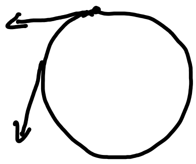
Exercise 1

S^n has a nonzero tangent vector field iff n is odd.

$n = 1$ $\quad v: S^1 \to \mathbb{R}^2$ is continuous



S^2

vecta field = tangent vecta field

$v: S^n \to \mathbb{R}^{n+1}$ is a nonzero vector field

$f: S^n \to S^n \qquad f(x) = \dfrac{v(x)}{\|v(x)\|} \qquad \|v(x)\| \neq 0$

On the interval $[0,\pi]$

$$h(x,t) = x \cdot \cos t + f(x) \cdot \sin t \qquad S^n \times [0,\pi] \to S^n$$

continuous

$x \perp f(x)$

$\|x \cos t + f(x) \sin t\|^2$

$= \|x\|^2 \cos^2 t + \|f(x)\|^2 \, |\sin t|^2 = \underline{1}$

$t = 0 \qquad h(x,0) = x \qquad id_{S^n}$

$t = \pi \qquad h(x,\pi) = -x \qquad id_{S^n} \sim -id_{S^n}$

$\deg id_{S^n} = \deg(-id_{S^n})$

$1 = (-1)^{n+1} \Rightarrow n$ is odd

n odd $\qquad S^{2n-1} \subseteq \mathbb{R}^{2n}$

$x_1, x_2, \dots x_{2n-1}, x_{2n}$

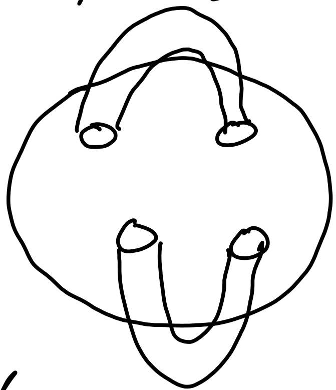
$$v(x_1, x_2, \dots x_{2n-1}, x_{2n}) = (x_2, -x_1, x_4, -x_3, \dots \quad x_{2n}, -x_{2n-1})$$

$$\langle x, v(x) \rangle = x_1 x_2 - x_2 x_1 + x_3 x_4 - x_4 x_3 + \dots \quad = 0$$

Exercise 2 Compute homology of two-dim. surfaces.

Two types of compact 2-dim. manifolds

M_g



S^2 2g holes

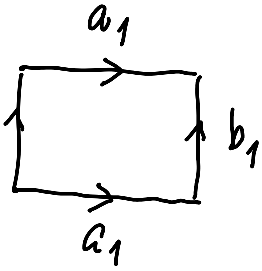
$M_g = S^2$ with g handles.

It has a structure of CW-complex.

Model for this structure

$$M_g = e^0 \cup e^1_1 \cup e^1_2 \dots \cup e^1_{2g} \cup e^2$$

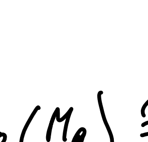
$g = 1$ torus



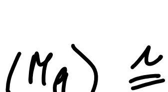
Hatcher

T^0 •

T^1



T^2



$$C^{CW}(M_g): \quad 0 \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{0} \overset{1}{\bigoplus_{1}^{2g}} \mathbb{Z} \xrightarrow{0} \overset{0}{\mathbb{Z}} \xrightarrow{0} 0$$

$H_0(M_g) \cong \mathbb{Z}$

$H_1(M_g) \cong \bigoplus_{1}^{2g} \mathbb{Z}$

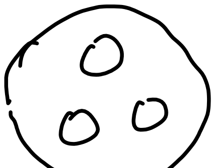
$H_2(M_g) \cong \mathbb{Z}$

$$d(e^2) = \underbrace{d_1}_{=0} e^1_1 + d_2 e^1_2 + \dots + d_{2g} e^1_{2g}$$

$\partial e_2 = S^1$

$d_1 = 0$

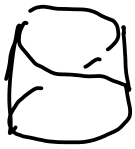
Nonorientable 2-dim surfaces N_g $g = 1, 2, 3, \dots$



