Contents

CHAPTER 1

What is differential topology?

The main subject is the study of topological (i.e. global) properties of smooth manifolds. Typical questions involve:

- Is it possible to embed every smooth manifold in some $\mathbb{R}^k$, $k \gg 0$?
- What properties of smooth maps are generic? A typical example is transversality, e.g. transversality to 0 is demonstrated by the following two examples
  pictures of $x^2(x-3)$ *(not generic) and* $(x+1)(x-1)(x-3)$ *(generic)*
  The second function is generic because changing it slightly does not change the situation. However a perturbation of the first function will either miss the x-axis or will have two "generic" intersections.
- Classification of manifolds. Already Riemann in the late 19^{th} century described all (closed) surfaces. They are classified by orientability and genus ("the number of holes").
  picture of a sphere with a handle One can read this information off homology.

  The next step is dimension 3. Poincaré was pursuing this question. He asked whether Betty numbers (the dimensions of the free parts of homology - i.e. all that people at that time knew of homology) where enough to distinguish (compact and oriented) non-diffeomorphic 3-manifolds. He showed that it is not. In fact even the homology groups do not suffice (an example supplied by Poincaré): if M is a manifold with a perfect group ($[\pi_1(M), \pi_1(M)] = \pi_1(M)$) then $H_1(M) = 0 = H_2(M)$ by Poincaré duality and the manifold has the same homology as S^3 (it is so-called homology sphere) but is not even homotopy equivalent to it as $\pi_1(S^3) = 0$. His next question was the following. Is every compact simply connected 3-manifold diffeomorphic to S^3? This is known as Poincaré conjecture and was solved only 100 years later by Perelman.

  Remark: one can always find a map $M^m \to S^m$ of degree 1. If $\pi_1(M) = 0$ and $H_*(M) \cong H_*(S^m)$ then this map is a homotopy equivalence.

  We have the following diagram relating a 3-manifold M and S^3

  diffeo $\Rightarrow$ homeo $\Rightarrow$ htpy equiv $\Rightarrow$ hlgy iso

  Concerning the reverse implications: the last one is valid for simply connected M, the middle one is what is the real Poincar'e conjecture. This is very difficult in dimension 3 (and also 4), easier for dimension ≥ 5 but still very hard. The reverse of the first implications holds in dimension 3 but is not true in higher dimensions.

For example on S^7 there are 28 distinct differentiable structures (when one is concerned about orientations, otherwise 15).

The conclusion should be now that the classification problem is too difficult.

Another idea of Poincaré (still in the 19^{th} century) was to consider a much weaker relation on compact manifolds: M is called bordant to N, denoted $M \simeq N$, if the disjoint union $M \sqcup N \cong \partial W$ is a boundary of some compact manifold W with boundary. For example $S^m \simeq \emptyset$ as $S^m = \partial D^{m+1}$.

Classify manifolds up to bordism! We denote the set of bordism classes of manifolds by $\mathfrak{N}_*$ and observe that it has a natural structure of a ring with respect to $\sqcup$ and $\times$. In 50's Thom showed that $\mathfrak{N}_*$ is isomorphic to homotopy groups $\pi_*(MO)$ of some space (or rather spectrum) MO. Moreover Thom was able to compute this ring and

$$\mathfrak{N}_* \cong \mathbb{Z}/2[x_n \mid n \geq 2, n \neq 2^t - 1]$$

where $\deg(x_n) = n$ i.e. x_n is represented by a manifold of dimension n. At the same time Pontryagin identified a similar bordism ring Ω_*^{fr} of the so-called framed manifolds with the stable homotopy groups of spheres, the main object of study of algebraic topology. The way to prove both these identifications is the same and is called the Pontryagin-Thom construction. This will be our first goal in the lecture.

CHAPTER 2

Embedding into Euclidean space

Manifold will always be meant to be smooth, Hausdorff and second countable[1].

DEFINITION 2.1. A topological space X is called *paracompact* if every open covering of X has a locally finite refinement.

Here $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if $\forall V \in \mathcal{V}\ \exists U \in \mathcal{U} : V \subseteq U$. A covering $\mathcal{U}$ is called *locally finite* if every point $x \in X$ has a neighbourhood V which intersects only a finite number of elements of $\mathcal{U}$, i.e. $\{U \in \mathcal{U} \mid U \cap V \neq \emptyset\}$ is finite.

THEOREM 2.2. *For a connected Hausdorff locally Euclidean space X the following conditions are equivalent*

a) *X is second countable*
b) *$X = \bigcup_{i=0}^{\infty} K_i$ where each K_{i+1} is a compact neighbourhood of K_i*
c) *X is paracompact*

PROOF. "a)$\Rightarrow$b)": We need the following

SUBLEMMA. If X is second countable then every open covering has a countable subcovering.

PROOF OF THE SUBLEMMA. Let $\mathcal{U}$ be a covering and $\{V_i\}_{i=0}^{\infty}$ a basis for the topology. For each V_i choose (if possible) $U \in \mathcal{U}$ such that $V_i \subseteq U$ and call it U_i. Then the claim is that $\{U_i\}_{i=0}^{\infty}$ is a subcovering. *nice picture* $\square$

Let us continue with the proof of the theorem now. Cover X by open sets U_i with compact closure and we can assume that this collection is countable. We construct K_i inductively starting with $K_0 = \bar{U}_0$. In the inductive step cover K_i by $U_{j_1}, \dots, U_{j_n}$ and form $K_{i+1} = \bar{U}_{i+1} \cup \bigcup_{k=1}^{n} U_{j_k}$.
"b)$\Rightarrow$c)": Let $\mathcal{U}$ be any covering. For each i let $U^i_{j_1}, \dots, U^i_{j_{n_i}}$ cover $K_i - \operatorname{int} K_{i-1}$. Then

$$\{U^i_{j_k} \cap (\operatorname{int} K_{i+1} - K_{i-1}) \mid i = 0, \dots; k = 1, \dots, n_i\}$$

does the job. *nice picture*
The reverse implications are similar. $\square$

DEFINITION 2.3. A (smooth) *partition of unity* subordinate to a covering $\mathcal{U}$ of M is a collection of smooth functions $\lambda_i : M \to \mathbb{R}$, $i \in I$, such that the following conditions are satisfied

- $\operatorname{supp} \lambda_i$ is contained in some set $U \in \mathcal{U}$ from the covering

[1] having a countable basis for its topology

- the collection $\{\operatorname{supp} \lambda_i \mid i \in I\}$ is locally finite
- $\sum \lambda_i = 1$ which makes sense thanks to the local finiteness property

REMARK. If λ_i is a partition of unity subordinate to $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$ then λ_i is also a partition of unity subordinate to $\mathcal{U}$.

REMARK. Choosing for each λ_i a definite $U \in \mathcal{U}$ for which $\operatorname{supp} \lambda_i \subseteq U$ we can add those λ_i's corresponding to a single U to get λ_U with $\operatorname{supp} \lambda_U \subseteq U$. In this way we get a partition of unity which is indexed by the covering $\mathcal{U}$ itself.

Now we can start proving the following theorem.

THEOREM 2.4. *Partitions of unity exist subordinate to any open covering.*

If I am not mistaken then $\rho(x) = e^{-\frac{1}{x+1}} e^{-\frac{1}{1-x}}$ should be a function which is positive on $(-1, 1)$ and zero elsewhere. Similarly we get a function $\rho(x_1) \cdots \rho(x_m) : \mathbb{R}^m \to \mathbb{R}$, positive on $(-1, 1)^m$ and zero elsewhere. *pictures*

LEMMA 2.5. *Let $K \subseteq U \subseteq \mathbb{R}^m$ with K compact and U open. Then there exists a smooth function $\lambda : \mathbb{R}^m \to \mathbb{R}_+ = [0, \infty)$ such that λ is positive on K and* $\operatorname{supp} \lambda \subseteq U$.

PROOF. Cover K by a finite number of open cubes V_i. Modifying ρ slightly we find λ_i which is positive on V_i and zero elsewhere. Then put $\lambda = \sum \lambda_i$. □

PROOF OF THEOREM 2.4. Enough to prove for $\mathcal{U}$ locally finite and with $U \in \mathcal{U} \Rightarrow \bar{U}$ compact and contained in a coordinate chart. This is so as for any $\mathcal{U}$ we can find a refinement of this form. Also we can assume that $\mathcal{U}$ is countable as any covering possesses a countable subcovering. Therefore let $\mathcal{U} = \{U_i\}_{i=0}^{\infty}$. We construct λ_i's inductively. We start with λ_0 any function which is positive on $U_0 - \bigcup_{i=1}^{\infty} U_i$ and with $\operatorname{supp} \lambda_0 \subseteq U_0$. We denote $V_0 := \lambda_0^{-1}(0, \infty)$. In the inductive step we choose a function λ_j which is positive on $U_j - \bigcup_{i=0}^{j-1} V_i - \bigcup_{i=j+1}^{\infty} U_i$ and with $\operatorname{supp} \lambda_j \subseteq U_j$. Then we denote $V_j := \lambda_j^{-1}(0, \infty)$ and the induction step is finished. Clearly the local finiteness is satisfied as $\mathcal{U}$ is supposed to be such. By construction $\sum_{i=0}^{\infty} \lambda_i > 0$ on M. To finish the proof we replace λ_j by $\frac{\lambda_j}{\sum_{i=0}^{\infty} \lambda_i}$. □

THEOREM 2.6. *Let M be a compact manifold. Then there exists an embedding of M into $\mathbb{R}^k$ for some $k \gg 0$.*

PROOF. Let $\varphi_i : U_i \to \mathbb{R}^m$, $i = 1, \ldots, k$, be a covering of M by coordinate charts (i.e. $\bigcup_{i=1}^{k} U_i = M$. Let λ_i, $i = 1, \ldots, k$ be a partition of unity subordinate to $\mathcal{U} = \{U_i\}_{i=1}^{k}$. The claim is that the map

$$j = (\lambda_1, \lambda_1 \cdot \varphi_1, \ldots, \lambda_k, \lambda_k \cdot \varphi_k) : M \to \mathbb{R}^{(m+1)k}$$

(with $\lambda_i \cdot \varphi_i$ the obvious extension - by zero - of the map with the same name from U_i to M). Clearly j is injective: if x and y are two points with $j(x) = j(y)$ then certainly there is i such that $0 \neq \lambda_i(x) = \lambda_i(y)$. Then in $\lambda_i(x) \cdot \varphi_i(x) = \lambda_i(y) \cdot \varphi_i(y)$ we can divide by this common value to conclude that $\varphi_i(x) = \varphi_i(y)$ and then refer to the injectivity of φ_i. The proof of injectivity of the tangential map follows the same idea: let $(x, v) \in T_xM$ be such that $j_*(x, v) = 0$. Then all $(\lambda_i)_*(x, v) = 0$ and therefore

$$0 = (\lambda_i \cdot \varphi_i)_*(x, v) = \lambda_i(x) \cdot (\varphi_i)_*(x, v) + (\lambda_i)_*(x, v) \cdot \varphi_i(x) = \lambda_i(x) \cdot (\varphi_i)_*(x, v)$$

Again one of the $\lambda_i(x)$ is nonzero and therefore $(\varphi_i)_*(x, v)$ must be zero. But as φ_i is an embedding this implies $v = 0$. $\square$

Aside. An embedding $j : M \hookrightarrow N$ is a diffeomorphism of M with a submanifold $j(M) \subseteq N$. Equivalently (HW) j is an injective immersion and a homeomorphism onto its image. Therefore for M compact the notions of an embedding and an injective immersion coincide.

It is possible to decrease k from the theorem to $2m + 1$. We need (an easy version of) Sard's theorem.

Definition 2.7. A subset $S \subseteq \mathbb{R}^m$ has measure 0 if for each $\varepsilon > 0$, S can be covered by a sequence C_i of cubes with

$$\sum_{i=0}^{\infty} \mathrm{Vol}(C_i) < \varepsilon$$

Remark. This notion is closed under countable unions.

Remark. If S has measure 0 then S does not contain any nonempty open subset.

Lemma 2.8. *Let $f : U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^m$, be a smooth map and $S \subseteq U$ a subset of measure 0. Then $f(S)$ has measure 0.*

Proof. Cut U into countably many compact cubes C_i On each C_i the derivatives of f are bounded, i.e.

$$|f'_u(x)| \leq K_i \qquad \forall x \in C_i, u \in S^{m-1}$$

nice picture Now we compute

$$|f(x + v) - f(x)| = \left| \int_0^1 f'_v(x + tv)\mathrm{d}t \right| \leq \int_0^1 |f'_v(x + tv)|\mathrm{d}t \leq \int_0^1 K_i|v|\mathrm{d}t = K_i|v|$$

In other words the lengths increases at most K_i-times, the image of a cube of side a lies in a cube of side $a\sqrt{m}K_i$. If $S \cap C_i$ is covered by cubes of total volume ε then $f(S \cap C_i)$ is covered by cubes of total volume $\varepsilon \cdot (\sqrt{m}K_i)^m$ which can be made arbitrarily small and thus $f(S \cap C_i)$ has measure 0. As $f(S) = \bigcup_i f(S \cap C_i)$ the proof is finished. $\square$

Corollary 2.9. *The property of having measure 0 is diffeomorphism invariant. In other words it does not depend on the chart.*

Therefore the following definition makes sense.

Definition 2.10. A subset $S \subseteq M$ of a smooth manifold M has measure 0 if it is so in each chart.

