CHAPTER 4

Averaging and Perturbation from a Geometric Viewpoint

In this chapter we describe some classical methods of analysis which are particularly applicable to problems in nonlinear oscillations. While these methods might be familiar to the reader who has studied nonlinear mechanics and perturbation theory, the present geometrical approach and the stress on obtaining approximations to Poincaré maps will probably be less familiar.

We start with the averaging method, originally due to Krylov and Bogoliubov [1934], which is particularly useful for weakly nonlinear problems or small perturbations of the linear oscillator. We show that, under suitable conditions, global information, valid on semi-infinite time intervals, can be obtained by this approach. Generally, in perturbation methods one starts with an (integrable) system whose solutions are known completely, and studies small perturbations of it. Since the unperturbed and perturbed vector fields are close, one might expect that solutions will also be close, but as we shall see, this is not generally the case, in that the unperturbed systems are often structurally unstable. As we have seen, arbitrarily small perturbations of such systems can cause radical qualitative changes in the structure of solutions. However, these changes are generally associated with limiting, asymptotic behavior and one does usually find that unperturbed and perturbed solutions remain close for *finite* times. Moreover, in this chapter we shall show that such finite time results, together with ideas from dynamical systems theory, do enable us to make deductions about the asymptotic behavior of solutions and the structure of the nonwandering set for the perturbed system.

Averaging is applicable to systems of the form

$$\dot{x} = \varepsilon f(x, t); \qquad x \in \mathbb{R}^n, \qquad \varepsilon \ll 1, \tag{4.0.1}$$

where f is T-periodic in t. In such a system the T-periodic forcing contrasts with the "slow" evolution of solutions on the average due to the $\mathcal{O}(\varepsilon)$ vector field. In the first four sections we will show how weakly nonlinear oscillators of the form

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, t) \tag{4.0.2}$$

can be recast in the standard form (4.0.1) and averaging applied. In this analysis we essentially deal with small perturbations of the linear oscillator $\ddot{x} + \omega^2 x = 0$, which is an example of an integrable Hamiltonian system.

We continue with a description of Melnikov's [1963] method for dealing with perturbations of general integrable Hamiltonian systems. Here one typically starts with a strongly nonlinear system,

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^{2n}, \tag{4.0.3}$$

and adds weak dissipation and forcing:

$$\dot{x} = f(x) + \varepsilon g(x, t). \tag{4.0.4}$$

While we are primarily concerned with periodically forced two-dimensional systems, we state the averaging results in the more general n-dimensional context, since they are no more difficult in that form. For Melnikov's method we restrict ourselves to two-dimensional problems, although some n-dimensional, and even infinite-dimensional, generalizations are available. We comment on extensions of this nature towards the end of the chapter. We also outline some of the theory of area preserving maps of the plane arising as Poincaré maps in time-periodic single degree of freedom Hamiltonian systems and in time-independent two degree of freedom systems.

4.1. Averaging and Poincaré Maps

There are many versions of the averaging theorem. Our account is based on the versions due to Hale [1969; Chapter V, Theorem 3.2], and Sanders and Verhulst [1982], who give a very full discussion from the viewpoint of asymptotics. We consider systems of the form

$$\dot{x} = \varepsilon f(x, t, \varepsilon); \qquad x \in U \subseteq \mathbb{R}^n, \quad 0 \leq \varepsilon \ll 1, \tag{4.1.1}$$

where $f\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ is C^r, $r \geq 2$, bounded on bounded sets, and of period $T > 0$ in t. We normally restrict ourselves to a bounded set $U \subset \mathbb{R}^n$. The associated *autonomous averaged system* is defined as

$$\dot{y} = \varepsilon \frac{1}{T} \int_0^T f(y, t, 0)\, dt \stackrel{\text{def}}{=} \varepsilon \bar{f}(y). \tag{4.1.2}$$

