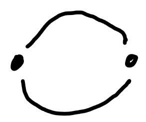
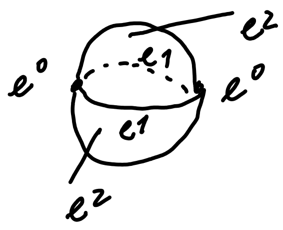
Exercise 1. *Show $\pi_k(S^\infty) = 0$ for all k, where S^∞ is $\operatorname{colim} S^n$. Using it show that S^∞ is homotopy equivalent to a point.*

$S^0 \subset S^1 \subset S^2 \subset S^3$

CW structure

$e^0 \cup e^0 \subset e^0 \cup e^0 \cup e^1 \cup e^1 \subset e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^2 \cup e^2 \subset \dots$





$S^\infty = \operatorname{colim} S^n$ CW-structure of S^∞ is given by this colimit

We prove that $\pi_k(S^\infty) = 0$.

$f\ S^k \to S^\infty$ S^k compact $f(S^k) \subset S^\infty$ is compact

$f(S^k)$ lies in finite union of cells

$\Rightarrow$ $f(S^k) \subset (S^\infty)^{(N)} = S^N$ N-skeleton

$k < N$

We know $\underline{\pi_k(S^N) = 0.}$

$g : S^k \to S^N$ is hom. to a cellular map $\bar{g} : S^k \to (S^N)^{(k)} = S^k$

$k < N$

$\bar{g} : S^k \to S^N \setminus \text{point}$

$\bar{g} \sim \text{const.}$

$f : S^k \to S^\infty$ $f \sim$ const in S^N

$f \sim$ const in S^∞ $[f] = 0$ in $\pi_k(S^\infty)$.

$\pi_k(S^\infty) = 0$

We use Whitehead to prove that S^∞ is homotopy equivalent to point

$$* \xrightarrow{i} S^\infty$$

$$\begin{array}{ccc} i_* \; \pi_k(*) & \longrightarrow & \pi_k(S^\infty) \\ \| & & \| \\ 0 & & 0 \end{array}$$

i_* is an iso in π_*, i is a weak hom. equivalence, $*$ and S^∞ are CW-complexes. Wh. Thm. gives that is a homotopy equivalence.

X is a CW-complex with $\pi_*(X) = 0$ then X is homotopy equivalent to a point.

Exercise 2. *Compute homotopy groups of $\mathbb{R}P^\infty$.*

$$\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \dots \quad \mathbb{R}P^n \subset \dots .$$

$$e^0 \cup e^1 \subset e^0 \cup e^1 \cup e^2 \subset \dots$$

$$\mathbb{R}P^\infty = \operatorname*{colim}_{n\to\infty} \mathbb{R}P^n \qquad \text{CW-complex}$$

$$\mathbb{Z}/2 \longrightarrow S^n \longrightarrow \mathbb{R}P^n$$

$$\begin{array}{ccl} x & \longrightarrow & [x] = [-x] \\ -x & & \end{array}$$

$$\mathbb{R}P^n = S^n/\mathbb{Z}/2$$

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d, equal] & S^n \arrow[r] \arrow[d, hook] & \mathbb{R}P^n \arrow[d, hook] \\
\mathbb{Z}/2 \arrow[r] & S^{n+1} \arrow[r] & \mathbb{R}P^{n+1}
\end{tikzcd}$$

Take colimits to get

$$\mathbb{Z}/2 \longrightarrow S^\infty \longrightarrow \mathbb{R}P^\infty \quad \text{fibration}$$

We use the long exact sequence of homotopy groups of fibration.

$$\begin{array}{ccccccc} \pi_{n+1}(S^\infty) & \longrightarrow & \pi_{n+1}(\mathbb{R}P^\infty) & \xrightarrow{\ \cong\ } & \pi_n(\mathbb{Z}/2) & \longrightarrow & \pi_n(S^\infty) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

$$\pi_n(\mathbb{Z}/2) = 0 \text{ for } n \geq 1$$

$$\pi_k(\mathbb{R}P^\infty) = 0 \quad k \geq 2$$

$$\pi_1(\mathbb{R}P^\infty) \cong \pi_0(\mathbb{Z}/2) \cong \mathbb{Z}/2 .$$

Exercise 3. *Show that the spaces $S^2 \times \mathbb{R}P^\infty$ and $\mathbb{R}P^2$ have the same homotopy groups but they are not homotopy equivalent.*