THEOREM 2.11 (Sard's theorem, easy version). *Let $f : M^m \to N^n$ be a smooth map with $m < n$. Then* $\operatorname{im} f$ *has measure 0 in N.*[2]

PROOF. This is a local problem and so we can assume

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

extend f to a map

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \xrightarrow{pr_1} \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$$

and observe that $\operatorname{im} f = f(\mathbb{R}^m \times 0)$ and $\mathbb{R}^m \times 0$ has measure 0 in $\mathbb{R}^n$. □

THEOREM 2.12. *Any compact smooth manifold M^m can be embedded into $\mathbb{R}^{2m+1}$.*

REMARK. M being compact is not important (but M being second countable is).

REMARK. In fact M can be embedded into $\mathbb{R}^{2m}$ but the proof given here could not work *picture of a trefoil knot where the projection as in the proof does not exist.*

PROOF. Let M be embedded in some $\mathbb{R}^k$. This is possible by Theorem 2.6. The strategy is to find a direction (or rather a line ℓ) in which to project to reduce the dimension k by 1.

$$M \overset{j}{\hookrightarrow} \mathbb{R}^k \xrightarrow{p} \mathbb{R}^k/\ell$$

The problem is to make sure that $p \circ j$ is still an embedding. This is possible if $k > 2m+1$:

- $p \circ j$ injective: consider the following map

$$\begin{aligned} M \times M - \Delta &\xrightarrow{g} \mathbb{R}P^{k-1} \\ (x, y) &\longmapsto [j(x) - j(y)] \end{aligned}$$

  The injectivity of $p \circ j$ is equivalent to $\ell \notin \operatorname{im} g$.
- $p \circ j$ immersion: consider the following map

$$\begin{aligned} STM &\xrightarrow{h} \mathbb{R}P^{k-1} \\ (x, v) &\longmapsto [j_*(x, v)] \end{aligned}$$

  Then equivalently $\ell \notin \operatorname{im} h$.

All together, $p \circ j$ is an embedding iff $\ell \notin \operatorname{im} g \cup \operatorname{im} h$. If $k > 2m + 1$ then $\operatorname{im} g \cup \operatorname{im} h$ has measure 0 by Theorem 2.11 and thus its complement is nonempty. □

REMARK. The same proof shows that M^m can be immersed into $\mathbb{R}^{2m}$.

[2]Here it is important that M is second countable. Taking $M = N$ but with discrete topology and $f = \mathrm{id}$ is an example where the theorem would fail otherwise.

CHAPTER 3

Tubular neighbourhoods

Our situation in this chapter is that we have a submanifold $M \subseteq N$ and we want to describe a neighbourhood of M in N in a nice way.

DIGRESSION (about Riemannian metrics).

THEOREM 3.1. *Every manifold possesses a Riemannian metric.*

PROOF. The simplest proof would be: embed M into $\mathbb{R}^{2m+1}$ and induce a metric from there. But we only proved the existence of an embedding for compact manifolds. We exhibit a different proof: let U_i be a covering of M by charts and λ_i a subordinate partition of unity. On U_i we can choose a Riemannian metric g_i using the chart. Then the required Riemannian metric on M is $\sum_i \lambda_i g_i$. This is possible because the set of Riemannian metrics in $T_x^*M \otimes T_x^*M$ is convex. $\square$

Every manifold M has a metric (is a metric space). A possible proof refers to a general statement that every second countable completely regular topological space is metrizable. We give a geometric proof of this fact for M connected. Let g be a Riemannian metric on M and define a length of a piecewise smooth curve $\gamma : [0,1] \to M$ by the formula

$$\ell(\gamma) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))\mathrm{d}t$$

where $\dot\gamma : [0,1] \to TM$ denotes the derivative of γ (evaluate the tangential mapping $\gamma_* : [0,1] \times \mathbb{R} \to TM$ on unit vectors $(x,1)$).

Now we can define a metric on M by a formula

$$d_g(x,y) = \inf\{\ell(\gamma) \mid \gamma : [0,1] \to M \text{ piecewise smooth}, \gamma(0) = x, \gamma(1) = y\}$$

To see that the topology induced by d_g is correct go local. Then in $\mathbb{R}^m$ the metric d_g is equivalent to the standard metric which is d_e for the standard Riemannian metric on $\mathbb{R}^m$. The equivalence comes from a relation $Ke \leq g \leq Le$ (for some $K, L > 0$) holding near some fixed point. The same relation then holds for the induced metrics: $Kd_e \leq d_g \leq Ld_e$ and so they induce the same topology.

NOTE. A geodesic γ is a solution to a certain second order differential equation

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

where ∇ is the Levi-Civita connection associated with the Riemannian metric.

For $(x, v) \in TM$ denote by

$$t \mapsto \varphi(t, x, v)$$

the geodesic starting at x with velocity v, i.e. $\dot{\gamma}(0) = (x, v)$. The curve $\varphi(t, x, v)$ is defined fot t in some neighbourhood of 0. Also $\varphi(t, x, v)$ only depends on tv, not on t and v separately

$$\varphi(t, x, v) = \varphi(1, x, tv) = \exp_x(tv)$$

under a notation $\exp_x v = \varphi(1, x, v)$. We get a smooth map

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & M \\ \cap\!\!| & & \\ \mathbb{R} \times TM & & \end{array}$$

with V an open neighbourhood of $0 \times TM$. If $C \subseteq M$ is a compact subset then $STM|_C$ is also compact and there exists $\varepsilon > 0$ such that $\varphi(t, x, v) = \exp_x tv$ is defined for all $x \in C$, $v \in T_xM$ with $|v| = 1$ and $|t| < \varepsilon$. In other words $\exp_x v$ is defined for all $x \in C$ and $v \in T_xM$ with $|v| < \varepsilon$.

Alltogether exp is defined on a neighbourhood of the zero section $M \subseteq TM$

$$\begin{array}{ccc} U & \xrightarrow{\exp} & M \\ \cap\!\!| & & \\ TM & & \end{array}$$

Consider the map

$$\begin{aligned} (\pi, \exp) : U &\longrightarrow M \times M \\ (x, v) &\longmapsto (x, \exp_x v) \end{aligned}$$

We claim that $(\pi, \exp)$ is a diffeomorphism near the zero section. Clearly when restrcted to the zero section $M \subseteq TM$ the map $(\pi, \exp)$ coincides with the diagonal embedding $\Delta : M \to M \times M$. We check what its differential at the zero section is. First we observe that at a zero section the tangent bundle of any vector bundle canonically splits

LEMMA 3.2. *Let $p : E \to M$ be a vector bundle. Then there is a short exact sequence of vector bundles over E*

$$0 \to p^*E \to TE \to p^*TM \to 0$$

which splits canonically at the zero section.

PROOF. First we define the maps in the sequence. We think of the pullback p^*TM as a subset of $E \times TM$ and write its elements as pairs. Similarly for p^*E but here we must be careful: the first coordinate serves as the point in the base whereas the second coordinate as the vector. The first map is

$$(u, v) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (u + tv)$$

This map is clearly injective and its image is called the *vertical* subbundle of TE. The second map in the sequence is induced by projection $p_* : TE \to TM$ to the base. Clearly this sequence is exact. Restricting to the zero section $M \subseteq E$ we obtain a short exact sequence

$$0 \to E \to TE|_M \to TM \to 0$$

which splits by the map $j_* : TM \to TE|_M$ induced by the inclusion $j : M \subseteq E$. Its image is called the *horizontal* subbundle. $\square$

Now we compute $(\pi, \exp)_*$ at the zero section. On the vertical subbundle

$$(\pi, \exp)_*((x, 0), (0, v)) = ((x, x), (0, v))$$

(this is $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (x, \exp_x tv)$) and on the horizontal subbundle

$$(\pi, \exp)_*((x, 0), (v, 0)) = ((x, x), (v, v))$$

(which is $\Delta_*(x, v)$). Such vectors clearly span $T_{(x,x)}(M \times M)$ and we can conclude that $(\pi, \exp)$ is an embedding of a neighbourhood of the zero section $M \subseteq TM$ onto a neighbourhood of the diagonal $\Delta(M) \subseteq M \times M$ by the following lemma.

Lemma 3.3. *Let $M \subseteq N$ be a compact submanifold, $f : N \to P$ a smooth map. If f is an injective immersion at M ($f|_M$ injective and $(f_*)|_M$ injective rather than $(f|_M)_*$ injective - not sufficient) then there is a neighbourhood U of N such that $f|_U$ is an embedding.*

Proof. Only need that there is a neighbourhood of M on which f is injective as M has a final system of compact neighbourhoods: if $f|_V$ is injective choose an open neighbourhood U of M for which $\bar{U}$ is a compact subset of V and then $f|_{\bar{U}}$ is a homeomorphism onto its image and thus so is $f|_U$. As we may also assume that f is an immersion on U, $f|_U$ is an embedding.

Therefore assume that there is no neighbourhood of M on which f is injective. Remembering that N is a metric space and exhibiting this for an $1/n$-neighbourhood of M we obtain sequences x_i and y_i of points in N such that $f(x_i) = f(y_i)$. We can also assume that both x_i and y_i converge, say to x and y respectively, both lying in M. As $f|_M$ is injective we must have $x = y$ and as f is an immersion at x necessarily $x_i = y_i$ for i big enough. $\square$

In fact with the same amount of effort one can prove a stronger result implying the same conclusion, namely that $(\pi, \exp)$ maps a neighbourhood of M in TM diffeomorphically onto a neighbourhood of $\Delta(M)$ in $M \times M$, even for non-compact manifolds:

Lemma 3.4. *Let $M \subseteq N$ be a smooth submanifold, $f : N \to P$ a smooth map. If f is a local diffeomorphism at M and $f|_M$ a homeomorphism onto its image then there is a neighbourhood U of N such that $f|_U$ is an embedding.*

Proof. Again we only need to show that f is injective on a neighbourhood of M. Cover M by open sets $U_i \subseteq N$ such that f is a diffeomorphism on a neighbourhood V_i of each $\bar{U}_i$. Since f is also a homeomorphism we can (after restriction) assume that U_i satifies $f(U_i) \cap f(M) = f(U_i \cap M)$ and further that $f(U_i)$ is locally finite. We denote $U = \bigcup U_i$ and consider the following subset of P:

$$W = \{y \in f(U) \mid \text{if } x, x' \in U \text{ both belong to } f^{-1}(y) \text{ then } x = x'\}$$

By construction $f(M) \subseteq V$ and it is enough to show that W is a neighbourhood of $f(M)$ as then the (well-defined!) inverse of f is smooth by the implicit function theorem. Therefore let $y \in W$. By local finiteness there is only a finite number of $f(U_i)$'s whose closure contains y, let us denote the corresponding indices by $i_1, \ldots, i_k$. In particular the unique (inside U) preimage of y lies in the intersection $\bar{U}_{i_1} \cap \cdots \cap \bar{U}_{i_k}$. We claim that the open set

$$[f(U) \cap f(V_{i_1} \cap \cdots \cap V_{i_k})] - \bigcup_{i \notin \{i_1 \ldots, i_k\}} \overline{f(U_i)}$$

is contained in W. This is because for z from this set the only way $f(x) = z$ could happen for $x \in U$ is that $x \in V_{i_1} \cap \cdots \cap V_{i_k}$ and as f is injective on this set there is only one possible choice of such x. $\square$

In general we consider now an embedding $j : M \hookrightarrow N$. A *normal bundle* $\nu(j)$ of j is the vector bundle $TM^\perp$, the orthogonal complement of TM inside j^*TN. This definition depends on the choice of metric but it is possible to give one which does not: $\nu(j) = j^*TN/TM$.

Remark. Let $0 \to E \to F \to F/E \to 0$ be a short exact sequence of vector bundles and choose a Riemannian metric on F. Then we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] & F \arrow[r] & F/E \arrow[r] & 0 \\ & & E^\perp \arrow[u] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

Therefore every short exact sequence of vector bundles (in general over a paracompact topological space) split.

Example 3.5. For the diagonal embedding $\Delta : M \to M \times M$ the normal bundle is

$$\nu(\Delta) = (TM \oplus TM)/TM \cong TM$$

Here TM sits in $TM \oplus TM$ diagonally. The orthogonal complement of TM is (if we impose orthogonality of the two summands TM)

$$\{(v, -v) \in TM \oplus TM \mid v \in TM\} \cong TM$$

Example 3.6. Let $j : M \hookrightarrow \mathbb{R}^k$ be an embedding of M into $\mathbb{R}^k$ and consider $\nu(j)$. For any other embedding $j' : M \hookrightarrow \mathbb{R}^{k'}$ and the corresponding $\nu(j')$ we get a relation

$$\nu(j) \oplus (\mathbb{R}^{k'} \times M) \cong \nu(j) \oplus TM \oplus \nu(j') \cong (\mathbb{R}^k \times M) \oplus \nu(j')$$

Such vector bundles $\nu(j)$, $\nu(j')$ are calles stably isomorphic (i.e. they are isomorphic after adding trivial vector bundles - of possibly different dimensions). We also associate to $\nu(j)$ a formal dimension $-m$, intuitively replacing the defining relation $\nu(j) \oplus TM \cong \mathbb{R}^k \times M$ by $\nu(j) \oplus TM = 0$ and therefore thinking of $\nu(j)$ as $-TM$.

CONSTRUCTION 3.7. Thinking of $\nu(j)$ as $TM^\perp$, exp provides a map

$$\exp : \mathring{D}_\varepsilon TM^\perp \longrightarrow N$$

an embedding of a neighbourhood $\mathring{D}_\varepsilon TM^\perp$ of the zero section of $TM^\perp$ as a neighbourhood of M. The proof is exactly the same as for $\Delta : M \to M \times M$. Note that this construction depends on the Riemannian metric on N.