In this situation we have

Theorem 4.1.1 (The Averaging Theorem). *There exists a C^r change of coordinates $x = y + \varepsilon w(y, t, \varepsilon)$ under which* (4.1.1) *becomes*

$$\dot{y} = \varepsilon \bar{f}(y) + \varepsilon^2 f_1(y, t, \varepsilon), \tag{4.1.3}$$

where f_1 is of period T in t. Moreover

(i) *If $x(t)$ and $y(t)$ are solutions of* (4.1.1) *and* (4.1.2) *based at x_0, y_0, respectively, at $t = 0$, and $|x_0 - y_0| = \mathcal{O}(\varepsilon)$, then $|x(t) - y(t)| = \mathcal{O}(\varepsilon)$ on a time scale $t \sim 1/\varepsilon$.*

(ii) *If p_0 is a hyperbolic fixed point of* (4.1.2) *then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,* (4.1.1) *possesses a unique hyperbolic periodic orbit $\gamma_\varepsilon(t) = p_0 + \mathcal{O}(\varepsilon)$ of the same stability type as p_0.**

(iii) *If $x^s(t) \in W^s(\gamma_\varepsilon)$ is a solution of* (4.1.1) *lying in the stable manifold of the hyperbolic periodic orbit $\gamma_\varepsilon = p_0 + \mathcal{O}(\varepsilon)$, $y^s(t) \in W^s(p_0)$ is a solution of* (4.1.2) *lying in the stable manifold of the hyperbolic fixed point p_0 and $|x^s(0) - y^s(0)| = \mathcal{O}(\varepsilon)$, then $|x^s(t) - y^s(t)| = \mathcal{O}(\varepsilon)$ for $t \in [0, \infty)$. Similar results apply to solutions lying in the unstable manifolds on the time interval $t \in (-\infty, 0]$.*

Remarks. Conclusions (ii) and (iii) generalize to more complicated hyperbolic sets. In particular, Hale [1969] shows that if (4.1.2) has a hyperbolic closed orbit Γ, then (4.1.1) has a hyperbolic invariant torus T_Γ. Generalizations to almost periodic functions f are also available (Hale [1969]). Conclusion (iii) implies that the averaging theorem can be used to approximate stable and unstable manifolds in bounded sets and generally to study the global structure of the Poincaré map of (4.1.1), as our examples will demonstrate.

Proof. We will sketch the first two parts of the proof using standard results from differential equations; for the last parts it is more convenient to use the ideas of Poincaré maps and invariant manifolds. We start by explicitly computing the change of coordinates. Let

$$f(x, t, \varepsilon) = \bar{f}(x) + \tilde{f}(x, t, \varepsilon) \tag{4.1.4}$$

be split into its mean, $\bar{f}$, and oscillating part $\tilde{f}$. Let

$$x = y + \varepsilon w(y, t, \varepsilon), \tag{4.1.5}$$

without yet choosing w. Differentiating (4.1.5) and using (4.1.1) and (4.1.4) we have

$$\begin{aligned}[I + \varepsilon D_y w]\dot{y} &= \dot{x} - \varepsilon \frac{\partial w}{\partial t} \\ &= \varepsilon \bar{f}(y + \varepsilon w) + \varepsilon \tilde{f}(y + \varepsilon w, t, \varepsilon) - \varepsilon \frac{\partial w}{\partial t},\end{aligned}$$

* γ_ε may be a trivial periodic orbit, $\gamma_\varepsilon(t) \equiv p_0$, cf. Example 1 on p. 171 below.

or

$$\dot{y} = \varepsilon[I + \varepsilon D_y w]^{-1}\left[\bar{f}(y + \varepsilon w) + \tilde{f}(y + \varepsilon w, t, \varepsilon) - \frac{\partial w}{\partial t}\right]. \quad (4.1.6)$$

Expanding (4.1.6) in powers of ε and choosing w to be the anti-derivative of $\tilde{f}$:

$$\frac{\partial w}{\partial t} = \tilde{f}(y, t, 0), \quad (4.1.7)$$

we obtain

$$\dot{y} = \varepsilon\bar{f}(y) + \varepsilon^2\left[D_y f(y, t, 0)w(y, t, 0) - D_y w(y, t, 0)\bar{f}(y) + \frac{\partial\tilde{f}}{\partial\varepsilon}(y, t, 0)\right] + \mathcal{O}(\varepsilon^3)$$

$$\stackrel{\text{def}}{=} \varepsilon\bar{f}(y) + \varepsilon^2 f_1(y, t, \varepsilon), \quad (4.1.8)$$

as required.

