MODEL NEURONS: FROM HODGKIN-HUXLEY TO HOPFIELD

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Model neural networks are built of model neurons. While real biological neurons exhibit extremely complex and rich behavior, neuronal dynamics must be considerably simplified to make networks analytically and computationally tractable. The most complete and realistic descriptions of neuronal behavior are based on analytic fits of detailed voltage clamp measurements of the voltage and time dependence of cell membrane currents. Many different membrane currents may be involved and the resulting models can be extremely complex. The simplest model of this type, and the one which will be considered here, is the classic Hodgkin-Huxley model [1] of the squid giant axon. Even in this case the relative complexity of the model makes it rather intimidating to imagine studying large networks built from Hodgkin-Huxley cells. On the other extreme is the binary, on/off (McCulloch-Pitts, Little or Hopfield) model cell used in most studies of attractor neural networks [2]. The simplicity of the binary neuronal model makes extremely detailed studies of network behavior possible but leaves much of the biological complexity behind. Clearly a first priority in improving the biological accuracy of network modeling is to consider networks built from more realistic model neurons.

Neuronal models falling between the extremes represented by the Hodgkin-Huxley and binary, Hopfield-like models do of course exist. Most prominent are the integrate and fire models [3] and the FitzHugh-Nagumo model [4]. These were constructed phenomenologically to match the basic behavior displayed by the Hodgkin-Huxley model and to a lesser degree by real neurons. The absence of a more logical derivation of these models makes it difficult to see their exact relationship to more complete descriptions and to understand the nature of the approximations which have tacitly been used. Here we will present a step by step reduction of the classic Hodgkin-Huxley model based on work done in collaboration with Eve Marder. The advantage of this derivation is that it is mathematically explicit and that it provides a variety of descriptions of increasing simplicity as more and more simplifying approximations are made.

The starting point of the derivation is the Hodgkin-Huxley model. Through a reduction procedure the Hodgkin-Huxley equations are approximated by a system of two first-order differential equations. This results in a model similar in spirit to that of FitzHugh and Nagumo, but considerably different from it in detail. Our derivation provides missing links between the Hodgkin-Huxley description and more phenomenological models while at the same time producing a simplified model which is more accurate than the FitzHugh-Nagumo model. A reduction similar in some ways to ours has been given previously [5] but our methods are both more accurate and more general.

Once the two-dimensional model has been derived there are two possible approaches for further reduction and simplification. One of these leads to either a linear or nonlinear integrate

and fire model while the other gives rise to a binary model with a time-dependent threshold [6,7]. Through our reduction procedure all of the parameters arising in these simplified models will be directly obtained from parameters of the original Hodgkin-Huxley description and all steps and approximations in the derivation will be explicit. In particular, we will arrive at a binary model of the general Hopfield type except that it includes hysteresis and a time-dependent threshold factor with dynamics completely determined by the underlying Hodgkin-Huxley description.

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THE HODGKIN-HUXLEY MODEL

The basic equation governing the dynamics of any neuron with spatially constant membrane potential V is given by conservation of electric charge,

$$C\frac{dV}{dt} = -F + I \tag{2.1}$$

where C is the cell capacitance, F the membrane current and I the sum of external and synaptic currents entering the cell, each per unit of cell membrane area. If the potential is not spatially constant an additional Laplacian term enters this equation making the analysis much more complicated. We will restrict our attention to the spatially constant or space clamped case although our results can be applied to the more general situation as well. The membrane capacitance C is typically 1 μFarad per square centimeter. In the Hodgkin-Huxley model the membrane current F arises primarily from the conduction of sodium and potassium ions through voltage dependent channels in the membrane. In addition, other ionic currents are described by an Ohmic leakage contribution. F is a function of V and of three time- and voltage-dependent conductance variables m, h and n,

$$F(V,m,h,n) = g_L(V - V_L) + g_K n^4(V - V_K) + g_{Na} h m^3 (V - V_{Na}) \tag{2.2}$$

where g_L = 0.3 mmho/cm^2, g_K = 36 mmho/cm^2, g_{Na} = 120 mmho/cm^2, V_L = -54.402 mvolt, V_K = -77 mvolt and V_{Na} = 50 mvolt.

The conductance variables m, h and n are both voltage and time dependent and by definition take values between zero and one. They approach asymptotic values $\overline{m}(V)$, $\overline{h}(V)$ and $\overline{n}(V)$ with time constants $\tau_m(V)$, $\tau_h(V)$ and $\tau_n(V)$ respectively,

$$\tau_m(V)\frac{dm}{dt} = \overline{m}(V) - m \quad \tau_h(V)\frac{dh}{dt} = \overline{h}(V) - h \quad \tau_n(V)\frac{dn}{dt} = \overline{n}(V) - n. \tag{2.3}$$