DEFINITION 3.8. A *tubular neighbourhood* of $j : M \hookrightarrow N$ is a pair (E, ι) where $p : E \to M$ is a smooth vector bundle and $\iota : E \to N$ is an embedding of E as a neighbourhood of M in such a way that $\iota|_M = j$ (here M means the zero section).

NOTE. Here E must be isomorphic to the normal bundle of j:

$$\begin{tikzcd} E \arrow[r] & TE|_M \arrow[r, "\iota_*"] & TN|_M \arrow[r, "p"] & TM^\perp \\ & TM \arrow[u] \arrow[ur, "j_*"'] & & \end{tikzcd}$$

with p the orthogonal projection. As $TE|_M$ is a direct sum $E \oplus TM$ and $p\iota_*|_{TM} = pj_* = 0$ the composition across the top row must be an isomorphism.

THEOREM 3.9. *Every compact submanifold $M \subseteq N$ has a tubular neighbourhood.*

PROOF. Use $TM^\perp \cong \mathring{D}_\varepsilon TM^\perp \xrightarrow{\exp} N$. □

In particular a tubular neighbourhood provides a retraction

$$r : \iota(E) \xrightarrow{\iota^{-1}} E \xrightarrow{p} M$$

a nice picture of the retraction

Now we show an interesting application of tubular neighbourhoods (and in particular of the above retraction).

3.A. Smoothening maps

Let M and N be compact manifolds and $\iota : \nu(N) \to \mathbb{R}^k$ a tubular neighbourhood of some embedding $N \hookrightarrow \mathbb{R}^k$. Our aim is to find for each continuous map $f_0 : M \to N$ a smooth approximation, i.e. a smooth map $f_1 : M \to N$ with a homotopy $f : I \times M \to N$ for which $f(0, -) = f_0$ and $f(1, -) = f_1$. In fact we will find a small homotopy f so that f_1 will be close to f_0 but we will make this precise only later.

THEOREM 3.10. *Approximations exist for every $f_0 : M \to N$. If f_0 is already smooth in a neighbourhood of a closed subset $K \subset M$ then the homotopy f can be chosen to be constant on a (possibly smaller) neighbourhood of K (and in particular $f_1 \equiv f_0$ near A).*

PROOF. Find $\varepsilon > 0$ such that every $B_\varepsilon(x), x \in N$ lies in the tubular neighbourhood of N. Cover M by an open covering $\mathcal{U}$ with the following properties:

- $U \in \mathcal{U} \Rightarrow \forall x, y \in U : |f_0(x) - f_0(y)| < \varepsilon$
- $U \in \mathcal{U} \Rightarrow$ either $U \subseteq M - K$ or $f_0|_U$ is smooth

Let λ_U be a subordinate partition of unity and let g_U be smooth maps $U \to \mathbb{R}^k$ satisfying

- $\forall x \in U : |g_U(x) - f_0(x)| < \varepsilon$
- if $U \cap K \neq \emptyset$ then $g_U = f_0$

Such a collection exists - it is either dictated by the second condition or, when $U \subseteq M - K$, one can choose $x_0 \in U$ and put $g_U(x) = f_0(x_0)$ constant. Define $g : I \times M \to \mathbb{R}^k$ by the following formula

$$g(t, x) = (1 - t)f_0(x) + t \sum_{U \in \mathcal{U}} \lambda_U(x) g_U(x)$$

Easily $\operatorname{im} g$ lies in the tubular neighbourhood and one can compose with the retraction r to obtain the required $f(t, x) = r(g(t, x))$. $\square$

3.B. Classification of 1-dimensional manifolds

The classification we present here works equally well for manifolds with boundary. First we define all the needed notions (and slightly more).

In the same way manifolds are modeled on the Euclidean space $\mathbb{R}^m$ manifolds with boundary are modeled on the halfspace

$$\mathbb{H}^m = \mathbb{R}_- \times \mathbb{R}^{m-1} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \leq 0\}$$

To do differential topology on manifolds with boundary it is important to define what a smooth map is.

Definition 3.11. A map $\varphi : \mathbb{H}^m \to \mathbb{H}^n$ is called *smooth* if all the partial derivatives (possibly one-sided) are continuous. This is the standard definition but for convenience our working (and equivalent) definition will be: φ is smooth if it can be locally extended to a smooth map $\mathbb{R}^m \to \mathbb{R}^m$. Again a map is called a diffeomorphism if it is smooth together with its inverse.

Definition 3.12. A *smooth manifold with boundary* is a Hausdorff second countable topological space M with a chosen equivalence class of atlases - a collection of charts satisfying

- each $\varphi : U \to \mathbb{H}^m$ is a homeomorphism of an open subset $U \subseteq M$ with an open subset of $\mathbb{H}^m$
- the transition maps $\psi\varphi^{-1} : \mathbb{H}^m \to \mathbb{H}^m$ (partially defined) are diffeomorphisms

The set ∂M of all points which in some (and then in any) coordinate chart lie in $\partial\mathbb{H}^m = \{0\} \times \mathbb{R}^{m-1}$ is called the *boundary* of M.

Definition 3.13. A tangent bundle TM of a manifold with boundary M is defined in the same way with $T\mathbb{H}^m = \mathbb{R}^m \times \mathbb{H}^m$, i.e. we allow all vectors even those pointing out of the manifold (therefore the kinetic definition of TM is not appropriate).

Definition 3.14. A subset $M \subseteq N$ of a manifold with boundary N is called a *neat submanifold* if there is a covering of M by charts $\varphi : U \to \mathbb{H}^n$ such that

$$\varphi(M \cap U) = \mathbb{H}^m \cap \varphi(U)$$

Here $\mathbb{H}^m \subseteq \mathbb{H}^n$ consists precisely of all $(x_1, \ldots, x_n)$ with only first m coordinates nonzero[1].

THEOREM 3.15. *Let $\varphi : M \to N$ be a smooth map, $\partial N = \emptyset$, and suppose that both φ and $\varphi|_{\partial M}$ are submersions near $\varphi^{-1}(y)$ where $y \in N$ is some fixed point. Then $\varphi^{-1}(y)$ is a neat submanifold.*

PROOF. This is a local problem so we can assume that $\varphi : \mathbb{H}^m \to \mathbb{R}^n$ and $y = 0$. The statement is classical on the interior of M, thus we assume that $x \in \partial\mathbb{H}^m$ is such that $\varphi(x) = 0$. As $\varphi|_{\partial\mathbb{H}^m}$ is a submersion at x we can choose among $x_2, \ldots, x_m$ an n-tuple of coordinates such that the derivatives of φ with respect to them give an isomorphism. We assume that these are $x_{m-n+1}, \ldots, x_m$. Writing now

$$\mathbb{H}^m \cong \mathbb{H}^{m-n} \times \mathbb{R}^n$$

we define a smooth map

$$\psi : \mathbb{H}^m \xrightarrow{(pr_1, \varphi)} \mathbb{H}^{m-n} \times \mathbb{R}^n \cong \mathbb{H}^m$$

where $pr_1 : \mathbb{H}^m \to \mathbb{H}^{m-n}$ denotes the projection onto the first $(m-n)$ variables. Easily $\mathrm{d}\psi(x)$ is an isomorphism and so by the inverse function theorem ψ is a diffeomorphism near x (but note that the inverse function theorem does not imply that the inverse ψ^{-1} takes $\mathbb{H}^m$ into $\mathbb{H}^m$, this follows from our definition of ψ). In the chart ψ the subset $\varphi^{-1}(0)$ becomes

$$\psi(\varphi^{-1}(y)) = \{(x_1, \ldots, x_{m-n}, 0, \ldots, 0) \in \mathbb{H}^m\}$$

□

A serious HW: every compact neat submanifold has a tubular neighbourhood.

From now on we will distinguish between compact manifolds without boundary - they will be called *closed* - and compact manifolds with boundary which we will refer to simply as compact manifolds.

Now we are ready to classify 1-dimensional manifolds.

THEOREM 3.16. *Every connected 1-dimensional manifold is diffeomorphic to exactly one of the following list*

- $[0, 1]$
- $[0, \infty)$
- $\mathbb{R}$
- S^1

(following Goodwillie we do not call $\emptyset$ connected as $\pi_0(\emptyset) \neq *$).

PROOF. Let M be a connected 1-dimensional manifold and choose a Riemannian metric on it. Let $v \in T_xM$ be a unit tangent vector and consider a geodesic determined by it:

$$\begin{aligned} \varphi : J &\longrightarrow M \\ t &\longmapsto \exp_x tv \end{aligned}$$

[1]Loosely speaking, a neat submanifold should be perpendicular to the boundary. In particular ∂M is not a neat submanifold of M although it is a submanifold (a notion which we will not define).

Here J is an interval in $\mathbb{R}$ but as M could have some boundary, the same could be true of J. As φ is a submersion ($\forall t \in J : |\dot{\varphi}(t)| = 1$) the image of φ is open and for the same reason it is also closed[2]. As M is connected φ is surjective. Suppose first that it is also injective. Then φ is a diffeomorphism and this gives us the first three possibilities. Assume now that for some $a < b$ we have $\varphi(a) = \varphi(b)$. As $\dot{\varphi}(a)$ and $\dot{\varphi}(b)$ are two unit vectors in the same 1-dimensional space $T_{\varphi(a)}M$ necessarily $\dot{\varphi}(a) = \pm\dot{\varphi}(b)$.

- $\dot{\varphi}(a) = -\dot{\varphi}(b)$: in this case $\varphi(a+t) = \varphi(b-t)$ which for $t = (b-a)/2$ gives a contradiction $\dot{\varphi}((a+b)/2) = -\dot{\varphi}((a+b)/2)$.
- $\dot{\varphi}(a) = \dot{\varphi}(b)$: in this case $\varphi(a+t) = \varphi(b+t)$ and φ is periodic with period $(b-a)$. Then necessarily $J = \mathbb{R}$. Let T be the smallest period of φ (which easily exists). Then φ induces a map

  $$\tilde{\varphi} : S^1 \cong \mathbb{R}/T\mathbb{Z} \longrightarrow M$$

  which is now injective and therefore a diffeomorphism.

□

Remark. Surjective submersions are quotients in the category of smooth manifolds. This is because surjective submersions have local sections.

$$\begin{tikzcd} M \arrow[r] \arrow[d, two heads, "\substack{\text{surjective}\\ \text{submersion}}"'] & P \\ N \arrow[ur, dashed] & \end{tikzcd}$$

a factorization exists
iff it exists in Set

In particular such quotients are diffeomorphic iff they have the same kernel. In the previous proof this could be used as an explanation why $S^1 \cong M$ in the last case - both are quotients of $\mathbb{R}$.

[2]One can partition M into images of geodesics, each of which is open.

CHAPTER 4

Sard's theorem

DEFINITION 4.1. Let $f : M \to N$ be a smooth map. A point $x \in M$ is called a *critical point* of f if $f_* : T_xM \to T_{f(x)}N$ has rank strictly lower than $n = \dim N$. The set of critical points is called a *critical set* of f and denoted Σ_f. A point $y \in N$ is called a critical value if it is an image of a critical point, $y \in f(\Sigma_f)$. Otherwise it is called a *regular value.*

THEOREM 4.2 (Sard). *The set of critical values has measure 0 in N.*

COROLLARY 4.3 (Brown). *The set of regular values of any smooth map $f : M \to N$ is dense in N.*

Before giving the proof of Sard's theorem we show an application.

THEOREM 4.4 (Brouwer). *There is no retraction $D^n \to \partial D^n = S^{n-1}$.*

PROOF. First we show that there is no smooth retraction. Suppose $r : D^n \to S^{n-1}$ is such a retraction. Take a regular value $y \in S^{n-1}$ (which exists by Sard's theorem). Then $r^{-1}(y)$ is a compact 1-dimensional neat submanifold of D^n and contains y as a boundary point. There must exist a second boundary point. But as $r(z) = z$ for all $z \in S^{n-1}$ the only $z \in S^{n-1}$ in $r^{-1}(y)$ is y itself, a contradiction. *picture*

To reduce to the smooth case we use smooth approximations. Let $r : D^n \to S^{n-1}$ be a continuous retraction. We define a new retraction $r' : D^n \to S^{n-1}$ by pasting together r on the disc of radius 1/2 and the radial projection on the annulus. *picture* We apply the relative version of the smoothening and find a smooth $r'' : D^n \to S^{n-1}$ with $r'' = r'$ near S^{n-1}. □

In the course of the proof of Sard's theorem we need a version of Taylor formula which is also of interest independently.

LEMMA 4.5. *Let $f : U \to \mathbb{R}^n$ be a smooth function defined on some convex subset U of $\mathbb{R}^m$. Then for any two points $x, y \in U$ the following formula holds*

$$f(y) = f(x) + \cdots + (T_x^k f)(x)(y-x)^{\otimes k} + \cdots + (T_x^l f)(y)(y-x)^{\otimes l}$$

(all $T_x^k f$ are applied to x except the last one, which is applied to y) where

$$\begin{aligned} T^k f : U \times U &\longrightarrow \hom\big((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n\big) \\ (x, y) &\longmapsto (T_x^k f)(y) \end{aligned}$$

is also smooth and $(T_x^k f)(x) = \frac{1}{k!} f^{(k)}$.