To obtain conclusion (i) we use a version of Gronwall's lemma:

Lemma 4.1.2 (cf. Coddington and Levinson [1955], p. 37). *If u, v, and $c \geq 0$ on $[0, t]$, c is differentiable, and*

$$v(t) \leq c(t) + \int_0^t u(s)v(s)\, ds,$$

then

$$v(t) \leq c(0) \exp \int_0^t u(s)\, ds + \int_0^t c'(s)\left[\exp \int_s^t u(\tau)\, d\tau\right] ds.$$

To prove the lemma, let $R(t) = \int_0^t u(s)v(s)\, ds$ and first show that $R' - uR \leq uc$. After integrating this differential inequality and some manipulation, including integration by parts, one obtains the result.

Now consider equations (4.1.2) and (4.1.3). Integrating and subtracting, we have

$$y_\varepsilon(t) - y(t) = y_{\varepsilon 0} - y_0 + \varepsilon \int_0^t [\bar{f}(y_\varepsilon(s)) - \bar{f}(y(s))]\, ds$$
$$+ \varepsilon^2 \int_0^t f_1(y_\varepsilon(s), s, \varepsilon)\, ds,$$

where $y_\varepsilon(t)$ is a solution of (4.1.3) based at $y_{\varepsilon 0}$. Letting $y_\varepsilon - y = \zeta$, L be the Lipschitz constant of $\bar{f}$ and C the maximum value of f_1, this becomes

$$|\zeta(t)| \leq |\zeta(0)| + \varepsilon L \int_0^t |\zeta(s)|\, ds + \varepsilon^2 Ct. \quad (4.1.9)$$

Applying Gronwall's lemma, with $c(t) = |\zeta(0)| + \varepsilon^2 Ct$ and $u(s) = \varepsilon L$, we have

$$|\zeta(t)| \leq |\zeta(0)|e^{\varepsilon Lt} + \varepsilon^2 C \int_0^t e^{\varepsilon L(t-s)}\, ds$$
$$\leq \left[|\zeta(0)| + \frac{\varepsilon C}{L}\right]e^{\varepsilon Lt}. \quad (4.1.10)$$

Thus, if $|y_{\varepsilon 0} - y_0| = \mathcal{O}(\varepsilon)$, we conclude that $|y_\varepsilon(t) - y(t)| = \mathcal{O}(\varepsilon)$ for $t \in [0, 1/\varepsilon L]$. Finally, via the transformation (4.1.5) we have

$$|x(t) - y_\varepsilon(t)| = \varepsilon w(y_\varepsilon, t, \varepsilon) = \mathcal{O}(\varepsilon)$$

and, using the triangle inequality

$$|x(t) - y(t)| \le |x(t) - y_\varepsilon(t)| + |y_\varepsilon(t) - y(t)|,$$

we obtain the desired result.

To prove (ii) we consider the Poincaré maps P_0, P_ε associated with (4.1.2) and (4.1.3). Rewriting these latter systems as

$$\dot{y} = \varepsilon\bar{f}(y); \qquad \dot{\theta} = 1, \tag{4.1.11}$$

$$\dot{y} = \varepsilon\bar{f}(y) + \varepsilon^2 f_1(y, \theta, \varepsilon); \qquad \dot{\theta} = 1, \tag{4.1.12}$$