The six functions $\tau_m(V)$, $\tau_h(V)$, $\tau_n(V)$, $\overline{m}(V)$, $\overline{h}(V)$ and $\overline{n}(V)$ are given by fairly complicated formulas which are the result of detailed fits to experimental data. For all three variables,

$$\tau_{(m,h,n)} = \frac{1}{\alpha_{(m,h,n)} + \beta_{(m,h,n)}} \qquad (\overline{m},\overline{h},\overline{n}) = \frac{\alpha_{(m,h,n)}}{\alpha_{(m,h,n)} + \beta_{(m,h,n)}} \tag{2.4}$$

where

$$\alpha_m = \frac{.1(V+40)}{1 - \exp[-.1(V+40)]} \qquad \alpha_h = .07\exp[-.05(V+65)] \tag{2.5}$$

$$\alpha_n = \frac{.01(V+55)}{1-\exp[-.1(V+55)]} \qquad \beta_m = 4\exp[-.0556(V+65)]$$

$$\beta_h = \frac{1}{1+\exp[-.1(V+35)]} \qquad \beta_n = .125\exp[-.0125(V+65)].$$

Here and in the following we suppress the units in formulas with the understanding that all potentials are in mvolts , all times in msec and all currents in μamps per cm^2. Although the Hodgkin-Huxley equations are quite complicated they mix in a rather minimal way. The cell potential V is affected by all three of the other dynamic variables m, h and n. However, m, h and n do not directly couple to each other, they only interact through V. This property will allow us to approximate the dynamics through the introduction of an auxiliary potential variable.

The functions $\overline{m}$, $\overline{h}$, $\overline{n}$ and all three τ's are plotted in Figure 1. Note that the time constant

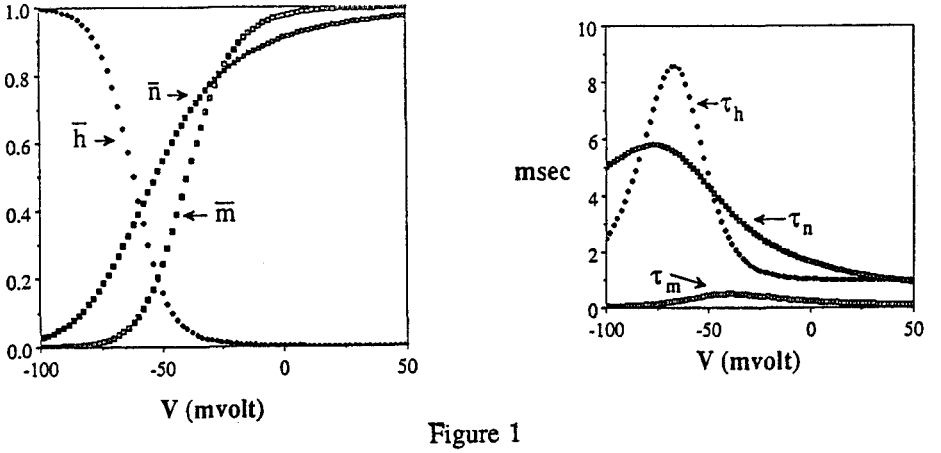

Figure 1

which governs the behavior of m is much smaller than those for h and n. Furthermore the time constants for h and n are roughly the same over most of the voltage range. These properties will be important for our reduction of the model.

The result of integrating the Hodgkin-Huxley equations are as follows. For $I = 0$, V remains at a resting potential of -65 mvolt. A positive current I of sufficient strength and duration will induce a sudden depolarization of the cell to about 50 mvolts followed rapidly by a hyperpolarization and then a slower recovery back to the resting level. This is of course an action potential. The time course of a typical action potential is shown in Figure 2. Note that the rise of the action potential is extremely rapid and that its fall has two parts, an initial downward ramp followed by a more rapid drop to a hyperpolarized potential. There is then a relatively long recovery back to the resting potential. The sudden rise of the action potential is caused by a rapid increase of the variable m which turns on a positive inward sodium current. The action potential is terminated as the variables h and n adjust more slowly to the change in membrane potential. The variable h decreases shutting off the sodium current which drove the upward swing of the action potential. At the same time an increase in n initiates a positive outward potassium current which hyperpolarizes the cell. The final recovery involves the readjustment of h and n back to their resting values.

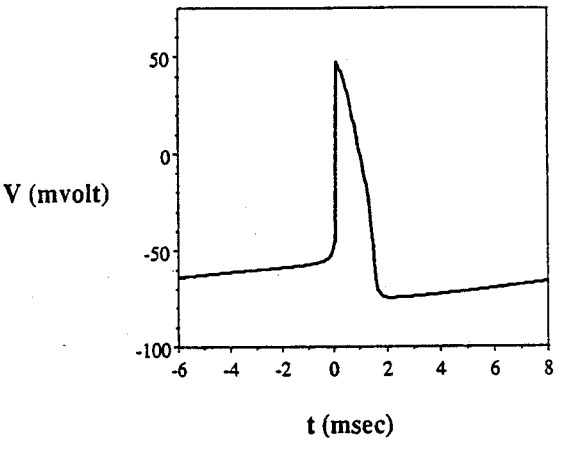

Figure 2

Three essential features are exhibited by the Hodgkin-Huxley model and these ideally should be present in any serious model of neuronal dynamics. They are 1) the action potential itself, 2) the refractory period after an action potential during which the slow recovery of the potassium and sodium conductance has a drastic impact on electrical properties and 3) the ability of the model cell to capacitively integrate incoming current pulses. These last two features are essential if a model cell is to react appropriately to synaptic inputs. The reduced two-dimensional model we discuss retains all three properties while the integrate and fire or binary models involve either some or complete loss of accuracy in describing property 2) or 3) respectively.

