$0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$a \longmapsto 2a$

$b \longmapsto b \bmod 2$

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

$a \longmapsto (a, 0)$

$(a, b) \longmapsto b$

$a \longleftarrow (a,b)$

$(0, b) \longleftarrow b$

$$0 \to A \underset{p}{\overset{f}{\rightleftarrows}} B \underset{q}{\overset{g}{\rightleftarrows}} C \to 0$$

$(1) \Rightarrow (2) \wedge (3)$

$(1)\quad p : B \to A \qquad p \circ f = id_A$

We want to define $q : C \to B \qquad g \circ q = id_C$

g is onto $\quad \forall c \in C \quad \exists b \in B \quad g(b) = c$

$\exists b' \in B \quad g(b') = c$

We prove $\quad b - fp(b) = b' - fp(b')$

$g(b - b') = 0 \quad b - b' \in \ker g = \operatorname{im} f \quad b - b' = f(a)$

$a \in A$

$$\big(b - fp(b)\big) - \big(b' - fp(b')\big) = \big(b - b'\big) - \big(fp(b - b')\big) =$$

$$= f(a) - f\underbrace{pf}_{id_A}(a) = f(a) - f(a) = 0$$

So we can define $\quad q(c) = b - fp(b)$

for any $b \in B$, $g(b) = c$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[l, bend left, "p"] \arrow[r, "g"] & C \arrow[l, bend left, "q"] \end{tikzcd}$$

$$g \circ q = id_C$$

$$g \circ q(c) = g \circ \big(b - fp(b)\big) = g(b) - \underbrace{gf}_{0}p(b)$$

$$= g(b) = c$$

We prove (3)

$$f \circ p(b) + q \circ g(b) = fp(b) + b - fp(b) = b$$

$$f \circ p + q \circ g = id_B \qquad (3)$$

$(3) \Rightarrow (1) \wedge (2)$

$$fp + qg = id_B \qquad / \circ f$$

$$fpf + q\underbrace{gf}_{0} = f$$

$$f\underbrace{pf} = f = f \circ id_A \quad f \text{ is inj}$$

$$\Downarrow$$

$$pf = id_A \qquad (1)$$

$$fp + qg = id_B \qquad \mid g \circ$$

$$\underbrace{gf}_{0}p + gqg = g$$

$$\underbrace{gq}g = id_C \circ g \qquad g \text{ is surj.}$$

$$\Downarrow \quad gq = id_C \qquad (2)$$

Homework $(2) \Rightarrow (1)$ and (3)

Exercise 6. *Let $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$ be a short exact sequence of chain modules. We have defined the connecting homomorphism $\partial_* : H_n(C) \to H_{n-1}(A)$ by the formula $\partial_*[c] = [a]$, where $\partial c = 0$, $f(a) = \partial b$ and $g(b) = c$. Show that this definition does not depend on a nor b.*

Short exact sequence of chain complexes

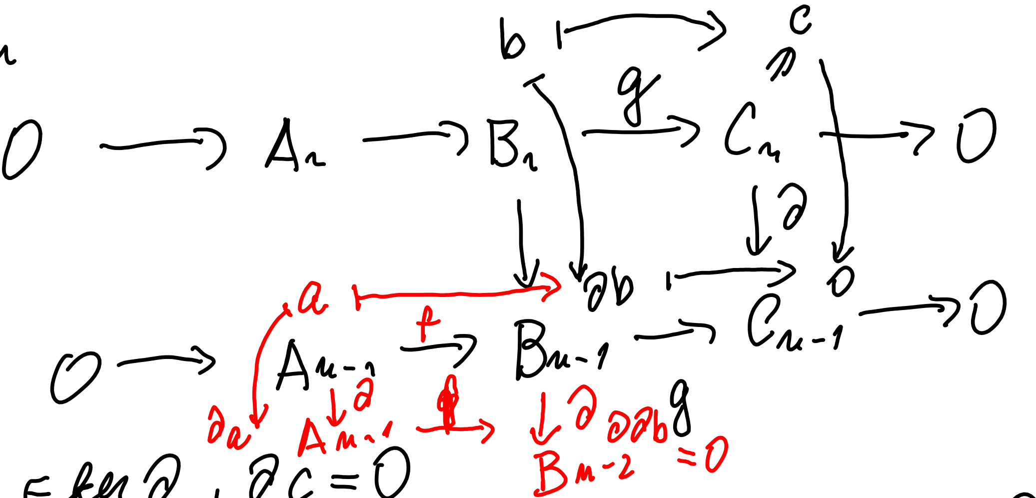
$$\begin{tikzcd}
 & \vdots & & & \\
0 \arrow[r] \arrow[d] & A_{n+1} \arrow[r, "f"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "g"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_n \arrow[r, "f"] \arrow[d, "\partial"] & B_n \arrow[r, "g"] \arrow[d, "\partial"] & C_n \arrow[d, "\partial_n"] & 0 \arrow[d] \\
0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} & 0 \\
 & \vdots & & &
\end{tikzcd}$$

Homology groups $H_n(C) = \dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$

There is a homomorphism (connecting homo)

$\partial_* : H_n(C_*) \longrightarrow H_{n-1}(A_*)$

Definition



$c \in C$, $c \in \ker \partial$, $\partial c = 0$

$\exists b \in B$ $\quad g(b) = c$ $\quad g(\partial b) = \partial g(b) = \partial c = 0$

$\partial b \in \ker g = \operatorname{im} f$ $\quad \exists!\, a \in A_{n-1}$ $\quad f(a) = \partial b$

We prove that $\partial a = 0$, $\quad f(\partial a) = \partial\partial b = 0$

f is inj. $\Rightarrow$ $\partial a = 0$.

We define $\partial_*[c] = [a]$.

This definition is correct

it does not depend on the choice of b and c

Suppose $b, b' \in B_n$ $\quad g(b) = g(b') = c$

$$
\begin{tikzcd}
a'' \arrow[r, mapsto] & b, b' \arrow[r, mapsto] & c \\
 & b-b' \arrow[r, mapsto] & 0 \\
A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d, "\partial"] & C_n \\
A_{n-1} \arrow[r, "f"] & B_{n-1} & \\
a, a' \arrow[r, mapsto] & \partial b, \partial b' & \\
\partial a'' \arrow[r, mapsto] & \partial b - \partial b' &
\end{tikzcd}
$$

$\partial a''$

$a - a' = \partial a''$
because f is inj.

$\partial_*[c] = [a] \qquad \partial_*[c] = [a']$

$[a] - [a'] = [a-a'] = [\partial a''] = 0 \in H_{n-1}(A_*)$

Independence of c

$$
\begin{tikzcd}
b' \arrow[rr, mapsto] & & c' & \\
B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 & \\
\partial b' \arrow[r, mapsto] & c & & \\
B_n \arrow[r] \arrow[d] & C_n & & \\
0 \arrow[r, mapsto] & \partial\partial b' = 0 & & \\
A_{n-1} \arrow[r] & B_{n-1} & &
\end{tikzcd}
$$

$c \in \operatorname{im} \partial$
$c = \partial c'$

$\partial_*[c] = [0]$
$= 0 \in H_{n-1}(A_*)$

The definition is correct

Conn. homom. plays an imp. role in the following sequence

$$H_{n+1}(C) \xrightarrow{\partial_*} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{f_*}$$

Thm This exact sequence is exact.

Homework to prove it.