where $(y, \theta) \in \mathbb{R}^n \times S^1$, and $S^1 = R/T$ is the circle of length T, we define a global cross section $\Sigma = \{(y, \theta) | \theta = 0\}$, and the first return or time T Poincaré maps* $P_0 \colon U \to \Sigma$, $P_\varepsilon \colon U \to \Sigma$ are then defined for (4.1.11), (4.1.12) in the usual way, where $U \subseteq \Sigma$ is some open set. Note that P_ε is ε^2-close to P_0 since T is fixed independent of ε. If p_0 is a hyperbolic fixed point for (4.1.2), then it is also a hyperbolic fixed point of $DP_0(p_0)$ since $DP_0(p_0) = e^{\varepsilon TDf(p)}$. Therefore, $\lim_{\varepsilon\to 0}(1/\varepsilon)[e^{\varepsilon TDf(p_0)} - Id] = TDf(p_0)$ is invertible. Since P_ε is ε-close to P_0, we also have $\lim_{\varepsilon\to 0}(1/\varepsilon)[DP_\varepsilon(p_0) - Id] = TDf(p_0)$. The implicit function theorem implies that the zeros of $(1/\varepsilon)[DP_\varepsilon(p_0) - Id]$ form a smooth curve $(p_\varepsilon, \varepsilon)$ in $\mathbb{R}^n \times \mathbb{R}$. The p_ε are fixed points of P_ε, and the eigenvalues of $DP_\varepsilon(p_\varepsilon)$ are ε^2 close to those of $DP_0(p_0)$ since $p_\varepsilon = p_0 + \mathcal{O}(\varepsilon)$ and $DP_\varepsilon(p_\varepsilon) = \exp[\varepsilon T(Df(p_\varepsilon) + \varepsilon^2 Df_1(p_\varepsilon))] = \exp[\varepsilon TDf(p_0)] + \mathcal{O}(\varepsilon^2)$. Thus (4.1.12) has a periodic orbit γ_ε ε-close to p_0, and via the change of coordinates (4.1.5), equation (4.1.1) has a similar orbit.

We remark that all that is required for the existence of a periodic orbit in (4.1.1) is the absence of any eigenvalues equal to one in the spectrum of $DP_0(p_0)$. However, the stability types of p_0 and γ_ε may not correspond if any eigenvalues of $DP_0(p_0)$ lie on the unit circle.

To prove (iii), suppose that (4.1.2) has a hyperbolic saddle point p_0 and consider solutions $y(t) \in W^s(p_0)$ and the corresponding solutions $y_\varepsilon(t) \in W^s(\gamma_\varepsilon)$ of the full system (4.1.3). The cases in which p_0 is a sink or source and W^s is replaced by W^u can be dealt with similarly. The proof is divided into two parts; an outer region in which the averaged vector field $\varepsilon\bar{f}(y)$ is large in comparison with the remainder term $\varepsilon^2 f_1(y, t)$, and an inner region in which the "perturbations" $\varepsilon^2 f_1$ and $\varepsilon\bar{f}$ are of comparable order. For more details see Sanders and Verhulst [1982]. We fix a δ neighborhood, U_δ, of p_0, so that, outside U_δ, we have $|\bar{f}(y))| \gg \varepsilon|f_1(y, t, \varepsilon)|$. As above, standard Gronwall estimates show that $|y_\varepsilon - y_0| = \mathcal{O}(\varepsilon)$ for times of order $1/\varepsilon$

* In P_0, the subscript "0" indicates that the $\mathcal{O}(\varepsilon^2)$ term is removed, *not that* $\varepsilon = 0$ in (4.1.11). The notation, P_0 for the $\mathcal{O}(\varepsilon)$ map and P_ε for the full map, will be used throughout this and the following three sections.

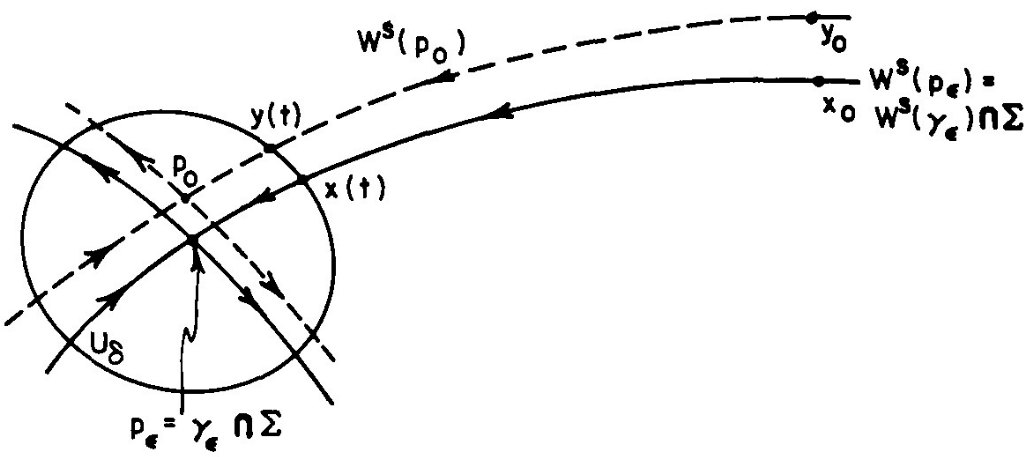

Figure 4.1.1. Validity of averaging on semi-infinite time intervals.