REDUCTION OF THE HOGDKIN-HUXLEY MODEL

The Hodgkin-Huxley model involves four time-dependent dynamical variables V, m, h and n. The four-dimensional nature of the phase space makes it difficult to visualize and intuitively understand the workings of this model. There is a tremendous conceptual advantage in reducing the model to two dimensions so that the phase space can be depicted in a straightforward manner. We therefore propose a procedure for reducing the number of dynamical variables from four to two. We do this in two steps. First, as discussed above the time scale associated with changes in m, τ_m, is much smaller than those associated with h and n. Thus, m will reach its asymptotic value $\overline{m}(V)$ much more rapidly than other changes in the model. If we are willing to give up some of the accuracy of the model over very short time scales we can replace m by its asymptotic value $\overline{m}(V)$ and ignore the differential equation for m entirely writing

$$m \approx \overline{m}(V) \tag{3.1}$$

and

$$F(V,m,h,n) \approx F(V,\overline{m}(V),h,n) \equiv \dot{F}. \tag{3.2}$$

The last equivalence signifies that when the symbol F is used below it stands for the function

$F(V, \overline{m}(V), h, n)$. The instantaneous approximation for m reduces the number of dynamic variables from four to three.

It would simplify things considerably if we could also replace h and n by their asymptotic values. However, if we did this we would destroy the ability of the model to generate action potentials since h and n would terminate the action potential as quickly as m could initiate it. Instead because of their longer time constants, these variables should lag behind, reaching their asymptotic values more slowly. This effect can be simulated by introducing an auxiliary voltage variable U and replacing h and n by their asymptotic values not at the potential V but rather at U. The dynamics of the U variable will cause it to lag behind V but to approach it asymptotically. Thus we write,

$$h \approx \overline{h}(U) \qquad n \approx \overline{n}(U) \tag{3.3}$$

so that

$$F(V, m, h, n) \approx F(V, \overline{m}(V), \overline{h}(U), \overline{n}(U)) \equiv f(V, U). \tag{3.4}$$

Again the last equivalence indicates that f refers to the function $F(v, \overline{m}(V), \overline{h}(U), \overline{n}(U))$. Because both $\overline{h}$ and $\overline{n}$ are monotonic functions the replacement of either one of the variables h or n by its asymptotic value at U would correspond to a simple change of variables. Indeed our reduction procedure is exact in this case. However, because we are replacing both h and n by a single variable U the reduction is approximate. In order to minimize the impact of this approximation we must choose the variable U carefully. Clearly we want the time dependence of U in f to mimic the time-dependence induced into F in the full model by the changing values of h and n. Thus, we equate time derivatives of F at constant V in the full and reduced models,

$$\frac{\partial F}{\partial h}\frac{dh(V)}{dt} + \frac{\partial F}{\partial n}\frac{dn(V)}{dt} = \left(\frac{\partial f}{\partial \overline{h}}\frac{d\overline{h}(U)}{dU} + \frac{\partial f}{\partial \overline{n}}\frac{d\overline{n}(U)}{dU}\right)\frac{dU}{dt}. \tag{3.5}$$

We now use the original formulas for dh/dt and dn/dt incorporating the approximations $h \approx \overline{h}(U)$ and $n \approx \overline{n}(U)$ so that

$$\tau_h(V)\frac{dh}{dt} \approx \overline{h}(V) - \overline{h}(U) \qquad \tau_n(V)\frac{dn}{dt} \approx \overline{n}(V) - \overline{n}(U). \tag{3.6}$$

Using these result we can solve the equal time-derivative condition for dU/dt in terms of V and U. This gives us a reduced two-dimensional version of the Hodgkin-Huxley model,

$$C\frac{dV}{dt} = -f(V, U) + I \tag{3.7}$$

and

$$\frac{dU}{dt} = g(V, U) \tag{3.8}$$

where

$$g(V, U) = \frac{A}{B} \tag{3.9}$$

with

$$A = \frac{\partial F}{\partial h}\left(\frac{\overline{h}(V) - \overline{h}(U)}{\tau_h(V)}\right) + \frac{\partial F}{\partial n}\left(\frac{\overline{n}(V) - \overline{n}(U)}{\tau_n(V)}\right) \tag{3.10}$$

and

$$B = \frac{\partial f}{\partial \overline{h}} \frac{d\overline{h}(U)}{dU} + \frac{\partial f}{\partial \overline{n}} \frac{d\overline{n}(U)}{dU}. \tag{3.11}$$

where $\partial F/\partial h$ and $\partial F/\partial n$ are to be evaluated at $h = \overline{h}(U)$ and $n = \overline{n}(U)$. Note that when $U = V$, $g(V,U) = 0$ so U will approach V asymptotically if it can.