S^2

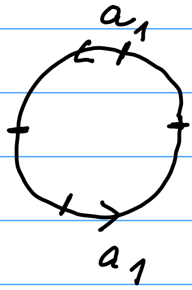
g holes

In every hole one attaches boundary of the Möbius band

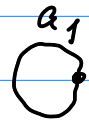
N_g  

S^2 with g Mobius bands attached

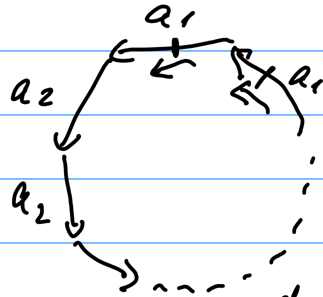
Model

$g = 1$ projective plane 

$\partial e_2 = S^1$



General g



$N_g = e^0 \cup e^1_1 \cup \dots \cup e^1_g \cup e^2$

$C^{CW}(N_g)$ $\qquad 0 \to \mathbb{Z} \xrightarrow{d_2} \bigoplus_1^g \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

$$d(e^2) = 2e^1_1 + 2e^1_2 + \dots + 2e^1_g$$

$H_0(N_g) \cong \mathbb{Z}$

$H_1(N_g) \cong \dfrac{\bigoplus_1^g \mathbb{Z}}{[2,2,2,\dots 2]} \cong \dfrac{\mathbb{Z}[e_1, e_2, \dots e_{g-1}, e_1 + e_2 \dots + e_g]}{2\mathbb{Z}[e_1 + e_2 + \dots + e_g]}$

$\cong \mathbb{Z}/2 \oplus \bigoplus_1^{g-1} \mathbb{Z}$

$H_2(N_g) \cong 0$

Exercise 3 $f: S^n \to S^n$ of degree k.
$X = D^{n+1} \cup_f S^n$, $p: X \to X/S^n \cong S^{n+1}$. Compute $H_*(X)$ and p_* in homology.

$X = D^{n+1} \cup_f S^n$ $f: S^n \to S^n$
degree $= k$

X CW-complex $k \neq 0$

$X = e^{n+1} \cup e^n \cup e^0$

$C^{CW}(X)$: $0 \xrightarrow{0} \underset{n+1}{\mathbb{Z}} \xrightarrow[k\times]{} \underset{n}{\mathbb{Z}} \xrightarrow{0} \underset{n-1}{0}$ $\underset{0}{\mathbb{Z}}$

$H_0(X) \cong \mathbb{Z}$

$H_n(X) \cong \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}/k$ $X^n = S^n$

$H_{n+1}(X) \cong 0$

projection $p: X \to X/S^n \cong S^{n+1}$

$p_* \quad \underset{\overset{\|}{0}}{H_{n+1}(X)} \to \underset{\overset{\|}{\mathbb{Z}}}{H_{n+1}(S^{n+1})}$ $p_* = 0$

$p_*: \underset{\cong \mathbb{Z}/k}{H_n(X)} \to H_n(S^{n+1}) \cong 0$ $p_* = 0$

Nevertheless, take homology with $\mathbb{Z}/k$ coefficients

$H_{n+1}(X) \cong \underline{\mathbb{Z}/k}$ $H_n(X) \cong \underline{\mathbb{Z}/k}$ $\underset{0}{(k\times)_*} = 0$

$C^{CW}(X; \mathbb{Z}/k)$ $\underset{n+1}{\mathbb{Z} \otimes \mathbb{Z}/k} \xrightarrow{k\times} \underset{n}{\mathbb{Z} \otimes \mathbb{Z}/k}$

$\to \mathbb{Z}/k \xrightarrow{0} \mathbb{Z}/k \to$

$p_* : H_{n+1}(X; \mathbb{Z}/k) \longrightarrow H_{n+1}(S^n; \mathbb{Z}/k)$

is identity, so nontrivial in homology.

C^{CW} :

$$\begin{tikzcd} \mathbb{Z}/k[e^{n+1}] \arrow[r, "0"] \arrow[d] & \mathbb{Z}/k[e^n] \arrow[dr] & e^0 \arrow[d] & X \\ \mathbb{Z}/k[e^{n+1}] \arrow[r] & 0 & e^0 & S^{n+1} \end{tikzcd}$$

In homology

$[e^{n+1}]$ is generator of $H_{n+1}(X;\mathbb{Z}/k)$ and p_* maps it in $[e^{n+1}]$ generator in $H_{n+1}(S^{n+1}, \mathbb{Z}/k)$.