outside U_δ. On the other hand, in U_δ the (local) stable manifold theorem guarantees that the stable manifold $W^s_{\text{loc}}(\gamma_\varepsilon)$ is ε, C^r close to $W^s_{\text{loc}}(p_0) \times [0, T]$. Moreover, within $W^s_{\text{loc}}(\gamma_\varepsilon)$ and $W^s_{\text{loc}}(p_0)$ solutions are contracting towards γ_ε and p_0, respectively, this contraction being dominated by an exponential term of the form $e^{-\lambda t}$. Using this fact, we can prove that, if y_ε and y_0 enter U_δ within $\mathcal{O}(\varepsilon)$, they remain within $\mathcal{O}(\varepsilon)$ for all forward time, Figure 4.1.1. Piecing the two estimates together and using the transformation (4.1.5) as above, we obtain the desired result. □

We note that Sanders [1980] and Murdock and Robinson [1980] (cf. Robinson [1981b]) give proofs of part (iii) of this theorem. In the proof of the last part of the theorem we are using the smooth dependence of (local) invariant manifolds on parameters. Thus statement (iii) also follows directly from the "big" invariant manifold theorem of Hirsch *et al.* [1977, Theorem 4.1].

4.2. Examples of Averaging

Example 1. Consider the scalar system

$$\dot{x} = \varepsilon x \sin^2 t. \tag{4.2.1}$$

Here $f(x, t, \varepsilon) = \bar{f}(x) + \tilde{f}(x, t, \varepsilon) = x/2 - (x/2)\cos 2t$ and we have

$$\frac{\partial w}{\partial t} = -\frac{y}{2}\cos 2t,$$

or

$$w = -\frac{y}{4}\sin 2t. \tag{4.2.2}$$

Note that the t-independent term which could appear in the anti-derivative is generally taken to be zero. From (4.1.8), the transformed system is

$$\dot{y} = \varepsilon\frac{y}{2} + \varepsilon^2\left[(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t)\left(-\frac{y}{4}\sin 2t\right) - (-\tfrac{1}{4}\sin 2t)\left(\frac{y}{2}\right)\right] + \mathcal{O}(\varepsilon^3),$$

or

$$\dot{y} = \varepsilon\frac{y}{2} + \varepsilon^2\frac{y}{16}\sin 4t + \mathcal{O}(\varepsilon^3). \tag{4.2.3}$$

Here the autonomous averaged equation is simply

$$\dot{y} = \varepsilon\frac{y}{2}. \tag{4.2.4}$$

The exact solution of (4.2.1) with initial value $x(0) = x_0$ is easily found to be

$$x(t) = x_0 e^{\varepsilon((t/2) - \sin(2t)/4)}. \tag{4.2.5}$$

Comparing this with the solution of the averaged equation

$$y(t) = y_0 e^{\varepsilon t/2}, \tag{4.2.6}$$

we see that

$$x(t) - y(t) = e^{\varepsilon t/2}[|x_0 - y_0| - \varepsilon x_0 \sin(2t)/4 + \mathcal{O}(\varepsilon^2)], \tag{4.2.7}$$

in agreement with conclusion (i) of the theorem. Here the hyperbolic source $y = 0$ of (4.2.4) corresponds to a trivial hyperbolic periodic orbit $x \equiv 0$ of (4.2.1), and, letting $t \to -\infty$ in (4.2.5)–(4.2.7), we see that $x(t), y(t) \to 0$ and hence $|x(t) - y(t)| \to 0$, in agreement with conclusions (ii) and (iii).