Simulations have shown that this reduced model is a good approximation of the full model. The easiest way to envision the dynamics given by Eqs. (3.7)-(3.11) is to plot the curves (isoclines) $dV/dt = -f + I = 0$ and $dU/dt = g = 0$ in the $U - V$ plane and to indicate the flows of U and V. This is done for $I = 0$ on the left side of Figure 3 below. The straight line corresponds to $U = V$ which makes $dU/dt = 0$. Along the curved line $dV/dt = 0$. The point $V = -65$ and $U = -65$ where these two curves intersect is the resting equilibrium point. The arrows indicate the flows toward these two curves. The right side of Figure 3 shows $dV/dt = 0$ isoclines for various positive values of the current I. The lowest curve corresponds to $I = 0$ and the left-hand portion of the curve rises as I is increased.

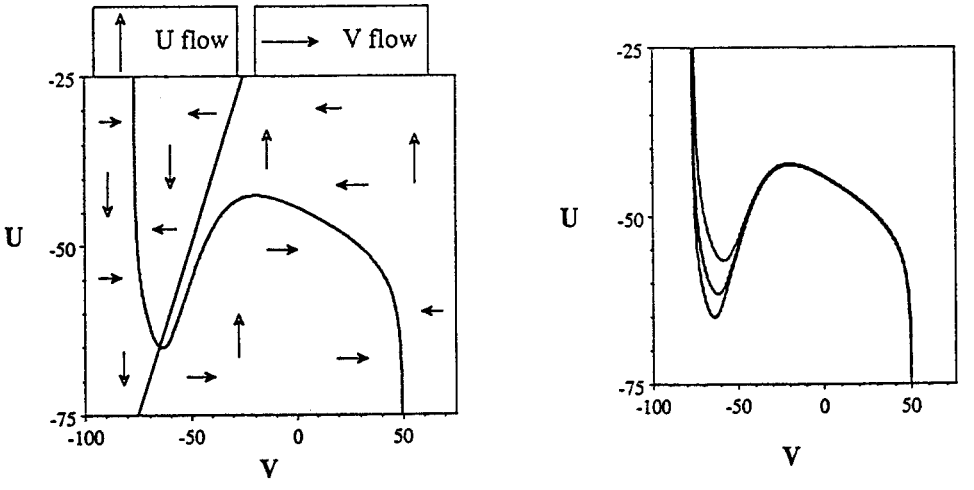

Figure 3

The model we have derived is, at least in spirit, similar to the FitzHugh-Nagumo model [4] which also involves two first-order differential equations. However, it has some essential and important differences. In the FitzHugh-Nagumo model g(V,U) is taken to be linear in both U and V while $f(V,U)$ is cubic in V and linear in U. The linear form for g is not such a bad approximation but the form assumed by FitzHugh and Nagumo for f considerably distorts the dynamics. In particular, as shown on the right side of Figure 3 the low voltage section of the $dV/dt = 0$ isocline moves up with positive current I. However, the high voltage portion of this curve is almost completely insensitive to external current. In the FitzHugh-Nagumo model the entire $dV/dt = 0$ isocline moves up by the same amount as a function of I and this causes some fairly severe misrepresentations of cell behavior. For example, the amplitude of the action potentials in the full Hodgkin-Huxley model and in our reduced model decrease

when the cell is pushed to a high firing rate by large injected current. In the FitzHugh-Nagumo model the amplitude is fairly independent of firing frequency. Furthermore no direct relation between parameters of the FitzHugh-Nagumo model and the underlying Hodgkin-Huxley model exists.

To indicate the behavior of the two-dimensional model we first consider the response of the system to constant positive current. Figure 4 shows the phase plane for zero current (lower curve) and for a positive current large enough to cause repetitive firing (I=10). The circular point on the lower curve is the resting point of the cell with zero current. This stable equilibrium lies at the point where the two isoclines shown intersect. When positive current is applied the low-voltage portion of the $dV/dt = 0$ isocline moves up until the intersection of the two isoclines falls well within the portion of the $dV/dt = 0$ isocline with positive slope. This point is unstable so the system goes into a limit cycle as shown by the arrows in Figure 4. This produces a train of action potentials.