PROOF. We construct $T^k f$ inductively starting with $k = 1$:

$$f(y) - f(x) = \big[f(x + t(y - x))\big]_0^1 = \left(\int_0^1 f'(x + t(y - x))\ \mathrm{d}t\right)(y - x)$$

and we denote $(T_x^1 f)(y) = \int_0^1 f'(x+t(y-x))\ \mathrm{d}t$. As a function of x and y this is a smooth map

$$T^1 f : U \times U \to \hom(\mathbb{R}^m, \mathbb{R}^n)$$

Iterating this construction we get $T_x^k f = T_x^1 \cdots T_x^1 f$ with

$$T^k f : U \times U \longrightarrow \hom((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n)$$

smooth and a Taylor formula[1]

$$f(y) = f(x) + \cdots + (T_x^k f)(x)(y - x)^{\otimes k} + \cdots + (T_x^l f)(y)(y - x)^{\otimes l}$$

Comparing the derivatives of the two sides we get $T_x^k(f)(x) = \frac{1}{k!} f^{(k)}$. □

PROOF OF SARD'S THEOREM. Firstly we can assume that $m \geq n$, the remaining cases are taken care of by Theorem 2.11. Also the problem is local so we can assume

$$\begin{array}{ccc} W & \xrightarrow{f} & \mathbb{R}^n \\ \cap\!| & & \\ \mathbb{R}^m & & \end{array}$$

We partition the critical set Σ_f according to the order of vanishing derivatives

$$\begin{aligned}
\Sigma_1 &= \{x \in \Sigma_f \mid \mathrm{d}f(x) \neq 0\} \\
\Sigma_2 &= \left\{ x \in \Sigma_f \;\middle|\; \begin{array}{c} \mathrm{d}f(x) = 0 \text{ but there is a nonvanishing} \\ \text{derivative of order at most } m/n \text{ at } x \end{array} \right\} \\
\Sigma_3 &= \{x \in \Sigma_f \mid \text{all derivatives of order at most } m/n \text{ vanish at } x\}
\end{aligned}$$

We need to show that each $f(\Sigma_i)$ has measure 0. This will be proved in 3 steps.

III. $f(\Sigma_3)$ has measure 0: let $C \subseteq W$ be a compact cube with side a. It is enough to show that $f(C \cap \Sigma_3)$ has measure 0. Let $x \in C \cap \Sigma_3$ and let y be another point in C. According to Lemma 4.5 we can write

$$f(y) = f(x) + (T_x^l f)(y)(y - x)^{\otimes l}$$

where l is the smallest integer greater than m/n. Thinking of $T^l f$ as a function

$$C \times C \times (\mathbb{R}^m)^l \longrightarrow \mathbb{R}^n$$

we get a bound $\left|(T_x^l f)(y)(v_1 \otimes \cdots \otimes v_l)\right| < K$ for all $x, y \in C$ and $v_i \in S^{m-1}$. Therefore

$$|f(y) - f(x)| = \left|(T_x^l f)(y)(y - x)^{\otimes l}\right| < K|y - x|^l \tag{4.1}$$

[1] What this formula roughly says is that after subtracting from f its Taylor polynomial at x of order $(l - 1)$ the remainder remains smooth even after "dividing" by $(y - x)^{\otimes l}$.

Now cover C by k^m cubes of side a/k, k some positive integer. Then $C \cap \Sigma_3$ is covered by at most k^m cubes of side a/k each containing a point from Σ_3. By our bound (4.1) the image $f(C \cap \Sigma_3)$ is covered by cubes of total volume at most

$$\mathrm{cst} \cdot k^m \cdot \left((a/k)^l\right)^n = \mathrm{cst} \cdot k^{-n(l-m/n)}$$

(at most k^m cubes in $\mathbb{R}^n$ of side at most $(a/k)^l$). As $k \to \infty$ this tends to 0.

This finishes the proof for $m = 1$ as then $n = 1$ and $\Sigma_f = \Sigma_3$ which is covered by step III. We proceed further by induction on m, i.e. we prove steps II and I assuming the whole theorem proved for all smaller m.

II. We show that $f(\Sigma_2)$ has measure 0. Let $x \in \Sigma_2$. There is a multiindex I and numbers i, j such that

$$\partial_{x_I} f_j(x) = 0$$
$$\partial_{x_i} \partial_{x_I} f_j(x) \neq 0$$

We will partition Σ_2 according to different I, i, j and denote these $\Sigma_2^{I,i,j}$. It is enough to show that the various parts $\Sigma_2^{I,i,j}$ have image of measure 0. Near the point $x \in \Sigma_2^{I,i,j}$ the function

$$\partial_{x_I} f_j : W \to \mathbb{R}$$

is a submersion and therefore $X = (\partial_{x_I} f_j)^{-1}(0)$ is near x a submanifold of dimension $m-1$. Clearly $\Sigma_2^{I,i,j} \subseteq \Sigma_{f|_X}$ (it is important here that we assume that *all* the first derivatives are zero!) and consequently

$$f(\Sigma_2^{I,i,j}) \subseteq f(\Sigma_{f|_X})$$

which has measure 0 by induction hypothesis.

I. We show that $f(\Sigma_1)$ has measure 0. Let $x \in \Sigma_1$ and assume for simplicity that $\partial_{x_1} f_1(x) = 0$ (otherwise change coordinates). Define a map

$$g = (f_1, x_2, \dots, x_m) : W \to \mathbb{R}^m$$

with differential

$$\begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_m} f_1 \\ 0 & & & \\ \vdots & & E & \\ 0 & & & \end{pmatrix}$$

Therefore g is a diffeomorphism on a neighbourhood U of x

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "g"', "\cong"] & \mathbb{R}^n \\ gU \arrow[ur, "\bar{f} = fg^{-1} = (x_1, \bar{f}_2, \dots, \bar{f}_n)"'] & \end{tikzcd}$$

Clearly $f(U \cap \Sigma_f) = \bar{f}(\Sigma_{\bar{f}})$ (the critical values are diffeomorphism invariant) and so it is enough to show that $\bar{f}(\Sigma_{\bar{f}})$ has measure 0. Denote the last $n-1$ coordinates of $\bar{f}$ by

$$h_{x_1}(x_2, \dots, x_m) := (\bar{f}_2, \dots, \bar{f}_n)(x_1, \dots, x_m)$$

Then we can write $\Sigma_{\bar{f}} = \bigcup_{x_1 \in \mathbb{R}} \{x_1\} \times \Sigma_{h_{x_1}}$ and similarly

$$\bar{f}(\Sigma_{\bar{f}}) = \bigcup_{x_1 \in \mathbb{R}} \{x_1\} \times h_{x_1}(\Sigma_{h_{x_1}})$$

Fubini theorem gives the following formula with μ denoting the measure

$$\mu(\bar{f}(\Sigma_{\bar{f}})) = \int \mu(h_{x_1}(\Sigma_{h_{x_1}}))\ \mathrm{d}x_1 = \int 0\ \mathrm{d}x_1 = 0$$

The second equality follows from the induction hypothesis as each h_{x_1} is a smooth map $\mathbb{R}^{m-1} \to \mathbb{R}^{n-1}$. $\square$

Remark. The version of Fubini theorem that we need here is rather easy to prove, first we state it: let C be a compact subset of $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-1}$. Denote by $\mathbb{R}^n_t$ the subset $\{t\} \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ and similarly $C_t = C \cap \mathbb{R}^n_t$. This version says that if each C_t has measure 0 then so does C.

To prove this let us cover, for each t, C_t by a countable family K_i of cubes of total volume ε. By compactness there is a small interval I_t around t such that the products $I_t \times K_i$ cover $C \cap (I_t \times \mathbb{R}^{n-1})$. The image of C under projection to the first coordinate is a compact subset and thus lies in a closed interval, say of length a. This interval is covered by the intervals I_t and one can pick from them a finite subset $I_{t_1}, \ldots, I_{t_k}$ of intervals of total length at most $2a$. Then the total volume of cubes we used to cover $C \cap (I_{t_i} \times \mathbb{R}^{n-1})$ is at most $2a\varepsilon$ and by varying ε can be dropped arbitrarily low.

CHAPTER 5

Transversality

Definition 5.1. Let there be given smooth maps as in the diagram

$$M \xrightarrow{f} N \xleftarrow{f'} M'$$

The maps f and f' are said to be *transverse* if for each $x \in M$, $x' \in M'$ with $y = f(x) = f'(x')$ the following holds

$$f_*(T_x M) + f'_*(T_{x'} M') = T_y N$$

We denote this fact by $f \pitchfork f'$.

picture

We will need a special case when f' is an embedding $j : A \hookrightarrow N$ of a submanifold.

$$\begin{tikzcd} & A \arrow[d, hook, "j"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Then f is said to be transverse to A, denoted $f \pitchfork A$, if it is transverse to the embedding j, i.e. if for each $x \in M$ with $f(x) \in A$

$$f_*(T_x M) + T_{f(x)} A = T_{f(x)} N$$

The following reformulation will be useful: for $x \in f^{-1}(A)$ consider the composition

$$T_x M \longrightarrow T_{f(x)} N \longrightarrow (TN/TA)_{f(x)} = \nu(j)_{f(x)}$$

with $\nu(j)$ the normal bundle of the embedding j. Then $f \pitchfork A$ iff this map is surjective for all $x \in f^{-1}(A)$.

Example. A submersion is transverse to every submanifold.

Example. Let $A = \{y\} \subseteq N$. Then $f \pitchfork A$ iff f is a submersion near (or at) $f^{-1}(y)$ iff y is a regular value of f.

In the proceeding the following notation will be very useful. A codimension of an embedding $j : A \hookrightarrow N$ is the difference

$$\operatorname{codim} j = \dim N - \dim A = \text{the dimension of the fibre of } \nu(j)$$

of the dimensions.

Example. Let $f : M \to N$ be any smooth map and $j : A \hookrightarrow N$. If $\dim M < \operatorname{codim} j$ then $f \pitchfork A$ is equivalent to $\operatorname{im} f \subseteq N - A$.

Proposition 5.2. *Let $f : M \to N$ be smooth and $j : A \hookrightarrow N$ a submanifold (which is neat in the case $\partial N \neq \emptyset$) such that $f \pitchfork A$ and $f|_{\partial M} \pitchfork A$. Then $f^{-1}(A)$ is a neat submanifold of the same codimension as A. Moreover the map g in the following diagram is a fibrewise isomorphism*

$$\begin{tikzcd} f^{-1}A \arrow[r, "f'"] \arrow[d, hook, "j'"'] & A \arrow[d, hook, "j"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \qquad \begin{tikzcd} \nu(j') \arrow[r, "g"] \arrow[d] & \nu(j) \arrow[d] \\ f^{-1}A \arrow[r, "f'"'] & A \end{tikzcd}$$

Remark. The slogan here is: "transverse pullbacks exist". More generally when $f \pitchfork f'$ then $M \times_A M'$ is a submanifold of $M \times M'$ and we get a pullback

$$\begin{tikzcd} M \times_N M' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M' \arrow[d, "f'"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Serious HW: prove this (by reducing to the proposition).

Proof. This is again a local problem and so we can assume

$$\mathbb{H}^m \xrightarrow{f=(f_1,f_2)} \mathbb{R}^{n-d} \times \mathbb{R}^d$$

with $A = \mathbb{R}^{n-d} \times \{0\}$. The condition $f \pitchfork A$ translates to the component f_2 being a submersion near $f^{-1}(A)$ and similarly for $f|_{\partial M} \pitchfork A$. Therefore locally $f^{-1}(A) = f_2^{-1}(0)$ is a neat submanifold by Theorem 3.15. $\square$

Remark. A neat submanifold is a submanifold (which we have not defined) $A \subseteq N$ such that $\partial A = A \cap \partial N$ and which is transverse to ∂N.

Sard's theorem says that most of the points are transverse to a given $f : M \to N$. We will now generalize this to submanifolds. In fact we will approach this problem from the other side: we will prove that "most" of the maps $f : M \to N$ are transverse to a given A.

a picture of how this is natural

Lemma 5.3. *Consider the following diagram with $f \pitchfork A$ and any smooth map $g : P \to M$.*

$$\begin{tikzcd} & f^{-1}(A) \arrow[r, "f'"] \arrow[d, hook, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, hook, "j"] \\ P \arrow[r, "g"'] & M \arrow[r, "f"'] & N \end{tikzcd}$$

Then $g \pitchfork f^{-1}(A)$ if and only if $fg \pitchfork A$.

PROOF. Clearly for $x \in P$ the conditions $fg(x) \in A$ and $g(x) \in f^{-1}(A)$ are equivalent. For such x consider the diagram

$$\begin{tikzcd} \nu(j')_{g(x)} \arrow[r, "\cong"] & \nu(j)_{fg(x)} \\ T_xP \arrow[u, "g_*"] \arrow[ur, "(fg)_*"'] & \end{tikzcd}$$

In this diagram g_* is surjective iff $(fg)_*$ is surjective. This completes the proof. $\square$

A serious HW: generalize this to arbitrary j (not necessarily an embedding).

REMARK (A very important remark!). Most of the statements we proved for compact manifolds work (with the same proof) for general manifolds near any given compact subset. Two (important!) examples are

- Let $C \subseteq M$ a compact subset of a smooth manifold M. Then there exists an embedding of some neighbourhood U of C in some $\mathbb{R}^k$, $k \gg 0$.
  For a proof, instead of covering M by a finite number of charts and producing a map, cover C and write down the same formula. It will be an embedding of some neighbourhood U.
- Let $C \subseteq M \overset{j}{\hookrightarrow} N$. Then some neighbourhood U of C has a tubular neighbourhood, i.e. there is an embedding
  $$\iota : \nu(j)|_U \to N$$
  extending the embedding j.
  Here exp is still defined and an embedding near C, i.e. on some $\mathring{D}_\varepsilon \nu(j)|_U$. *picture*

We are now ready to prove our main theorem.