In this case $p_* \; H_{n+1}(X; \mathbb{Z}/k) \to H_{n+1}(S^{n+1}; \mathbb{Z}/k)$ is an identity

Euler characteristic of X

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X)$$

Exercise 4 Prove

$$\chi(C_*) = \chi(H_*(C_*))$$

$$\chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

$$H_i = \frac{Z_i}{B_i} \qquad \frac{\text{cycles}}{\text{boundaries}} \qquad \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$$

$$B_i \subseteq C_i$$

$$\operatorname{rank} H_i(C_*) = \operatorname{rank} Z_i - \operatorname{rank} B_i$$

$$0 \to Z_i \hookrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0 \qquad \text{short exact seq}$$

$$\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1}$$

$$\chi(C_*) = \operatorname{rank} C_0 - \operatorname{rank} C_1 + \operatorname{rank} C_2 - \operatorname{rank} C_3 + \dots$$

$C_0 = Z_0$

$$= \operatorname{rank} Z_0 - (\operatorname{rank} Z_1 + \operatorname{rank} B_0) + (\operatorname{rank} Z_2 + \operatorname{rank} B_1)$$
$$- (\operatorname{rank} Z_3 + \operatorname{rank} B_2) + \dots$$

$$= (\operatorname{rank} Z_0 - \operatorname{rank} B_0) - (\operatorname{rank} Z_1 - \operatorname{rank} B_1)$$
$$+ (\operatorname{rank} Z_2 - \operatorname{rank} B_2) - (\qquad)$$

$$= \operatorname{rank} H_0 - \operatorname{rank} H_1 + \operatorname{rank} H_2 - \dots$$

$$= \chi(H(C_*))$$

Lefschetz number $f: X \to X$

$$H_i f: H_i(X)/\mathrm{Tor} \to H_i(X)/\mathrm{Tor}$$

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f$$

Lefschetz Thm

If $L(f) \neq 0$, then $f: X \to X$ has a fixed point.

Exercise 5: $f: D^n \to D^n$ has always a fixed point.
$f: \mathbb{R}P^n \to \mathbb{R}P^n$, n even, has always a fixed point

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$f: D^n \to D^n$ $\quad H_i(D^n) \cong \mathbb{Z}$ $i = 0$, 0 otherwise

$\operatorname{tr} H_i f$ = trace of the matrix $a_{11} + a_{22} + \dots + a_{nn}$

$L(f) = 1 \neq 0$

f has a fixed point

$$H_0(D^n) \to H_0(D^n)$$
$$f_0: \mathbb{Z} \xrightarrow{id} \mathbb{Z}$$

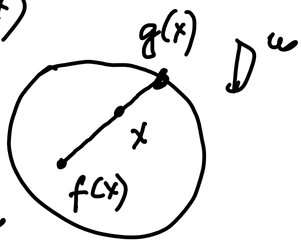
(1) $\operatorname{tr} id = 1$

Brauer theorem, $\quad$ Proof $\quad S^{n-1}$ is not a retract of D^n

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[dr, "id"'] & D^n \arrow[d, "r"] \\
& S^{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{\mathbb{Z}}{H_{n-1}(S^{n-1})} \arrow[r] \arrow[d, "id"] & \overset{0}{H_{n-1}(D^n)} \arrow[dl] \\
\mathbb{Z}\ H_{n-1}(S^{n-1}) &
\end{tikzcd}$$

impossible

$f: \underline{D^n \to D^n}$ without a fixed point $\quad x \neq f(x)$



$g: D^n \to S^{n-1}$ cont.