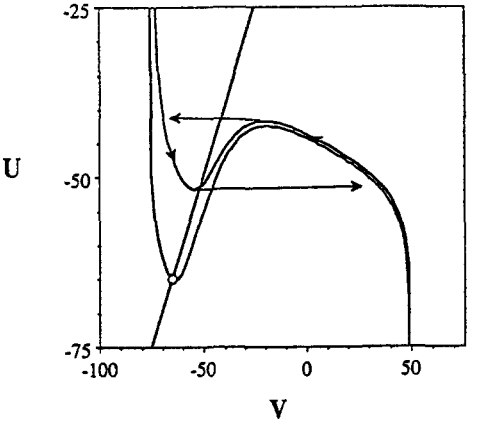

Figure 4

Another way to exhibit the dynamics of the reduced model is to subject a model cell which is initially at the resting potential $V = -65$ and $U = -65$ to (square) current pulses of various amplitudes and durations. Figures 5 and 6 show responses to such pulses. Along with the $dV/dt = 0$ and $dU/dt = 0$ isoclines, the phase-space trajectory is shown with data points marked at equal time intervals and arrows indicating the sense of the motion. The initial line of data points in all the figures shown indicates the behavior of V and U during the application of the current pulse while the curved trajectories give the subsequent response. The left side of Figure 5 shows the effect of a small pulse of current. This causes an upward shift in V that is insufficient to move V out of the region where $dV/dt < 0$ (compare with Figure 3). As a result V moves back down to its resting value and no action potential is fired. The loop in the return trajectory indicates that the U variable has also responded to the current pulse although this response is less dramatic than that for V. The right side of Figure 5 shows the effects of a stronger current pulse which is nevertheless still too weak to produce an action potential. Here, the dynamics of the U variable plays an essential role in preventing the firing of an action potential. Note that the current pulse is sufficiently strong to move V

into the region to the right of the $dV/dt = 0$ isocline where $dV/dt > 0$. However, no action potential results because the increase in the U variable causes the trajectory to curve back into the region where $dV/dt < 0$.

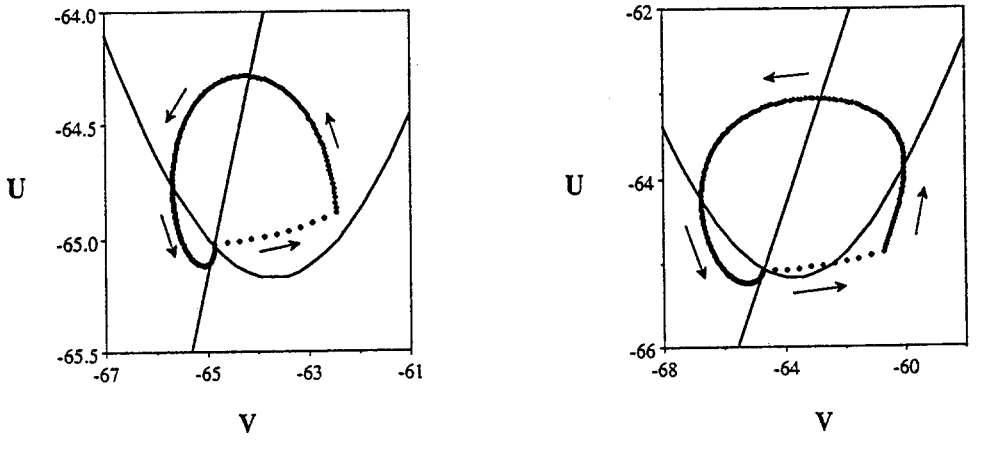

Figure 5

Figure 6 shows the effects of a current pulse sufficiently strong to produce an action potential. Here, V is pushed far enough into the unstable $dV/dt > 0$ region so that it climbs all the way up to the high-voltage portion of the $dV/dt = 0$ isocline. The initial trajectory is shown at left and the complete loop from the resting potential, through the action potential and back to the resting potential again is shown at right. It is interesting to note that the phase plane trajectory stays quite close to the $dV/dt = 0$ isocline except initially near the resting potential.