THEOREM 5.4. *Let M be a compact smooth manifold and $A \subseteq N$ a manifold, both A and N boundaryless. Then for any $f_0 : M \to N$ there exists a smooth homotopy $f : I \times M \to N$ such that $f(0,-) = f_0$ and $f(1,-) \pitchfork A$. Moreover if f_0 is transverse to A near some closed subset $K \subseteq M$ then f can be chosen constant near K.*

PROOF. Embed a neighbourhood of $f_0(M) \subseteq N$ in some $\mathbb{R}^k$ with a tubular neighbourhood $\iota : \nu(N)|_U \xrightarrow{\cong} V \subseteq \mathbb{R}^k$. First assume that $K = \emptyset$ and define

$$F : \mathbb{R}^k \times M \longrightarrow \mathbb{R}^k$$
$$(v, x) \longmapsto f_0(x) + v$$

For a small v we land in the tubular neighbourhood V and can define

$$G : \mathring{D}_\varepsilon \times M \xrightarrow{F} V \xrightarrow{r} U \subseteq N$$

This is clearly a submersion and in particular $G \pitchfork A$. Assuming this we consider $G^{-1}(A)$ and a regular value v for its projection to $\mathbb{R}^k$.

$$G^{-1}(A) \subseteq \mathbb{R}^k \times M \longrightarrow \mathbb{R}^k$$

We claim that for this regular value

$$G(v,-) : M \to N$$

is transverse to A. This is proved by examining the following diagram.

$$\begin{tikzcd}
 & G^{-1}(A) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, hook] \\
\{v\} \times \mathring{D}_\varepsilon \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathring{D}_\varepsilon \times M \arrow[r, "G"'] \arrow[d, "pr"] & N \\
\{v\} \arrow[r, hook] & \mathring{D}_\varepsilon &
\end{tikzcd}$$

Moreover formula $(t,x) \mapsto G(tv,x)$ then defines the required homotopy.

Now we describe what has to be changed when $K \neq \emptyset$. As was observed it is enough to alter F so that it is independent of v near K but still transverse to A. To do this choose a function $\lambda : M \to [0,1]$ for which $\lambda^{-1}(0)$ is a neighbourhood of K where f_0 is transverse to A. Then $\widetilde{F} : (v,x) \mapsto f_0(x) + \lambda(x)v$ is the required alteration: at poits (v,x) with $\lambda(x) > 0$ it is still a submersion while, if $\lambda(x) = 0$ then necessarily $\mathrm{d}\lambda(x) = 0$ too and $\widetilde{F}_*(T_v\mathbb{R}^k \times T_xM) = (f_0)_*(T_xM)$. $\square$

Remark. Again we have a "near a compact subset" version for a compact subset $C \subseteq M$ of a noncompact manifold M: there is a homotopy $g : I \times V \to N$ defined on a neighbourhood V of C with $g(0,-) = f_0|_V$ and with $g(1,-) \pitchfork A$ (and also relative version with K: then f is constant near $K \cap V$). Now we describe a variant of this version.

Choose a function $\rho : M \to [0,1]$ such that $\rho \equiv 1$ on a neighbourhood U of C and with support in V. Now we can define a global homotopy

$$\begin{aligned} f : I \times M &\longrightarrow N \\ (t,x) &\longmapsto g(\rho(x)t, x) \end{aligned}$$

This homotopy has the following properties

- $f(0,-) = f_0$
- $(f(1,-)|_U) \pitchfork A$
- f is constant near K and outside of a neighbourhood V of C (which in fact can be chosen arbitrarily)

Using this version we can prove

Theorem 5.5. *Let $E \to M$ be a smooth bundle over a compact manifold M. Let $E' \subseteq E$ be a subbundle (locally a subset of the form $F' \times U \subseteq F \times U$). For every section $s_0 : M \to E$ there exists a homotopy of sections $s : I \times M \to E$ with $s(0,-) = s_0$ and $s(1,-) \pitchfork E'$. Again if s_0 is transverse to E' near some closed subset K then s can be chosen constant near K.*

Proof. Cover M by open subsets $U_1, \dots, U_k$ over which we have trivializations

$$\begin{tikzcd} E|_{U_i} \arrow[rr, "\cong"] \arrow[dr] & & F \times U_i \arrow[dl] \\ & U_i & \end{tikzcd}$$

identifying $E'|_{U_i}$ with $F' \times U_i$. Now observe that a section (f, id) of a trivial bundle $F \times U \to U$ is transverse to $F' \times U$ iff f is transverse to F'. In this way we translate the situation over each U_i to one where Theorem 5.4 (or rather the remark following it) could be applied.

Let λ_i be a partition of unity subordinate to $U_1, \dots, U_k$. According to Theorem 5.4 there are homotopies of sections $s^i : I \times M \to E$ with the following properties

- $s^1(0, -) = s_0$ and $s^{j+1}(0, -) = s^j(1, -)$ for $j \geq 1$
- each $s^j(1, -) \pitchfork E'$ near $\operatorname{supp} \lambda_j$
- s^j is constant near $K \cup \bigcup_{i<j} \operatorname{supp} \lambda_i$

The concatenation of $s^1, \dots, s^k$ is the required homotopy. $\square$

Serious HW: show that the relation of being smoothly homotopic (relative K, i.e. by a homotopy constant near K) is transitive.

An application to vector bundles is the following.

5.A. Classifying vector bundles

We study the set of (linear) monomorphisms $\mathbb{R}^n \to \mathbb{R}^k$ by studying its complement[1]. Denote by $\hom_r(\mathbb{R}^n, \mathbb{R}^k)$ the subset of $\hom(\mathbb{R}^n, \mathbb{R}^k)$ of maps of rank r.

Lemma 5.6. *$\hom_r(\mathbb{R}^n, \mathbb{R}^k)$ is a smooth submanifold of $\hom(\mathbb{R}^n, \mathbb{R}^k)$ of codimension $(n-r)(k-r)$.*

Proof. Let $\varphi_0 \in \hom(\mathbb{R}^n, \mathbb{R}^k)$ and assume that

$$\varphi_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$$

with the block A_0 of size $r \times r$ invertible. Then $\varphi_0 \in \hom_r(\mathbb{R}^n, \mathbb{R}^k)$ iff $D_0 = C_0 A_0^{-1} B_0$. Therefore on a neighbourhood of such φ_0 every $\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has A invertible and we get a chart

$$\begin{aligned} \mathrm{GL}(r) \times \hom(\mathbb{R}^{n-r}, \mathbb{R}^r) \times \hom(\mathbb{R}^r, \mathbb{R}^{k-r}) &\longrightarrow \hom_r(\mathbb{R}^n, \mathbb{R}^k) \\ (A, B, C) &\longmapsto \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} \end{aligned}$$

[1]More precisely we study connectivity of the set of monomorphisms by studying the codimension of its complement.

with codimension equal to the size of the D-block. As after after a linear change of coordinates on $\mathbb{R}^n$ and $\mathbb{R}^k$ we can assume that the topleft $r \times r$ block of any element of $\hom_r(\mathbb{R}^n, \mathbb{R}^k)$ is invertible this finishes the proof. □

We can now reprove the existence of an inverse of a vector bundle, BUT we get a relative version which will be crucial in our classification.

CONSTRUCTION 5.7. Let $E \to M$ be a vector bundle. Then $\hom(E, \mathbb{R}^k)$ is a vector bundle with fibre $\hom(E_x, \mathbb{R}^k)$ over x. In fact it is canonically isomorphic to $E^* \oplus \cdots \oplus E^*$ (k-times). We will also need its subbundle $\hom_{\mathrm{inj}}(E, \mathbb{R}^k)$ consisting of all injective homomorphisms.

PROPOSITION 5.8. *Let $E \to M$ be a smooth vector bundle over a compact manifold M with the dimension of the fibre n. Then every smooth section*

$$s : \partial M \to \hom_{\mathrm{inj}}(E, \mathbb{R}^k)|_{\partial M}$$

extends to a section over M provided $k \geq m + n$.

PROOF. Extend s arbitrarily to a section

$$\tilde{s} : M \to \hom(E, \mathbb{R}^k)$$

This is possible using partition of unity and the fact that the fibre is convex (contractible). Now we can homotope $\tilde{s}$ to a section t which is transverse to all $\hom_r(E, \mathbb{R}^k)$, $0 \leq r < n$. This is done using Theorem 5.5 and an easy fact that these are subbundles. But as $m < (n-r)(k-r)$ for all such r the transversality is equivalent to t not meeting $\hom_r(E, \mathbb{R}^k)$. Therefore

$$t : M \to \hom_{\mathrm{inj}}(E, \mathbb{R}^k)$$

□

REMARK. Such a section is equivalent to an inclusion of vector bundles

$$E \hookrightarrow \mathbb{R}^k \times M$$

Any complementary subbundle (e.g. $E^\perp$) gives an inverse of E. We will be interested in yet another reformulation of a section of $\hom_{\mathrm{inj}}(E, \mathbb{R}^k)$.

The fibre of $\hom_{\mathrm{inj}}(E, \mathbb{R}^k)$ is $\hom_{\mathrm{inj}}(\mathbb{R}^n, \mathbb{R}^k) = V_{k,n}$, the *Stiefel manifold* of n-frames in $\mathbb{R}^k$ (n-tuples of linearly independent vectors). Now $V_{k,n}$ has a natural transitive action of $\mathrm{GL}(n)$ with the orbit space $G_{k,n}$, the *Grassmann manifold* of all n-dimensional linear subspaces in $\mathbb{R}^k$. The projection map is identified with

$$\begin{aligned} V_{k,n} &\longrightarrow G_{k,n} \\ (e_1, \ldots, e_n) &\longmapsto \mathrm{span}(e_1, \ldots, e_n) \end{aligned}$$

There is a description of $G_{k,n}$ as a homogenous space $O(k)/(O(n) \times O(k-n))$. This identification is constructed from the obvious transitive action of $O(k)$ on $G_{k,n}$. By acting on the standard subspace $\mathbb{R}^n \subseteq \mathbb{R}^k$ we obtain a map

$$O(k) \to G_{k,n}$$

sending a matrix to the span of its first n columns (which is clearly smooth). It is easy to identify the stabilizer of $\mathbb{R}^n$ as the group of matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

This is what is understood by $O(n) \times O(k-n)$. The above map $O(k) \to G_{k,n}$ therefore factorizes through an immersion

$$O(k)/(O(n) \times O(k-n)) \longrightarrow G_{k,n}$$

which is then easily seen to be injective and thus a diffeomorphism. What will be important for us is that $G_{k,n}$ is a closed manifold.

There is a canonical n-dimensional vector bundle $\gamma_{k,n}$ over $G_{k,n}$ sometimes also called tautological[2]

$$\begin{tikzcd}
\gamma_{k,n} \arrow[r, equal] \arrow[d, "\substack{\text{vector}\\\text{bundle}}"'] & \{(P,x) \mid P \in G_{k,n}, x \in P\} \arrow[d, hook] \arrow[dr, "j"] & \\
G_{k,n} & G_{k,n} \times \mathbb{R}^k \arrow[l] \arrow[r] & \mathbb{R}^k
\end{tikzcd}$$

Lemma 5.9. *There is a bijection*

$$\{\textit{fibrewise isomorphisms } E \xrightarrow{f'} \gamma_{k,n}\} \xrightarrow{\cong} \{\textit{sections of } \hom_{\mathrm{inj}}(E, \mathbb{R}^k)\}$$

sending f' to the composition $jf' : E \to \mathbb{R}^k$ thought of as a section of $\hom_{\mathrm{inj}}(E, \mathbb{R}^k)$.

Proof. Given a section t the corresponding f' must be

$$E_x \ni u \longmapsto (\operatorname{im} t(x), t(x)(u))$$

$$\begin{tikzcd}
E \arrow[r, "f'"] \arrow[d] & \gamma_{k,n} \subseteq G_{k,n} \times \mathbb{R}^k \arrow[d] \\
M \arrow[r, "f"'] & G_{k,n}
\end{tikzcd}$$

and it is enough to show that $\operatorname{im} t : M \to G_{k,n}$ is smooth. Locally (when $E = \mathbb{R}^n \times U \to U$ is trivial) the section t is given by a smooth map $U \to \hom_{\mathrm{inj}}(\mathbb{R}^n, \mathbb{R}^k) = V_{k,n}$ and im agrees with the canonical projection $V_{k,n} \to G_{k,n}$. $\square$

Remark. More generally for a principal G-bundle $P \to B$ and a G-space X there is a bijection between sections of the associated bundle $P \times_G X \to B$ and G-equivariant maps $P \to X$. This specializes to the previous lemma by taking $X = V_{k,n}$ after one replaces the vector bundles by the respective principal $\mathrm{GL}(n)$-bundles.

[2] An alternative definition goes as follows. The projection $V_{k,n} \to G_{k,n}$ is in fact a principal $\mathrm{GL}(n)$-bundle and $\gamma_{k,n}$ is the associated vector bundle $\gamma_{k,n} = V_{k,n} \times_{\mathrm{GL}(n)} \mathbb{R}^n$.