EXERCISE 4.2.1. Study the system $\dot{x} = -\varepsilon x \cos t$ by the method of averaging. Does it have a hyperbolic limit set? Compare the averaged and exact solutions.

EXERCISE 4.2.2. Repeat the averaging analysis for $\dot{x} = \varepsilon(-x + \cos^2 t)$. In particular check the validity of conclusions (ii) and (iii) of the theorem.

EXERCISE 4.2.3. Study the nonlinear systems

$$\dot{x} = \varepsilon(x - x^2)\sin^2 t$$

and

$$\dot{x} = \varepsilon(x \sin^2 t - x^2/2)$$

by the method of averaging. What do you notice about their solutions?

EXAMPLE 2 (Weakly nonlinear forced oscillations). In many weakly nonlinear oscillator problems, the second-order equation to be studied takes the form

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t), \tag{4.2.8}$$

where f is T periodic in t. In particular, if f is sinusoidal with frequency $\omega \approx k\omega_0$, we have a system close to a *resonance of order* k. In such a situation, our expectation of finding an almost sinusoidal response of frequency ω/k prompts the use of the invertible van der Pol transformation, which recasts (4.2.8) into the form (4.1.1) which can then be averaged. We set

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad A = \begin{bmatrix} \cos\left(\frac{\omega t}{k}\right) & -\frac{k}{\omega}\sin\left(\frac{\omega t}{k}\right) \\ -\sin\left(\frac{\omega t}{k}\right) & -\frac{k}{\omega}\cos\left(\frac{\omega t}{k}\right) \end{bmatrix}, \tag{4.2.9}$$

$$A^{-1} = \begin{bmatrix} \cos\left(\frac{\omega t}{k}\right) & -\sin\left(\frac{\omega t}{k}\right) \\ -\frac{\omega}{k}\sin\left(\frac{\omega t}{k}\right) & -\frac{\omega}{k}\cos\left(\frac{\omega t}{k}\right) \end{bmatrix},$$

under which (4.2.8) becomes

$$\dot{u} = -\frac{k}{\omega}\left[\left(\frac{\omega^2 - k^2\omega_0^2}{k^2}\right)x + \varepsilon f(x, \dot{x}, t)\right]\sin\left(\frac{\omega t}{k}\right),$$

$$\dot{v} = -\frac{k}{\omega}\left[\left(\frac{\omega^2 - k^2\omega_0^2}{k^2}\right)x + \varepsilon f(x, \dot{x}, t)\right]\cos\left(\frac{\omega t}{k}\right), \tag{4.2.10}$$

in which x, $\dot{x}$ can be written as functions of u, v, and t via (4.2.9). If $\omega^2 - k^2\omega_0^2 = \mathcal{O}(\varepsilon)$, then (4.2.10) is in the correct form for averaging.

As a specific example, we take the standard Duffing equation which is considered in almost every text on nonlinear oscillations:

$$\ddot{x} + \omega_0^2 x = \varepsilon[\gamma \cos \omega t - \delta \dot{x} - \alpha x^3], \tag{4.2.11}$$

where $\omega_0^2 - \omega^2 = \varepsilon\Omega$, i.e., we are close to order one resonance. Setting $k = 1$ in (4.2.9) we obtain the transformed system

$$\dot{u} = \frac{\varepsilon}{\omega}[\Omega(u \cos \omega t - v \sin \omega t) - \omega\delta(u \sin \omega t + v \cos \omega t)$$
$$+ \alpha(u \cos \omega t - v \sin \omega t)^3 - \gamma \cos \omega t] \sin \omega t,$$

$$\dot{v} = \frac{\varepsilon}{\omega}[\Omega(u \cos \omega t - v \sin \omega t) - \omega\delta(u \sin \omega t + v \cos \omega t)$$
$$+ \alpha(u \cos \omega t - v \sin \omega t)^3 - \gamma \cos \omega t] \cos \omega t. \tag{4.2.12}$$