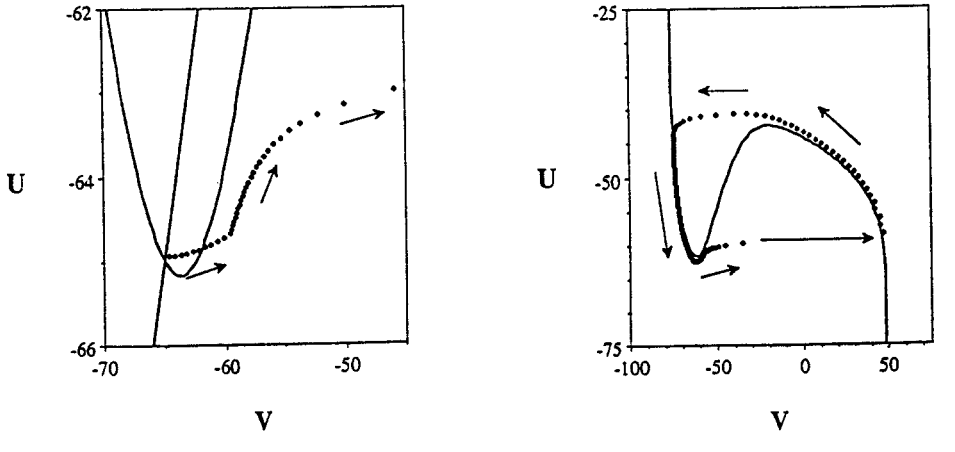

Figure 6

FURTHER REDUCTION OF THE MODEL

We will now show how the two-dimensional model can be further reduced to yield some even simpler models. Clearly with two differential equations in the model an obvious simplification is to ignore either one or the other. The differential equation for V reflects the capacitive properties of the cell, as the presence of the capacitance C would suggest. The differential equation for U on the other hand reproduces the time dependence of the membrane conductance. The capacitive behavior reflected in the V equation is responsible for the integrative behavior of the cell while the time-dependent conductances given by the U equation produce the refractory period. Thus, to proceed we must be willing to give up an accurate description of one or the other of these two essential features. As we will see, integrate and fire models ignore the U equation and must incorporate refractoriness in an ad hoc and crude manner. Binary models on the other hand ignore the V equation and thus do not simulate the integrative and time-delay behavior which capacitance provides. Nevertheless, each of these models rather accurately reproduces the remaining dynamics and thus provides an extremely useful if partial description of neuronal behavior.

A standard procedure for reducing a set of two nonlinear equations like (3.7)-(3.11) (if such a reduction is in fact possible) is to treat one of the variables as instantaneous. For example if dV/dt is much greater than dU/dt the V variable will rapidly move to a value which makes $dV/dt = 0$ and the longer time scale dynamics of the system will be governed by the constraint $dV/dt = -f + I = 0$ and the slower behavior given by the differential equation for U. Indeed this is the case through most of the dynamic range of V and U. This is evident in Figure 6 where we noted that the phase-space trajectory stays quite close the $dV/dt = 0$ isocline throughout the action potential. In fact throughout most of the phase plane $|f| >> |g|$, so that V is constrained to stay close to the $dV/dt = 0$ isocline most of the time. However, to accurately predict the firing pattern of a cell we are most interested in what happens near the resting potential and firing threshold and here the situation is more complex. Figure 7 shows $-f(V, -65)$ and $g(V, -65)$ in the relevant region of V. In this region f and

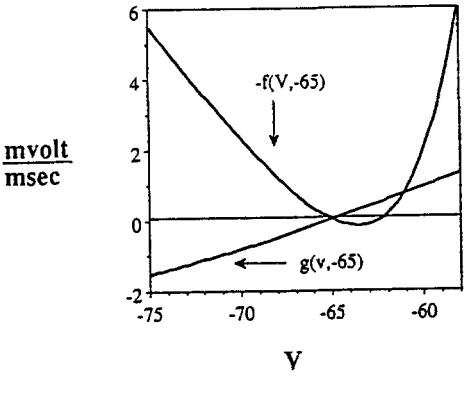

Figure 7

g are of comparable magnitude. Below the resting potential the magnitude of f tends to be bigger. Thus in this region V can be considered instantaneous and the dynamics is determined by the isocline constraint $-f + I = 0$ and the equation for dU/dt. However for potentials slightly above the resting potential f is actually smaller the g. This region is essential for triggering an action potential and we see that ignoring the capacitive effects incorporated in the equation for dV/dt here would be a mistake. A reduction of the two differential equations involves a trade-off. If the region between the resting potential and the firing threshold is of most interest then the V dynamics dominates. This is the approach taken by integrate and fire models as we discuss next. The approximation $dV/dt = -f + I = 0$ which removes V as an independent dynamical variable leaving only the differential equation for U is valid throughout most of the range of V except in the threshold region. This approximation gives rise to the binary models.

INTEGRATE AND FIRE MODELS

Integrate and fire models can approximately duplicate the capacitive behavior of a cell, but they treat the refractory properties very roughly at best. To derive an integrate and fire model from the reduced model of the last section we ignore the dynamics of the U variable. This is essentially a large C approximation of Eqs. (3.7)-(3.11). Since the time-dependence of the membrane conductance is not reproduced adequately the model cannot itself produce action potentials. Rather it is assumed that when the membrane potential exceeds some threshold value the result is the firing of an action potential. Likewise, since the U dynamics is ignored no refractory period arises in the model but this is sometimes included crudely by specifying that the V dynamics freezes for a certain time period following an action potential.

The elimination of the dynamics of the U variable can be done in two different ways. First we can set $U = V$ permanently making $dU/dt = 0$. The remaining dynamics is then just

$$C\frac{dV}{dt} = -f(V,V) + I. \tag{5.1}$$

As can be seen from the left side of Figure 8 $f(V,V)$ is roughly linear and this further approximation is often made. The result is the linear integrate and fire model. To write it we