If in the diagram

$$\begin{tikzcd} E \arrow[r, "f'"] \arrow[d] & \gamma_{k,n} \arrow[d] \\ M \arrow[r, "f"'] & G_{k,n} \end{tikzcd}$$

the homomorphism f' is a fibrewise isomorphism we call $f : M \to G_{k,n}$ a *classifying map* for E. This is because in such a situation $E \cong f^*\gamma_{k,n}$ is determined by f.

Theorem 5.10. *Let M be a closed(?) smooth manifold. There is a bijection*

$$\begin{gathered} \{\textit{homotopy classes of maps } M \xrightarrow{f} G_{k,n}\} \\ \big\downarrow\cong \\ \{\textit{isomorphism classes of vector bundles } E \to M\} \end{gathered}$$

sending f to the pullback bundle $f^*\gamma_{k,n}$ provided $k > m + n$.

Proof. We will first show that this mapping is well-defined (i.e. $f_0 \sim f_1$ implies $f_0^*\gamma_{k,n} \cong f_1^*\gamma_{k,n}$), a fact usualy proved topologically (in much greater generality). First observe that it is enough to show that for every vector bundle $p : E \to I \times M$, $E|_{\{0\}\times M} \cong E|_{\{1\}\times M}$. For then a homotopy $f : I \times M \to G_{k,n}$ yields a pullback bundle $f^*\gamma_{k,n} \to I \times M$ with $f^*\gamma_{k,n}|_{\{t\}\times M} = f_t^*\gamma_{k,n}$.

$$\begin{tikzcd} f^*\gamma_{k,n}|_{\{t\}\times M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f^*\gamma_{k,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_{k,n} \arrow[d] \\ \{t\} \times M \arrow[r, hook] \arrow[rr, bend right, "f_t"'] & I \times M \arrow[r, "f"] & G_{k,n} \end{tikzcd}$$

Our idea will be to compare the two restrictions by the flow of some vector field. We start with $I \times M$ where we can compare[3] the two ends via a canonical vector field

$$Y(t, x) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} (s, x)$$

Let $X \in \Gamma(TE)$ be a vector field on E such that

- $X|_{I\times M} = Y$ (agrees with Y on the zero section)
- $p_*X = Y$ (projects to Y)

It is easy to find such X locally, on $\mathbb{R}^n \times J \times U$ with $J \subseteq I$, $U \subseteq \mathbb{R}^n$, we take

$$X(v, t, x) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} (v, s, x)$$

[3]The comparison is reflected in the fact that the flow of Y (which is $\mathrm{Fl}_s^X(t, x) = (s + t, x)$) produces a diffeomorphism $\mathrm{Fl}_1^Y = \mathrm{id} : \{0\} \times M \to \{1\} \times M$.

As both conditions are affine we can construct X globally using a partition of unity and get a flow

$$\begin{tikzcd} \mathrm{Fl}^X : W \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & E \\ \mathbb{R} \times E & \end{tikzcd}$$

(note that W is not open as E has boundary). Using our second condition on X

$$p\,\mathrm{Fl}^X(t,v) = \mathrm{Fl}^Y(t,p(v)) = (t + p_1(v), p_2(v))$$

Therefore restricting to $W \cap (\mathbb{R} \times E_{\{0\}\times M})$ we get

$$\begin{tikzcd} \varphi : W' \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & E \\ {[0,1] \times E|_{\{0\}\times M}} & \end{tikzcd}$$

now with W' open. Also W' contains $[0,1] \times (\{0\} \times M)$, therefore a neighbourhood of the form

$$[0,1] \times \mathring{D}_\varepsilon E|_{\{0\}\times M}$$

In particular restricting to this disc bundle we finally obtain

$$\psi = \varphi(1,-) : \mathring{D}_\varepsilon E|_{\{0\}\times M} \to E_{\{1\}\times M}$$

This is a diffeomorphism onto its image (it is defined by a flow, its inverse is given by following the flow for time -1). Note that ψ is not necessarily linear but we will linearize it (by taking a derivative) using its 2 properties

- $\psi(\{0\} \times M) = \{1\} \times M$ (preserves the zero section)
- $v \in \mathring{D}_\varepsilon E_{(0,x)} \Rightarrow \psi(v) \in E_{(1,x)}$ (fibrewise over identity)

Taking the derivative at the zero section we get

$$\begin{tikzcd} T(E|_{\{0\}\times M})|_{\{0\}\times M} \arrow[r, phantom, "\cong"] \arrow[d, "\psi_*"'] & E|_{\{0\}\times M} \oplus TM \arrow[d, "\begin{pmatrix} \alpha & 0 \\ 0 & \mathrm{id} \end{pmatrix}"] \\ T(E|_{\{1\}\times M})|_{\{1\}\times M} \arrow[r, phantom, "\cong"] & E|_{\{1\}\times M} \oplus TM \end{tikzcd}$$

with $\alpha : E|_{\{0\}\times M} \to E|_{\{1\}\times M}$ the required isomorphism.

Now we continue with proving the bijectivity of the mapping. We saw in Proposition 5.8 and Lemma 5.9 that it is surjective and thus it is enough to show that any two classifying maps $f_0, f_1 : M \to G_{k,n}$ for E are homotopic. Consider

$$\begin{tikzcd} (I \times E)|_{\partial(I\times M)} \arrow[rr] \arrow[dr] \arrow[dd] & & \gamma_{k,n} \arrow[dd] \\ & I \times E \arrow[ur, dashed] \arrow[dd] & \\ \partial(I \times M) \arrow[rr] \arrow[dr] & & G_{k,n} \\ & I \times M \arrow[ur, dashed] & \end{tikzcd}$$

Here the map $\partial(I\times M) \to G_{k,n}$ is f_i on $\{i\}\times M$. As they are both classifying maps for E the map (fibrewise isomorphism) on the top exists. According to Lemma 5.9 this determines a section of $\hom_{\mathrm{inj}}(I \times E, \mathbb{R}^k)$ over $\partial(I \times M)$ which can be extended by Proposition 5.8 to a section over $I \times M$ which in return provides an extension $f : I \times M \to G_{k,n}$. $\square$

CHAPTER 6

Degree of a map

Let $f : M \to N$ be a smooth map between closed oriented manifolds of the same dimension n. The degree $\deg(f, y)$ is defined in the following way:

- pick a regular value $y \in N$ of f
- observe that $f^{-1}(y)$ is a discrete compact subset of N, hence finite

$$f^{-1}(y) = \{x_1, \dots, x_k\}$$

- at each x_i the derivative of f is an isomorphism $f_{*x_i} : T_{x_i}M \to T_yN$ and define the local degree of f at x_i by

$$\deg_{x_i} f = \begin{cases} 1 & f_{*x_i} \text{ preserves orientation} \\ -1 & f_{*x_i} \text{ reverses orientation} \end{cases}$$

- define $\deg(f, y) = \sum \deg_{x_i} f$

Observe that changing the orientation of M to the opposite one changes the sign of $\deg(f, y)$, similarly for the orientation of N. Also observe that if M and N are non-oriented, the degree makes sense mod 2.

To proceed further we need to know how boundary of a manifold inherits an orientation. Let W be an oriented manifold with boudnary. The orientation of ∂W is characterized by the following: at $x \in \partial W$ choose any outward pointing $e \in T_xW$. Then a basis $(e_1, \dots, e_n)$ in $T_x(\partial W)$ is positive iff $(e, e_1, \dots, e_n)$ is positive in T_xW. Therefore a boundary of $\mathbb{H}^{n+1} = \mathbb{R}_- \times \mathbb{R}^n$ with its standard orientation is $\mathbb{R}^n$ with its standard orientation.

Example 6.1. $\partial(I \times M) = (\{1\} \times M) \cup (-\{0\} \times M)$ where the minus sign in front of an oriented manifold denotes the same manifold with the opposite orientation.

Lemma 6.2. *Let $f : W^{n+1} \to N^n$ be a smooth map between compact oriented manifolds, $\partial N = \emptyset$. Let $y \in N$ be a regular value for both f and $f|_{\partial W}$. Then* $\deg(f|_{\partial W}, y) = 0$.

Proof. $f^{-1}(y)$ is a compact neat 1-dimensional submanifold, therefore a disjoint union of circles (which must lie in the interior) and neatly embedded intervals. If $\gamma : I \to W$ is one of them then $\dot\gamma(0)$ is outward iff $\dot\gamma(1)$ is inward. We have the following picture

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] & \{y\} \arrow[d, hook] \\ W \arrow[r] & N \end{tikzcd} \qquad \begin{tikzcd} I \times T_yN = \nu(\gamma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T_yN \arrow[d] \\ I \arrow[r] & \{y\} \end{tikzcd}$$

We can think of $\nu(\gamma)$ as a neighbourhood of γ in W and, possibly after reversing γ, even with the orientations agreing. Then

$$\partial(\nu(\gamma)) = (\{1\} \times T_yN) \cup (-\{0\} \times T_yN)$$

As the derivative of f at γ in the directions "orthogonal" to γ agrees with the projection $I \times T_yN \to T_yN$ (the map of the normal bundles) it is the same (namely id) on both ends but for opposite orientations of the source. In particular

$$\deg_{\gamma(0)} f|_{\partial W} + \deg_{\gamma(1)} f|_{\partial W} = 0$$

Summing over all such intervals γ we obtain the result. □

LEMMA 6.3. *Let $f_0, f_1 : M \to N$ be two smooth homotopic maps (M and N closed of the same dimension) such that $y \in N$ is a regular value for both of them. Then* $\deg(f_0, y) = \deg(f_1, y)$.

PROOF. Start with any homotopy

$$\tilde{f} : I \times M \to N$$

we can find an approximation $f : I \times M \to N$ transverse to $\{y\}$ with $f \equiv \tilde{f}$ near $\partial(I \times M)$. Thus both f and $f|_{\partial(I\times M)}$ have $\{y\}$ as a regular value and so

$$\deg(f|_{\partial(I\times M)}, y) = 0$$

As $\partial(I \times M) = (\{1\} \times M) \cup (-\{0\} \times M)$ this means

$$0 = \deg(f|_{\partial(I\times M)}, y) = \deg(f_1, y) - \deg(f_0, y)$$

□

THEOREM 6.4. *Let $f : M \to N$ be a smooth map between closed manifolds of the same dimension with N connected. If y_0, y_1 are two regular values of f then*

$$\deg(f, y_0) = \deg(f, y_1)$$

We denote this number by $\deg f$*. It only depends on the homotopy class of f.*

PROOF. The heart of the proof is (a strong form of) homogenity of M. Namely, there exists a smooth homotopy

$$\varphi : I \times M \to N$$

with the following properties (where $\varphi_t = \varphi(t, -)$):

- each φ_t a diffeomorphism (such φ is then called a diffeotopy)
- $\varphi_0 = \mathrm{id}$ and $\varphi_1(y_1) = y_0$

Assuming for now that it exists construct a homotopy

$$\begin{aligned} F : I \times M &\longrightarrow N \\ (t, x) &\longmapsto \varphi(t, f(x)) \end{aligned}$$

Again $F_0 = f$ and $F_1 = \varphi_1 f$ and so y_0 is a regular value for $F|_{\partial(I\times M)}$.

$$\begin{tikzcd} & \{y_1\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{y_0\} \arrow[d] \\ M \arrow[r, "f"] & N \arrow[r, "\varphi_1"] & N \end{tikzcd}$$

Therefore $\deg(f, y_0) = \deg(F_0, y_0) = \deg(F_1, y_0) = \deg(\varphi_1, y_0)\cdot\deg(f, y_1)$ but

$$\deg(\varphi_1, y_0) = \deg(\varphi_0, y_0) = 1$$

as $\varphi_0 = \mathrm{id}$. The homotopy invariance follows from the previous lemma.

Now we return to the existence of the diffeotopy φ. First we find such a diffeotopy locally, but as we would like to extend it, it should be compactly supported (i.e. nonconstant only on a compact subset). Let $y \in \mathbb{R}^n$ and assume for simplicity that $y \in D^n$. Choose a smooth function $\rho : \mathbb{R}^n \to [0,1]$ with $\rho \equiv 1$ near D^n and with compact $\operatorname{supp}\rho$. This function will extend the standard translation on D^n into a compactly supported diffeotopy. Namely take a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ sending x to $-\rho(x)\cdot y$. As X is compactly supported the flow

$$\mathrm{Fl}^X : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$$

is globally defined. Now we can take $\varphi = \mathrm{Fl}^X|_{[0,1]\times\mathbb{R}^n}$ as $X = -y$ on D^n implies $\varphi(1, y) = 0$. If this was in a chart we can extend φ by a constant homotopy to get a global diffeotopy. Now we consider the following equivalence relation:

$$z \simeq z' \Leftrightarrow \exists \text{ a smooth diffeotopy from id to a map sending } z' \text{ to } z$$

The local construction ensures that the orbits are open and therefore closed (as the complement is the union of the other orbits). Thus there is only one orbit. $\square$

Remark. Let M^n be a closed oriented connected smooth manifold. Then the set of homotopy classes of maps $M \to S^n$ is in a bijective correspondence with $\mathbb{Z}$ via the degree map

$$\deg : [M, S^n] \xrightarrow{\cong} \mathbb{Z}$$

We will prove this result in a much greater generality in the next chapter.