Averaging (4.2.12) over one period $T = 2\pi/\omega$, we obtain

$$\begin{aligned} \dot{u} &= \frac{\varepsilon}{2\omega}\left[-\omega\delta u - \Omega v - \frac{3\alpha}{4}(u^2 + v^2)v\right] \stackrel{\text{def}}{=} \varepsilon f_1(u, v), \\ \dot{v} &= \frac{\varepsilon}{2\omega}\left[\Omega u - \omega\delta v + \frac{3\alpha}{4}(u^2 + v^2)u - \gamma\right] \stackrel{\text{def}}{=} \varepsilon f_2(u, v) \end{aligned} \tag{4.2.13}$$

or, in polar coordinates; $r = \sqrt{u^2 + v^2}$, $\phi = \arctan(v/u)$:

$$\begin{aligned} \dot{r} &= \frac{\varepsilon}{2\omega}[-\omega\delta r - \gamma \sin\phi], \\ r\dot{\phi} &= \frac{\varepsilon}{2\omega}\left[\Omega r + \frac{3\alpha}{4}r^3 - \gamma\cos\phi\right]. \end{aligned} \tag{4.2.14}$$

Perturbation methods carried to $\mathcal{O}(\varepsilon)$ give precisely the same result (cf. Nayfeh and Mook [1979], Section 4.1.1).

Recalling the transformation

$$x = u(t)\cos\omega t - v(t)\sin\omega t \equiv r(t)\cos(\omega t + \phi(t)),$$

we see that the slowly varying amplitude, r, and phase, ϕ, of the solution of (4.2.11) are given, to first order, by solutions of the averaged system (4.2.14). It is therefore important to find the equilibrium solutions of or fixed points of (4.2.14), which, by the averaging theorem and the transformation (4.2.9), correspond to steady, almost sinusoidal solutions of the original equation. Fixing α, δ, and γ and plotting the fixed points $\bar{r}$, $\bar{\phi}$ of (4.2.14) against Ω or ω/ω_0, we obtain the frequency response curve familiar to engineers: see Figure 4.2.1. We shall consider the "jump" bifurcation phenomenon below. For more details on the Duffing equation see Nayfeh and Mook [1979], or, for bifurcation details, Holmes and Rand [1976]. The stability types of the branches of steady solutions shown in Figure 4.2.1 are obtained by consideration of the eigenvalues of the linearized averaged equation, and we invite the reader to check our assertions.

In Figure 4.2.2 we show typical phase portraits for (4.2.13)–(4.2.14), obtained by numerical integration for parameter values for which three hyperbolic fixed points coexist. In Figure 4.2.3(a) we reproduce the stable and unstable manifolds of the saddle point, under the orientation reversing transformation (4.2.9) (with $k = 1$) applied at $t = 0$: $x = u$, $\dot{x} = -\omega v$. According to Theorem 4.1.1, these manifolds should approximate the stable and unstable manifolds of the Poincaré map of the full system (4.2.11), which we show in Figure 4.2.3(b). These latter manifolds were also computed numerically. We note that the agreement is good, but remark that it deteriorates as ω moves away from ω_0 (Ω increases). For more examples, see Fiala [1976].