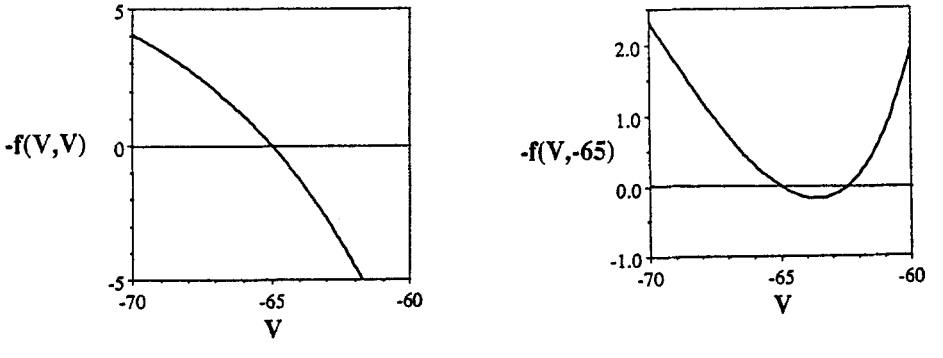

Figure 8

re-express V as

$$V = v - 65 \tag{5.2}$$

so that

$$C\frac{dv}{dt} = -\frac{v}{R} + I \tag{5.3}$$

where by fitting the curve $f(V, V)$ we find R = 0.8 kOhm-cm^2.

The approximation $U = V$ is actually a pretty terrible one to make. As can be seen in the phase-space curves for the two-dimensional model, U is never very close to V except at the resting value of -65. A more sensible approximation is to set U = -65 and leave it there. Then, V is governed by the equation

$$C\frac{dV}{dt} = -f(V, -65) + I. \tag{5.4}$$

The curve $f(V, -65)$ is given in the right side of Figure 8. This curve is quite far from linear, in particular it passes through zero twice. The higher value of V for which $f = 0$ defines the threshold potential since for all higher potentials up to the peak of the action potential $dV/dt > 0$. Fitting this curve in the relevant region gives a nonlinear integrate and fire model again written in terms of v,

$$C\frac{dv}{dt} = -.250v + .083v^2 + .008v^3 + I \tag{5.5}$$

with $v_{threshold} = 2.5$. Note that the parameters for both these integrate and fire models are given directly by the underlying Hodgkin-Huxley model through the reduction procedure and simple curve fitting.

BINARY MODELS

As discussed above the membrane potential V tends to stay quite close to the $dV/dt = 0$ isocline most of the time throughout most of the phase space. Thus, another approach to further reduction is to assume that the neuron has small enough capacitance C so that the differential equation for V can be replaced with the constraint equation

$$f(V, U) = I. \tag{6.1}$$

In other words, we assume that the capacitance is small enough so that changes of V caused by changes in the current I and in U can be approximated as instantaneous. We can then solve the constraint equation for V as a function of U. The resulting values of V versus U are plotted at the left in Figure 9. Note that V is a multi-valued function of U. To correctly define the inverse we can introduce a binary variable S which keeps track of which branch of the curve is being used. In Figure 9 an upper region given by $S = +1$ and a lower region given by $S = -1$ are indicated. For this figure $I = 0$. The correspondence between this S variable and the binary variable in Hopfield-type models should be obvious. $S = +1$ when the membrane potential takes high values characteristic of an action potential while $S = -1$ when the cell is not firing and near its resting potential. The region between those indicated by $S = +1$ and $S = -1$ does not need to be considered because V will never enter this region. As the arrows indicate V will jump instantaneously between the $S = \pm 1$

regions. This is an instantaneous approximation to the beginning and end (specifically the second rapidly dropping part of the end) of an action potential and is a result of our ignoring the cell capacitance. Since these processes are extremely rapid this is actually quite a good approximation.

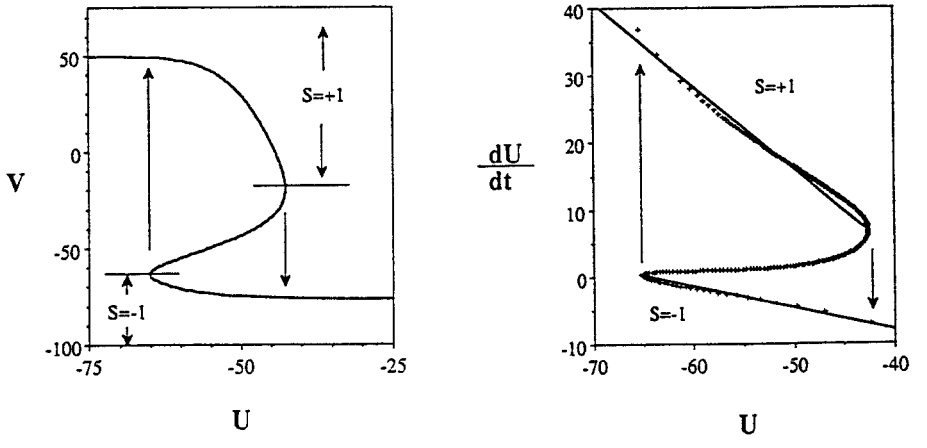

Figure 9

From the curve shown on the left side of Figure 9, V can be eliminated in favor of S and U. Then dU/dt depends on U and S rather than on U and V. This is shown as the hatched line in the right-hand panel of Figure 9. Also shown by solid lines in this figure are linear fits to dU/dt in the regions $S = +1$ and $S = -1$ which are used to further simplify the model. As can be seen these linear fits are quite good. A non-zero current can be included by noting that for $S = +1$ current has no appreciable impact while for $S = -1$ it effectively shifts the U variable (see the right side of Figure 3). In particular we find that to a good approximation

$$\frac{dU}{dt} = -.3(U + 65 - .6I) \tag{6.2}$$

for $S = -1$ and

$$\frac{dU}{dt} = -1.3U - 50 \tag{6.3}$$

for $S = +1$. The worst part of this approximation is the fact that the differential equation for U is linear in I when in fact the response becomes nonlinear if the current is too large.