CHAPTER 7

Pontryagin-Thom construction

We start with a geometric reformulation of the degree isomorphism

$$\deg : [M, S^n] \xrightarrow{\cong} \mathbb{Z}$$

as for different dimensions the simple counting of preimages could not work. Nevertheless in this reformlulation we do assume $m = n$. The preimage of a regular point of a map $f : M \to S^n$ is a finite subset of M with signs - describing wether f preserves or reverses orientation at the point in question.

$$\begin{tikzcd}
{[M, S^n]} \arrow[rr, "\deg"] \arrow[d] & & \mathbb{Z} \\
\{\text{finite subsets of } M \text{ with signs}\}/\sim \arrow[urr, "\varepsilon"'] & &
\end{tikzcd}$$

Having in mind that the horizontal map should be an isomorphism and that we want in our geometric reformulation to have the vertical map an isomorphism, so must be the map denoted by ε which just add the signs of all the points. This requirement determines the equivalence relation but we want a more geometric description.

It is interesting to describe the Pontryagin-Thom construction in this setting. It plays an essential role in constructing the inverse of the vertical map. Starting with a finite subset X of M with signs attached to them we choose disjoint coordinate discs around each of them and then collapse the complement of their interiors to get the first map in

$$M \xrightarrow{\text{P-T constr.}} \bigvee_X S^n \xrightarrow{\text{class. of VB}} S^n$$

The second arrow just maps each copy S^n onto S^n by a map of degree equal to the sign associated to that point.

Let us now start with the general setting. Let M and N be closed smooth manifolds. We saw the interply between maps and submanifolds: if $f : M \to N$ is transverse to a submanifold $A \hookrightarrow N$ then $f^{-1}(A) \subseteq M$ is again a submanifold. For homotopies we get

Lemma 7.1. *If $f_0 \sim f_1$ and both are transverse to A then $f_0^{-1}(A)$ is cobordant to $f_1^{-1}(A)$ (in M - to be explained after the proof).*

Proof. Let $f : I \times M \to N$ be a homotopy from f_0 to f_1. If it is not smooth we can smoothen it rel $\partial(I \times M)$ where it already is smooth[1]. Also we can make it transverse to A rel $\partial(I \times M)$. Then $f^{-1}(A)$ is a neat submanifold of $I \times M$ and $\partial(f^{-1}(A)) = \big(\{0\} \times f_0^{-1}(A)\big) \cup \big(\{1\} \times f_1^{-1}(A)\big)$ as for neat submanifolds in general $\partial B = (\partial W) \cap B$. $\square$

Definition 7.2. Two compact submanifolds $B_0, B_1 \subseteq M$, $\partial B_0 = \partial B_1 = \partial M = \emptyset$, are called *cobordant in* M if there exists a neat submanifold $B \subseteq I \times M$ such that $\partial B = \big(\{0\} \times B_0\big) \cup \big(\{1\} \times B_1\big)$.

Question. Is it possible to find $A \subseteq N$ of codimension d in such a way that the mapping t below is a bijection?

$$t : \begin{Bmatrix} \text{homotopy classes of} \\ \text{maps } f : M \to N \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{cobordism classes of submanifolds} \\ B \subseteq M \text{ of codimension } d \end{Bmatrix}$$

Here start with $[f]$, make it transverse to A and define $t([f]) = [f^{-1}(A)]$.

To answer the question we need a construction

Definition 7.3. Let $E \to M$ be a vector bundle. Then the *Thom space* $\mathrm{Th}(E)$ of E is

$$\mathrm{Th}(E) = DE/SE$$

the quotient of the associated disc bundle by its boundary, the sphere bundle. $\mathrm{Th}(E)$ is only a topological space (not a manifold) but easily the composition

$$E \cong \mathring{D}E \hookrightarrow DE \to DE/SE = \mathrm{Th}(E)$$

is a topological embedding and so $\mathrm{Th}(E) = E \cup \{\infty\}$ is a smooth manifold away from ∞. Also if M is compact then $\mathrm{Th}(E)$ is the one-point compactification of E.

Remark. There is an alternative way of defining $\mathrm{Th}(E)$. Choosing a metric on E (which is only important in the smooth case anyway) we can now replace the typical fibre $\mathbb{R}^d$ by the one-point compactification $(\mathbb{R}^d)^+ = S^d \subseteq \mathbb{R}^d \times \mathbb{R}$ where $O(d)$ acts by $O(d) \cong O(d) \times 1 \subseteq O(d+1)$. In this way we obtain a sphere bundle E^+. As the inverse of the stereographic projection $\mathbb{R}^d \hookrightarrow S^d$ is $O(d)$-equivariant it induces an embedding

$$E \hookrightarrow E^+$$

Now E^+ has two sections, the zero section and $\infty : M \to E^+$. We define $\mathrm{Th}(E) = E^+/\infty(M) = E \cup \{\infty\}$. The main property of $\mathrm{Th}(E)$, namely that $\mathrm{Th}(E) - M \simeq *$ can now be easily verified: as the sphere $(\mathbb{R}^d)^+$ is symmetric in the $\mathbb{R}$-direction (as an $O(d)$-space). Therefore $\mathrm{Th}(E) - M \cong E/M$ and a contraction of this comes from the homotopy

$$\begin{aligned} h : I \times E &\longrightarrow E \\ (t, (x, v)) &\longmapsto (x, tv) \end{aligned}$$

[1] What we need is a homotopy which is smooth *near* $\partial(I \times M)$. Such a homotopy can be constructed from f by pasting the concatenating the constant homotopy on f_0 with f and the constant homotopy on f_1. *picture*

which easily passes to E/M.

Now we can answer the question: the solution is

$$N = \mathrm{Th}(\gamma_{k,d}) \quad \text{and} \quad A = G_{k,d} \subseteq \gamma_{k,d}, \text{ as the zero section}$$

Now we will explain why this should be a good choice. If we have a map which is transverse to A and take pullback

$$\begin{tikzcd} f^{-1}(A) \arrow[r, "f'"] \arrow[d, hook, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, hook, "j"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} \nu(j') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu(j) \arrow[d] \\ f^{-1}(A) \arrow[r, "f'"] & A \end{tikzcd}$$

This means that the normal bundle of *any* submanifold $B \subseteq M$ must be a pullback of $\nu(j)$ and the best choice is $\nu(j) = \gamma_{k,d}$, the universal bundle. To construct the inverse of t we need the Pontryagin-Thom construction.

Let $j : B \hookrightarrow M$ be a submanifold with a tubular neighbourhood

$$\iota : \nu(j) \hookrightarrow M$$

Also denote $U = \operatorname{im} \iota$. The Pontryagin-Thom construction is the map

$$j^! : M \to \mathrm{Th}(\nu(j))$$

defined by the following

- $j^!|_U : U \xrightarrow{\iota^{-1}} \nu(j) \to \nu(j)/(\nu(j) - \mathring{D}\nu(j)) \cong D\nu(j)/S\nu(j) = \mathrm{Th}(\nu(j))$
- $j^!|_{M-U}$ is constant onto ∞.

picture This clearly defines a continuous map $j^! : M \to \mathrm{Th}(\nu(j))$. In fact $j^!$ is smooth on a neighbourhood of B. As we prefer to think of the Thom space as $\mathrm{Th}(\nu(j)) = \nu(j) \cup \{\infty\}$ we will now alter the definition of $j^!$ to

$$j^!|_U : U \xrightarrow{\iota^{-1}} \nu(j) \hookrightarrow \mathrm{Th}(\nu(j))$$

(obtained by replacing $\nu(j)$ with $\mathring{D}\nu(j) \cong \nu(j)$, the result remains continuous). An advantage of this point of view is that any fibrewise isomorphism $E \to E'$ of vector bundles induces a continuous map $\mathrm{Th}(E) \to \mathrm{Th}(E')$ which is even smooth away from ∞.

Aside. One can give a more symmetric description of the map $j^!$ by embedding M into $\mathbb{R}^k$. Choosing a tubular neighbourhood of $B \subseteq \mathbb{R}^k$ inside some tubular neighbourhood of $M \subseteq \mathbb{R}^k$ the Pontryagin-Thom construction for the embedding $B \subseteq \mathbb{R}^k$ factors

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, two heads] \arrow[dr] & \mathrm{Th}(\nu(M)) \arrow[d, dashed, "j^!"] \\ & \mathrm{Th}(\nu(B)) \end{tikzcd}$$

The Poincaré duality can be expressed as $\mathrm{Th}(\nu(M))$ being dual to M. This construction then describes the dual map to j (in homology this map goes to cohomology by the Poincaré duality then continues with the induced map j^* in cohomology and finally returns to homology by Poincaré duality again).

THEOREM 7.4 (Thom). *The map*

$$t : \begin{Bmatrix} \textit{homotopy classes of maps} \\ f : M \to \mathrm{Th}(\gamma_{k,d}) \end{Bmatrix} \longrightarrow \begin{Bmatrix} \textit{cobordism classes of submanifolds} \\ B \subseteq M \ \textit{ of codimension } d \end{Bmatrix}$$

is bijective provided $k > n$.

To define t let us call a continuous map $f : M \to \mathrm{Th}(\gamma_{k,d})$ transverse to $G_{k,d}$ if it is smooth near $f^{-1}(G_{k,d})$ and transverse to $G_{k,d}$ in this neighbourhood.

LEMMA 7.5. *Every $f : M \to \mathrm{Th}(\gamma_{k,d})$ is homotopic to a map which is transverse to $G_{k,d}$ (plus a relative version).*

PROOF. Make f smooth near $f^{-1}(D\gamma_{k,d})$ and make sure that no points outside of $f^{-1}(\mathring{D}\gamma_{k,d})$ get mapped into $G_{k,d}$ by the perturbed map. Then make f transverse to $G_{k,d}$ near $f^{-1}(D\gamma_{k,d})$ and again not mapping points outside of $f^{-1}(\mathring{D}\gamma_{k,d})$ into $G_{k,d}$. $\square$

With this lemma in mind we define t from the theorem by choosing a representative f of the homotopy class with $f \pitchfork G_{k,d}$ and then taking $f^{-1}(G_{k,d})$. The relative version of the lemma then guarantees that this prescription is independent of the choice of f: if two such are homotopic, we can assume this homotopy to be transverse to $G_{k,d}$ and thus giving the required cobordism.

PROOF. First we prove surjectivity. Let $j : B \hookrightarrow M$ be a submanifold of codimension d and considet the Pontryagin-Thom construction

$$g : M \xrightarrow{j^!} \mathrm{Th}(\nu(j)) \xrightarrow{\mathrm{Th}(f')} \mathrm{Th}(\gamma_{k,d})$$

where the second map comes from classifying $\nu(j)$:

$$\begin{tikzcd} \nu(j) \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_{k,d} \arrow[d] \\ B \arrow[r, "f"] & G_{k,d} \end{tikzcd}$$

The composition g is transverse to $G_{k,d}$ with $g^{-1}(G_{k,d}) = B$.

The proof of injectivity is similar in the spirit but more complicated. Let $B_0 = f_0^{-1}(G_{k,d}) \sim f_1^{-1}(G_{k,d}) = B_1$ be cobordant in M. We proceed in few steps:

- First we reduce to the case $B_0 = B_1$. Let $j : B \hookrightarrow I \times M$ be a cobordism,
$$\partial B = \big(\{0\} \times B_0\big) \cup \big(\{1\} \times B_1\big)$$
and let $\iota : \nu(j) \hookrightarrow I \times M$ be its tubular neighbourhood. Considering the Pontryagin-Thom construction for j gives
$$I \times M \xrightarrow{j^!} \mathrm{Th}(\nu(j)) \longrightarrow \mathrm{Th}(\gamma_{k,d})$$

and thus a homotopy $g_0 \sim g_1$ where $B_i = g_i^{-1}(G_{k,d})$ Therefore it is enough to show $g_i \sim f_i$.

- We assume $B = f_0^{-1}(G_{k,d}) = f_1^{-1}(G_{k,d})$ with both $f_i \pitchfork G_{k,d}$. There are pullback diagrams

$$\begin{tikzcd} B \arrow[r, "f_i'"] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & G_{k,d} \arrow[d, hook] \\ M \arrow[r, "f_i"] & N \end{tikzcd} \qquad \begin{tikzcd} \nu(j) \arrow[r, "f_i''"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_{k,d} \arrow[d] \\ B \arrow[r, "f_i'"] & G_{k,d} \end{tikzcd}$$

This implies from the classification theorem that $f_0' \sim f_1'$. Moreover this homotopy is covered by a homomorphism of vector bundles

$$\begin{tikzcd} I \times \nu(j) \arrow[r, "f''"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_{k,d} \arrow[d] \\ I \times B \arrow[r, "f'"] & G_{k,d} \end{tikzcd}$$

Our next goal is to extend this homotopy to M.

- The next step is to extend the above homotopy to a tubular neighbourhood of $I \times B$. The main tool will be a linearization of maps between vector bundles. First we explain what happens in the case of a single vector space. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism preserving 0. We can write

$$f(x) = f(0) + (T_0^1 f)(x) \cdot x$$

with $T_0^1 f : \mathbb{R}^d \to \hom(\mathbb{R}^d, \mathbb{R}^d)$ smooth. The "zoom in" homotopy

$$h : (t, x) \mapsto 1/t \cdot f(tx) = (T_0^1 f)(tx) \cdot x$$

It follows from the first description that each h_t is a diffeomorphism whereas from the second description it is obvious that h is smooth and $h_0 = \mathrm{d}h(0)$. We summarize this in

Theorem 7.6. *Every diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ preserving 0 is homotopic through such maps to its differential at 0.*

Serious HW: Show how the last theorem implies $\mathrm{Diff}(\mathbb{R}^d, 0) \simeq \mathrm{GL}(d)$ (this requires understanding the topology on smooth maps).