Computation of hom. groups

$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$$

$$\pi_n(S^2 \times \mathbb{R}P^\infty) \cong \pi_n(S^2) \times \pi_n(\mathbb{R}P^\infty)$$

$$\pi_1(S^2 \times \mathbb{R}P^\infty) \cong \pi_1(\mathbb{R}P^\infty) \cong \mathbb{Z}/2$$

$n \geq 2$ $\pi_n(S^2 \times \mathbb{R}P^\infty) \cong \pi_n(S^2)$ $\pi_i(S^2) = 0$ $i = 0, 1$

$\pi_2(S^2) \cong \mathbb{Z}$

$$\mathbb{Z}/2 \to S^2 \to \mathbb{R}P^2$$

$$\underset{\substack{\| \\ 0}}{\pi_n(\mathbb{Z}/2)} \to \pi_n(S^2) \xrightarrow{\cong} \pi_n(\mathbb{R}P^2) \to \underset{\substack{\| \\ 0}}{\pi_{n-1}(\mathbb{Z}/2)}$$

$\boxed{n \geq 2}$ $\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2) \cong \pi_n(S^2 \times \mathbb{R}P^\infty)$

$n = 1$ $$\underset{\substack{\| \\ 0}}{\pi_1(S^2)} \to \pi_1(\mathbb{R}P^2) \xrightarrow{\cong} \pi_0(\mathbb{Z}/2) \to \underset{\substack{\| \\ 0}}{\pi_0(S^1)}$$

$$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$$

$$\pi_1(\mathbb{R}P^2) \cong \pi_1(S^2 \times \mathbb{R}P^\infty)$$

$\to$ $H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/\langle \alpha^3 \rangle$ $\alpha \in H^1$

$$H^*(S^2 \times \mathbb{R}P^\infty; \mathbb{Z}/2) \cong H^*(S^2; \mathbb{Z}/2) \otimes H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$$

$\beta \in H^2$ $\qquad\qquad\qquad\qquad$ $\mathbb{R}P^n \quad x^{n+1} = 0$

$$\cong \mathbb{Z}/2[\beta]/\langle \beta^2 \rangle \otimes \mathbb{Z}/2[x]$$

We see that these rings cannot be isomorphic.

$$H^n(\mathbb{R}P^2; \mathbb{Z}/2) = 0 \quad \text{for} \quad n \geq 3$$

$$H^n(\mathbb{R}P^\infty \times S^2; \mathbb{Z}/2) = \mathbb{Z}/2 \quad \text{for} \quad n \geq 3$$

$\Rightarrow$ $\mathbb{R}P^2$ cannot be hom. equivalent to $\mathbb{R}P^\infty \times S^2$.

Exercise 4. *Extension lemma: Let (X, A) be a pair of CW-complexes, Y a space with $\pi_{n-1}(Y, y_0) = 0$ whenever there is a cell of dimension n in $X - A$. Then every map $f\colon A \to Y$ can be extended to a map $F\colon X \to Y$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\ X \arrow[ur, "F"'] & \end{tikzcd}$$

$\pi_{n-1}(Y) = 0$

$\exists\, e^n \in X \setminus A$

By induction using CW-structure of X

$$X_{-1} = A$$
$$X_0 = A \cup X^{(0)}$$
$$X_k = A \cup X^{(k)}$$

We will extend f from X_{-1} to X_0 to X_1 etc.

$F_k : X_k \longrightarrow Y \qquad F_k = F_{k-1}$ on X_{k-1}

$F_{-1} = f$

$F_0/_A = f \qquad F_0(x)$ is arbitrary point in Y

$x \in X_0 \setminus A$

Suppose we have F_{k-1}, let us define F_k. We have to find F_k on every k-cell

$e^k_\alpha \in X_k \setminus X_{k-1} \qquad \varphi : (D^k, S^{k-1}) \longrightarrow (X_k, X_{k-1})$

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi"] & X_{k-1} \arrow[d, "F_{k-1}"] \\ & Y \end{tikzcd}$$

$\pi_{k-1}(Y) = 0$

$\Rightarrow \quad \varphi/S^{k-1} \sim$ const.

$$\begin{tikzcd} D^k \times \{1\} \cup S^{k-1} \times I \arrow[rr, "\text{const} \cup F_{k-1} \circ \varphi/S^{k-1} \sim \text{const.}"] \arrow[d, hook] & & Y \\ D^k \times I \arrow[urr, "H"'] & & \end{tikzcd}$$

HEP

$H \quad \Gamma \quad \sim$ const

$H(1) =$ const

$H(0) = F_k$

We do the same on all k-cells

we get $\quad F_k : X_k \longrightarrow Y$

$F(x) = F_k(x) \quad$ for $\quad x \in X_k$

$F : X \longrightarrow Y$ extends $f : A \longrightarrow Y$.