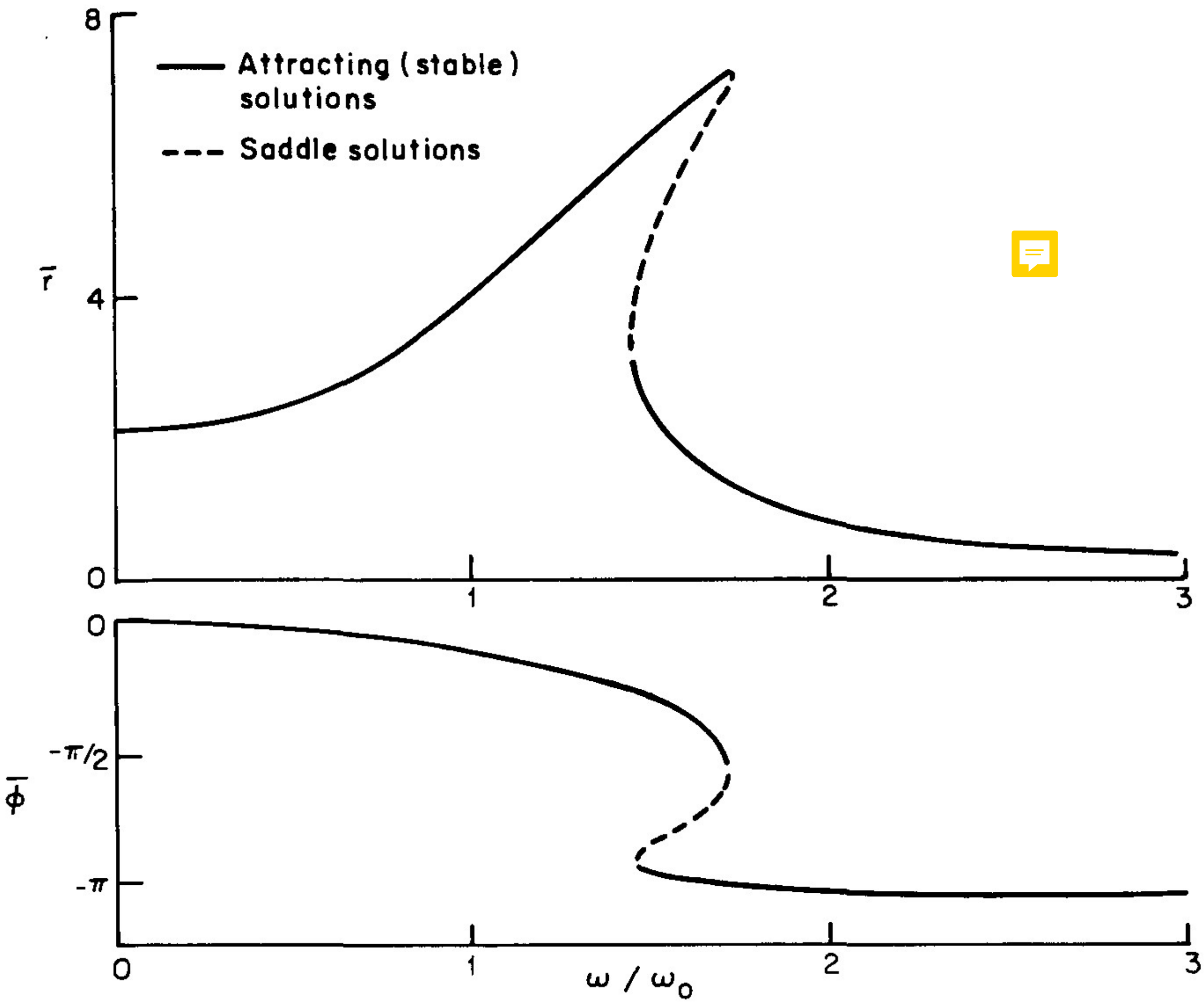

Figure 4.2.1. Frequency response function for the Duffing equation: $\varepsilon\alpha = 0.05$, $\varepsilon\delta = 0.2$, $\varepsilon\gamma = 2.5$.

EXERCISE 4.2.4. Carry out averaging for the "original" van der Pol equation

$$\ddot{x} + \frac{\alpha}{\omega}(x^2 - 1)\dot{x} + x = \alpha\gamma \cos \omega t,$$

with $1 - \omega^2 = \alpha\sigma = \mathcal{O}(\alpha) \ll 1$. Show that the averaged equation takes the form given in (2.1.13) with $\beta = \alpha\gamma$. Check as many of the assertions made in Section 2.1, concerning the averaged system, as you can.

EXERCISE 4.2.5. Consider equation (4.2.11) with $\varepsilon\gamma = \bar{\gamma} = O(1)$ and $\omega \approx 3\omega_0$ and apply the method of averaging to study the subharmonics of order three. In this case the transformation (4.2.9) should be replaced by

$$\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} x + B\cos(\omega t - \phi) \\ \dot{x} - \omega B \sin(\omega t - \phi) \end{pmatrix},$$

where

$$A = \begin{pmatrix} \cos\left(\frac{\omega t}{3}\right) & -\frac{3}{\omega}\sin\left(\frac{\omega t}{3}\right) \\ -\sin\left(\frac{\omega t}{3}\right) & -\frac{3}{\omega}\cos\left(\frac{\omega t}{3}\right) \end{pmatrix}. \tag{4.2.15}$$