The final binary model involves two equations, one to maintain the value of the S variable and the other a differential equation for U obtained by combining the two equations for dU/dt given above. We will keep track of S by demanding that it flip from -1 to +1 when U goes through the value -66 from above and from +1 to -1 when U passes through -43 from below. To simplify the final formulas we make the change of variables

$$U = 11.5u - 65 \quad i = .03I \tag{6.4}$$

Then the binary model is given by

$$S = sign[S + .9 - u + (1 - S)i] \tag{6.5}$$

with

$$\frac{du}{dt} = -(.8 + .5S)u + 1.5(1 + S) + .3(1 - S)i \tag{6.6}$$

The equation (6.5) is similar to that of a binary, Hopfield model with three important extra properties. 1) The presence of the S variable in the sign function provides hysteresis. 2) The u variable acts as a time-dependent threshold determined by the differential equation (6.6). 3) The current i only couples when $S = -1$. The addition of these three features lifts the binary model to a level where it is accurately reflecting the underlying dynamics of the full Hodgkin-Huxley model except of course that capacitive effects are not included. The additional complications introduced to achieve this are really quite minimal. In particular the differential equation for u is linear so it can be solved easily and analytic results can be obtained. A model similar to (6.5) and (6.6) has been studied extensively [7]. It is possible to derive analytic expressions for firing rates, phase response curves, firing delays, phase locking regions and many other responses of the cell to constant and time varying external currents.

A discrete time version of the model can be generated by turning Eq. (6.5) into a dynamic map and by integrating Eq. (6.6) over one time step Δt,

$$S(t + \Delta t) = sign[S(t) + .9 - u(t) + (1 - S(t))i(t)] \tag{6.7}$$

with

$$u(t + \Delta t) = \overline{u} + (u(t) - \overline{u}) \exp[-(.8 + .5S(t))\Delta t] \tag{6.8}$$

where

$$\overline{u} = \frac{1.5(1 + S(t)) + .3(1 - S(t))i(t)}{.8 + .5S(t)}. \tag{6.9}$$

Systems of binary model cells can be constructed by using the standard expression

$$i = \frac{1}{2} \sum_{j=1}^{N} J_j(S_j + 1) \tag{6.10}$$

for the synaptic current where the sum is over other cells coupled to the neuron being studied and the J_j are synaptic weights. This has also been done in a related model [7].

CONCLUSIONS

Following a well-defined reduction procedure we have constructed three types of reduced models. The two-dimensional model incorporates both the effects of cell capacitance and of time-dependent membrane conductances and so is the most accurate of the three. Integrate and fire models can be derived only at the expense of weakening the treatment of time-dependent conductances and most importantly refractoriness. Nevertheless these provide quite a good description near the resting and threshold potentials. By ignoring cell capacitance we obtained an interesting binary model which incorporates quite accurately aspects of the time-dependent conductances through hysteresis and through a time-dependent threshold factor. An addition difference between this derived binary model and those more commonly in use is that synaptic current only couples to the model cell when it is in the non-firing $S = -1$ state. In all three cases, parameters of the reduced models were obtained directly from the underlying full Hodgkin-Huxley model. An advantage of our reduction approach is that it

allows us to include the effects of other membrane currents besides those included in the original Hodgkin-Huxely description. For example we can determine how additional currents affect the time-dependent threshold of the binary model. This is presently being done.

REFERENCES

1. Hodgkin, A.L. and Huxley, A.F. (1952) *J. Physiol. (London)* **117**. 500.
2. McCulloch, W.S. and Pitts, W. (1943) *Bull. Math. Biophys.* **5**, 115; Little, W.A. (1975) *Math Biosci.* **19**, 101; Hopfield, J.J. (1982) *Proc. Natl. Acad. USA* **79**, 2554.
3. For a review see: Tuckwell (1988) H. *Introduction to Theoretical Neurobiology* (Cambridge University Press, Cambridge).
4. FitzHugh, R. (1961) *Biophys. J.* **1**, 445; Nagumo, J.S., Arimoto, S. and Yoshizawa, S. (1962) *Proc. IRE* **50**, 2061.
5. Krinskii, V.I. and Kokoz, Y.M. (1973) *Biofizika* **18**, 533 and 937.
6. Horn, D. and Usher, M. (1989) *Phys. Rev.* **A40**, 1036.
7. Abbott, L.F. (1990) *J. Phys. A* (in press); Proc. Natl. Acad. Sci USA (submitted).