Exercise 5. *Long exact sequence of the fibration (Hopf) $S^1 \to S^3 \to \mathbb{C}P^1 = S^2$.*

$$S^3 \to \mathbb{C}P^1 = S^2$$
$$\cap$$
$$\mathbb{C}^2$$
$$(z_1, z_2) \longmapsto \frac{z_1}{z_2} \in \mathbb{C} \cup \{\infty\} = S^2 = \mathbb{C}P^1$$

Compute first

$\pi_n(S^1)$

$$\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$$

Long sequence

$\pi_n(\mathbb{Z}) \cong 0$ for $n \geq 1$.

$\pi_n(\mathbb{R}) \cong 0$ for all n

$\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$

$\pi_1(S^1) \cong \pi_0(\mathbb{Z}) \cong \mathbb{Z}$

$\pi_0(S^1) \cong 0$

LES of $S^1 \to S^3 \to S^2$

$n \geq 3$
$$\pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1)$$
$$0 \qquad \pi_n(S^3) \xrightarrow{\cong} \pi_n(S^2) \qquad 0$$

for $n \geq 3$

next tutorial we prove that $\underline{\pi_k(S^k) \cong \mathbb{Z}}$ $\cong H_k(S^k)$

$$\mathbb{Z} \cong \pi_3(S^3) \cong \pi_3(S^2)$$

NONTRIVIAL and different $H_3(S^2) = 0$

$$\begin{tikzcd}
 & & \mathbb{Z} \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\
\pi_2(S^1) \arrow[r] \arrow[d, equal] & \pi_2(S^3) \arrow[r] \arrow[d, equal] & \pi_2(S^2) \arrow[r, "\cong"] & \pi_1(S^1) \arrow[d] \\
0 & 0 & & \pi_1(S^3) = 0
\end{tikzcd}$$

$\pi_2(S^2) \cong \mathbb{Z}$ without more general $\pi_4(S^4) \cong \mathbb{Z}$.

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r] \arrow[d, equal] & \pi_1(S^3) \arrow[r] \arrow[d, equal] & \pi_1(S^2) \arrow[r] \arrow[d, equal] & \pi_0(S^1) \arrow[r] \arrow[d, equal] & \pi_0(S^3) \arrow[d, equal] & \\
\mathbb{Z} & 0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

We get $\pi_2(S^2) = \mathbb{Z}$

and $\pi_3(S^2) = \mathbb{Z}$ under assumpt. that $\pi_3(S^3) \cong \mathbb{Z}$.

Exercise 6. *Using homotopy groups show that $\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^n$, $n > k \geq 1$.*

$n > k$

$$\mathbb{R}P^k \hookrightarrow \mathbb{R}P^n$$

$\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^n$

Suppose

$$\begin{tikzcd} \mathbb{R}P^k \arrow[r, hook, "i"'] \arrow[rr, bend right, "id"'] & \mathbb{R}P^n \arrow[r, "r"'] & \mathbb{R}P^k \end{tikzcd}$$

$$\begin{tikzcd} \pi_*(\mathbb{R}P^k) \arrow[r, "i_*"] \arrow[dr, "id"'] & \pi_*(\mathbb{R}P^n) \arrow[d, "r_*"] \\ & \pi_*(\mathbb{R}P^k) \end{tikzcd}$$

$\boxed{1 < k < n}$

$\pi_k(\mathbb{R}P^n) = 0$ from LES of fibr.

$1 < k$

$$\mathbb{Z}/2 \to S^n \to \mathbb{R}P^k$$

$$0 \cong \pi_k(S^n) \cong \pi_k(\mathbb{R}P^k)$$

$$\mathbb{Z}/2 \to S^k \to \mathbb{R}P^k$$

$$\mathbb{Z} \cong \pi_k(S^k) \cong \pi_k(\mathbb{R}P^k)$$

$1 < k < n$

$$\begin{tikzcd} \mathbb{Z} \cong \pi_k(\mathbb{R}P^k) \arrow[r] \arrow[dr, "id"'] & \pi_k(\mathbb{R}P^n) \cong 0 \arrow[d] \\ & \pi_k(\mathbb{R}P^k) \cong \mathbb{Z} \end{tikzcd}$$