$g|_{S^{n-1}} = id_{S^{n-1}}$ retraction

$f : \mathbb{RP}^n \longrightarrow \mathbb{RP}^n$ $\quad$ n even

$$H_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2 & 0<i<n \\ & i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

To $\quad H_*/_{Tor} \cong \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{otherwise} \end{cases}$

$L(f) = \mathrm{tr}(id : \mathbb{Z} \to \mathbb{Z}) = 1$

f has a fixed point. It is not true for n odd $\quad H_i(\mathbb{RP}^n) = \mathbb{Z}$ $\quad i=0$, $i=n$

$\mathrm{tr}(f_* : H_n(\mathbb{RP}^n) \longrightarrow H_n(\mathbb{RP}^n))$

$f : X \longrightarrow X$

$f_i \quad \underbrace{C_i X}_{\text{some kind of homology}} \longrightarrow C_i X$

C^{CW} $\quad X$ CW-complex

$f : X \to X$ $\quad$ f is cellular map

$f(X^n) \subseteq X^n$ // every cell $e \in X^n$ is mapped into another cell in X^n

Exercise 6 M smooth compact manifold.

Prove, if there is a nonzero tangent vector field on M, then $\chi(M) = 0$.

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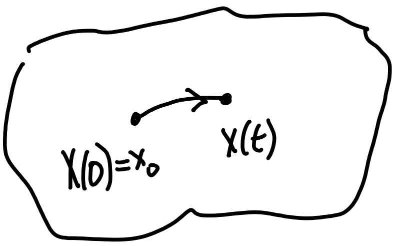
only opposite is more difficult tangent

$\Rightarrow$



M v : nonzero tangent vector

v

$$\dot{X}(t) = v(X(t))$$

$$X(0) = x_0$$

M compact

$\exists\, t > 0$ $[0,t]$, v

$X(0) \neq X(t)$

$X(0) = id_M$ $id_M \sim f$ f has no fixed point

$X(t) = f$ $L(id_M) = L(f) = 0$

$(-1)^i \operatorname{tr} id : H_i(M) \to H_i(M)$ ($\mathbb{Z}^k \to \mathbb{Z}^k$, $\operatorname{tr}(id) = k$, $k = \operatorname{rank} \mathbb{Z}^k$

$\mathbb{Z}$
$\mathbb{Z}$
$\mathbb{Z}$

$\underline{\operatorname{tr} id = \operatorname{rank} H_i(M)}$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$0 = L(f) = L(id) = \chi(M)$$

$\chi(M) = 0$ is necessary for the existence of nonzero vector field.

Exercise 7 Use $\mathbb{Z}/2$ coefficients to show that every map $f: S^n \to S^n$ satisfying $f(-x) = -f(x)$ has an odd degree.

$f: S^n \to S^n$ is odd

$$f(-x) = -f(x)$$

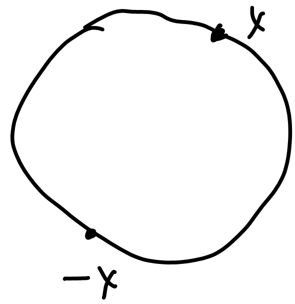
$\Longrightarrow \deg f$ is odd number.

We will leave it for the next!

Exercise 8 Borsuk-Ulam Thm. Every map $g : S^n \to \mathbb{R}^n$ has a point x such that
$$g(x) = g(-x).$$

One measures temperature and press on the earth

$\exists x$ and $-x$
with the same
temperature and press.

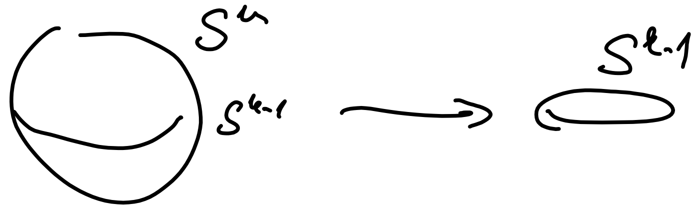


Proof by contradiction $\quad g(x) \neq g(-x) \quad \forall x \in S^n$

$$f : S^n \to S^{n-1} \qquad f(x) = \frac{g(x) - g(-x)}{\| g(x) - g(-x) \|} \in S^{n-1}$$

$$f(-x) = \frac{g(-x) - g(x)}{\| g(-x) - g(x) \|} = - \frac{g(x) - g(-x)}{\| g(x) - g(-x) \|} = - f(x)$$

f is odd



f / S^{n-1}

Previous prop. says deg f = odd

f/S^n_+ is homotopic to a constant map

f/S^{n-1}

deg f/S^{n-1} = deg const $= 0$
odd

contradiction