contradiction

$k = 1 < n$

$$\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) \cong \mathbb{Z}$$

$$\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$$

We get

$$\begin{tikzcd}
\mathbb{Z} \cong \pi_1(\mathbb{R}P^1) \arrow[r] \arrow[dr, "id"'] & \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2 \arrow[d] \\
& \pi_1(\mathbb{R}P^1) \cong \mathbb{Z}
\end{tikzcd}$$

a contradiction

Exercise 7. *Consider the map $q\colon S^1\times S^1\times S^1 \to S^3$ defined as a map $S^1\times S^1\times S^1 \to D^3/S^2$ where D^3 is a small disk in the triple torus which is the identity in the interior of D^3 and constant on its complement. Further, consider the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}P^1$ (described in Hopf fibration $S^1 \to S^3 \to S^2$). Compute q_* and $(pq)_*$ in homotopy groups. Show that pq is not homotopic to a constant map.*

$$\begin{array}{ccc} & & S^1 \\ & & \downarrow \\ q: S^1\times S^1\times S^1 & \longrightarrow & S^3 \\ & & \downarrow p \\ & & S^2 \end{array} \qquad p(z_1,z_2) = \frac{z_1}{z_2}$$

$S^1\times S^1\times S^1$ is a 3-dim manifold

Take a 3-dim disk $D^3 \hookrightarrow S^1\times S^1\times S^1$

$$\begin{aligned} q(x) &= x \quad \text{for } x \in \operatorname{int} D^3 \\ &= * \quad \text{for all other } x \end{aligned} \qquad \downarrow \quad D^3/\underbrace{\partial D^3}_{*} \cong S^3$$

Compute q_*, $(pq)_*$ in homotopy groups.

Show that pq is not homotopic to a constant map.

$S^1\times S^1\times S^1$ $\quad \pi_1 \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

$\pi_n \cong 0 \quad n \geq 2$

S^3 $\quad \pi_1(S^3) \cong 0$

$\pi_2(S^3) \cong 0$

$\pi_n(S^3) \cong ? \quad n \geq 3$

$q_*\ \pi_*(S^1\times S^1\times S^1) \longrightarrow \pi_*(S^3)$ is zero map

S^2 $\quad \pi_1(S^2) \cong 0$

$\pi_2(S^2) \cong \mathbb{Z}$

$(pq)_*\ \pi_*(S^1\times S^1\times S^1) \to \pi_*(S^2) \quad \pi_n(S^2) \cong ? \quad$ for $n \geq 3$

$= 0$

pq is not homotopic to a constant map.

Contradiction: $pq \sim \text{const}$.

$$\begin{tikzcd} S^1\times S^1\times S^1 \arrow[r, "q"] \arrow[dr, "pq \sim \text{const}"'] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$$

$$\begin{tikzcd} & & S^1 \arrow[d, "i"] \\ (S^1)^3\times\{0\} \arrow[r, "q"] \arrow[d] \arrow[urr, "j = H(1)"] & S^3 \arrow[r, phantom] & S^3 \arrow[d, "p"] \\ S^1\times S^1\times S^1\times I \arrow[rr, "h"'] \arrow[urr, "H"] & & S^2, s_0 \end{tikzcd}$$

$h(0) = pq \qquad h(1) = \text{const}$

$H(0) = q$

$pH(1) = \text{const}_{s_0}$

$H(1) = p^{-1}(s_0) \cong S^1$

$$\begin{tikzcd} & H_3(S^1) \cong 0 \arrow[d, "i_*"] \\ \mathbb{Z} \cong H_3(S^1\times S^1\times S^1) \arrow[ur, "j_*"] \arrow[r, "q_*"] & H_3(S^3) \cong \mathbb{Z} \end{tikzcd}$$

We need to compute q_* in homology.

$\mathbb{Z} \underset{\psi}{\cong} H^3(M, M\setminus x) \longrightarrow H^3(M \cdot)$

$$\begin{tikzcd} \text{int}\, D^3 \subset S^1\times S^1\times S^1 \arrow[d, "\text{id}"'] \arrow[r, phantom] & S^1\times S^1\times S^1 \arrow[d, "q"] & \mu_x \arrow[r, mapsto] \arrow[d] & {[S^1\times S^1\times S^1]} \arrow[d, squiggly] \\ \text{int}\, D^3 \arrow[r, hook] & S^3 = D^3 \cup * & \mu_x \arrow[r, mapsto] & {[S^3]} \end{tikzcd}$$

$q_*[S^1\times S^1\times S^1] = [S^3]$

$$\begin{tikzcd} q_* \quad \mathbb{Z} \arrow[r] \arrow[dr, "\text{id}"'] & 0 \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

A CONTRADICTION.