Now we will proceed with a fibrewise version: let $f : \mathbb{R}^{m'} \times \mathbb{R}^d \longrightarrow \mathbb{R}^m \times \mathbb{R}^d$ be a smooth map with the following two properties:

$$f(\mathbb{R}^{m'} \times 0) \subseteq \mathbb{R}^m \times 0 \quad \text{and} \quad f \pitchfork \mathbb{R}^m \times 0$$

We do the linearization fibrewise:

$$(t, (x, v)) \mapsto (f_1(x, tv), 1/t \cdot f_2(x, tv))$$

which is again smooth. This prescription makes sense between any vector bundles and we get

THEOREM 7.7. *Every smooth map $f : E' \to E$ between total spaces of vector bundles of the same dimension preserving and transverse to the zero section is homotopic through such maps to the vector bundle homomorphism f''.*

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, hook, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, hook, "j"] \\ E' \arrow[r, "f"] & E \end{tikzcd} \qquad \begin{tikzcd} E' \cong \nu(j') \arrow[r, "f''"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu(j) \cong E \arrow[d] \\ M' \arrow[r, "f'"] & M \end{tikzcd}$$

Now we return to the proof of injectivity. Denoting by U some tubular neighbourhood of B where both f_i are smooth the vector bundle homomorphism f'' from the previous step is a homotopy between linearizations f_i'' of $f_i|_U$. All together on U we have a homotopy, which we denote by f, as a concatenation of

$$f_0 \sim f_0'' \sim f_1'' \sim f_1$$

provided by the linearizations and the homotopy f''.

- The last step in the proof is to extend the homotopy f to M. Here we will make use of the contractibility of $\mathrm{Th}(\gamma_{k,d}) - G_{k,d}$. Namely we are given a map

$$\big(\{0\} \times M\big) \cup \big(I \times U\big) \cup \big(\{1\} \times M\big) \xrightarrow{f_0 \cup f \cup f_1} \mathrm{Th}(\nu(\gamma_{k,d}))$$

which we want to extend to $I \times M$. Replacing U by the subset corresponding to $D\nu(j)$ we can cut out the interior $\mathring{D}\nu(j)$ to obtain

$$N = I \times (M - \mathring{D}\nu(j)))$$

a manifold with corners (to be defined later). Our problem can then be formulated as extending the map $f : \partial N \to \mathrm{Th}(\gamma_{k,d}) - G_{k,d}$ to N. As $\mathrm{Th}(\gamma_{k,d}) - G_{k,d}$ is contractible $f \sim \infty$ is homotopic to the constant map with value ∞. This constant map can be extended to N easily so the proof is finished by the next theorem.

$\square$

DEFINITION 7.8 (Manifolds with corners). Manifolds with ccrneres are modelled on $(\mathbb{R}_-)^m$, the rest is "the same" with a couple of remarks. Unlike for manifolds with boundary, manifolds with corners are closed under products. That is why the manifold N from the proof of Thom's theorem is a manifold with corners. There are also certain disadvantages: e.g. tangent vectors are not represented even by one-sided curves. More seriously, the boundary of a manifold with corners is not a manifold

THEOREM 7.9. *Let N be a compact manifold with corners. Then $\partial N \hookrightarrow N$ has a homotopy lifting property.*

PROOF. Let X be an inward pointing vector field on N with no zeroes on ∂N. Locally this is easy $X = (-1, \dots, -1)$ on $(\mathbb{R}_-)^m$, to get X globally glue the local ones by the partition of unity. The flow of X is defined on

$$\mathrm{Fl}^X : \mathbb{R}_+ \times N \to N$$

Restrict to ∂N to get $\varphi : \mathbb{R}_+ \times \partial N \to N$. Observe that φ is injective: if $\varphi(t,x) = \varphi(t',x')$ with $t \leq t'$ then $\mathrm{Fl}^X(t'-t,x) = x'$ which is impossible as X is pointing inwards. Also $\operatorname{im}\varphi$ is an open neighbourhood of ∂N. This is easily seen in a chart. *picture*

The neighbourhood of ∂N homeomorphic to $\mathbb{R}_+ \times \partial N$ is called a collar (if N only had a boundary then even diffeomorphic). Start with $f : (\{0\} \times N) \cup (I \times \partial N) \longrightarrow X$. Using a retraction $I \times \mathbb{R}_+ \to (\{0\} \times \mathbb{R}_+) \cup (I \times \{0\})$ we can extend f to a map

$$f' : (\{0\} \times N) \cup (I \times \underbrace{(\mathbb{R}_+ \times \partial N)}_{\text{the collar of } \partial N})$$

Now choose $\lambda : \mathbb{R}_+ \to I$ such that $\lambda(0) = 1$ and $\operatorname{supp}\lambda$ is compact and put

$$f''(t,x) = f'(\rho(x) \cdot t, x)$$

where $\rho : N \to I$ is a function defined by

$$\begin{aligned} \rho|_{\mathbb{R}_+ \times \partial N} &= \lambda \circ pr_1 \\ \rho|_{N - (\mathbb{R}_+ \times \partial N)} &\equiv \lambda \circ 0 \end{aligned}$$

□

7.A. Possible modifications of the Thom's theorem

- Get rid of the "in M" setting to classify manifolds of dimension d up to an abstract cobordism. For that we need a manifold where all d-dimensional manifolds embed and similarly for their cobordisms. S^{2d+2} will do but we need further tools, namely the relative embedding theorem.
- Replace $\mathrm{Th}(\gamma_{k,d})$ by the Thom space of a universal vector bundle for vector bundles with a structure. Examples of these are:
  - $\mathrm{Th}(\tilde{\gamma}_{k,d})$ to get cobordism classes of submanifolds $B \subseteq M$ with an orientation of the normal bundle. It M was oriented this is equivalent to an orientation of B. Here $\tilde{\gamma}_{k,d}$ is a pullback of the universal as in the diagram

$$\begin{tikzcd} \tilde{\gamma}_{k,d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_{k,d} \arrow[d] \\ \tilde{G}_{k,d} \arrow[r] & G_{k,d} \end{tikzcd}$$

    with $\tilde{G}_{k,d} \to G_{k,d}$ the double covering obtained by considering the d-dimensional subspaces of $\mathbb{R}^k$ together with an orientation. For $\tilde{G}_{k,d}$ we have

$$[M, \tilde{G}_{k,d}] \cong \left\{ \begin{matrix} \text{isomorphism classes of oriented} \\ \text{vector bundles over } M \end{matrix} \right\}$$

– $S^d = \mathrm{Th}(\mathbb{R}^d \to *)$ to get cobordism classes of submanifolds with (a cobordism class of) a trivialization of the normal bundle. This is because the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^d \arrow[d] \\ M \arrow[r] & * \end{tikzcd}$$

provides a trivialization of the vector bundle $E \cong M \times \mathbb{R}^d$. In particular $[M^n, S^n]$ is in bijection with finite subsets of M with orientations of the tangent spaces at the points. Is M is oriented this is equivalent to a sign. Two such subsets of a connected manifold M are cobordant iff the total signs agree (if M is oriented) or if the number of points agree mod 2 (if M is non-orientable). This identifies

$$[M^n, S^n] \cong \begin{cases} \mathbb{Z} & M \text{ oriented} \\ \mathbb{Z}/2 & M \text{ non-orientable} \end{cases}$$

Serious HW: fill in details.

– complex, symplectic structures

CHAPTER 8

Index of a vector field

The idea behind the index is roughly the same as with degrees: if $X \in \Gamma(TM)$ we count zeros of X with appropriate signs.

More generally we define an intersection number. Let M, N be smooth manifolds with M closed and $j : A \hookrightarrow N$ a closed submanifold with $m + a = n$, i.e. $\dim M = \operatorname{codim} j$. Assume that $f : M \to N$ is transverse to A. Then $f^{-1}(A)$ is a discrete subset of M, hence finite. If we also assume that M, N and A are oriented then for $x \in f^{-1}(A)$ we can say whether the composition

$$\begin{tikzcd} T_xM \arrow[r, "f_*"] \arrow[rr, bend right, "\cong"'] & T_{f(x)}N \arrow[r, two heads] & T_{f(x)}N/T_{f(x)}A = \nu(j)_{f(x)} \end{tikzcd} \tag{8.1}$$

preserves or reverses orientation and put $\#_x(f, A) = \pm 1$ accordingly. This is because $\nu(j)$ inherits orientation from N and A. Namely, if $(e_1, \dots, e_a, e_{a+1}, \dots, e_n)$ is positive in $T_{f(x)}N$ and $(e_1, \dots, e_a)$ is positive in $T_{f(x)}A$ then $([e_{a+1}], \dots, [e_n])$ is positive in $\nu(j)_{f(x)}$. We define the intersection number $\#(f, A)$ of f with A to be

$$\sum_{x \in f^{-1}(A)} \#_x(f, A)$$

Remark. If $A = \{y\}$ with its canonical orientation then $\#(f, A) = \deg(f, y)$.

Theorem 8.1. *If $f_0, f_1 : M \to N$ are two homotopic maps, both transverse to A then $\#(f_0, A) = \#(f_1, A)$.*

Proof. Make the homotopy transverse to A and take $f^{-1}(A)$, a union of intervals and circles. For the intervals compare the local intersection numbers at the two boundary points. $\square$

Therefore the intersection number provides a map

$$[M, N] \xrightarrow{\#(-,A)} \mathbb{Z}$$

given by first approximating a map by one transverse to A and then taking the intersection number.

Definition 8.2. Let $X \in \Gamma(TM)$ be a vector field and assume that $X \pitchfork M$ ($M \subseteq TM$ as the zero section). The index $\operatorname{Ind} X$ of X is defined to be $\#(X, M)$ (under the assumption that M is oriented).

Corollary 8.3. *If X_0, X_1 are any two vector fields with $X_0 \pitchfork M$, $X_1 \pitchfork M$ then $\#(X_0, M) = \#(X_1, M)$.*

Proof. The required homotopy $X_0 \sim X_1$ is given by $(1-t)X_0 + tX_1$. $\square$

We denote the local index at $x \in X^{-1}(0)$ by $\operatorname{Ind}_x X$. To compute it we choose a coordinate chart around x and write X in this chart

$$X : U \to U \times \mathbb{R}^m$$

Composing with the projection

$$f : U \xrightarrow{X} U \times \mathbb{R}^m \xrightarrow{pr} \mathbb{R}^m$$

the local index is precisely the degree of this composition $\operatorname{Ind}_x X = \deg_x(f)$ as the differential $\mathrm{d}f(x)$ equals the composition (8.1) defining the local intersection number.

Observe that by changing the orientation of U we correspondingly change the orientation of $\mathbb{R}^m$ and thus the index $\operatorname{Ind}_x X$ does not depend on the orientation.

As sections transverse to a subbundle exist by the bundle version of the transversality theorem the index $\operatorname{Ind} X$ depends only on M (only if M is closed!) and we denote it from now on by $\chi(M)$ as we will prove shortly that it equals the Euler characteristics of M. First we describe $\chi(M)$ as a (unique) obstruction to the existence of a nonvanishing (nowhere zero) vector field: obviously if there exists $X \in \Gamma(TM)$ nowhere zero then $\chi(M) = 0$ as there are no intersections to count.

Theorem 8.4. *Let M be a connected closed*[1] *manifold. If $\chi(M) = 0$ then there exists a nonvanishing vector field on M.*

Proof. We will give the proof in several steps.

Step I. "reduce to a local problem": Let $X \in \Gamma(TM)$ be any section transverse to M and let $X^{-1}(0) = \{x_1, \dots, x_k\}$. We choose a coordinate disc D and move the points x_i inside D one by one. There is a diffeomorphism $\varphi_1 : M \xrightarrow{\cong} M$ sending x_1 into D. In the next step we use a compactly supported[2] diffeomorphism

$$\varphi_2 : M - \{\varphi_1(x_1)\} \xrightarrow{\cong} M - \{\varphi_1(x_1)\}$$

sending x_2 into D. Having a compact support, φ_2 extends to a diffeomorphism

$$\varphi_2 : M \xrightarrow{\cong} M$$

preserving $\varphi_1(x_1)$. The composition

$$\varphi = \varphi_k \circ \cdots \circ \varphi_1$$

[1]This assumption cannot be removed. All noncompact manifolds possess a nonvanishing vector field, as well as all manifolds with boundary.

[2]$\operatorname{supp} \psi = \{x \mid \psi(x) \neq x\}$

is a diffeomorphism of M mapping $X^{-1}(0)$ into D. Therefore

$$\begin{tikzcd} Y = \varphi_* \circ X \circ \varphi^{-1} : M \arrow[r] & TM \\ M \arrow[u, "\varphi"] \arrow[r, "X"] & TM \arrow[u, "\varphi_*"'] \end{tikzcd}$$

satisfies $Y^{-1}(0) \subseteq D$ and we reduced to a local problem.

Step II. "reduce index of Y to a degree": In a chart we consider

$$f : U \to \mathbb{R}^m \qquad Y = (\mathrm{id}, f) : U \to U \times \mathbb{R}^m$$

Recall that $\operatorname{Ind}_{x_i} X = \deg_{x_i} f$ where $x_i \in Y^{-1}(0)$. For simplicity we put $x_i = 0$. Choose a small disc D_i around x_i not containing any other points of $Y^{-1}(0)$. We claim that for the map

$$\begin{aligned} g : \partial D_i &\longrightarrow S^{m-1} \\ y &\longmapsto \frac{f(y)}{|f(y)|} \end{aligned}$$

$\operatorname{Ind}_0 Y = \deg g$. This is because on D_i we have a homotopy

$$f_t(y) = 1/t \cdot f(ty)$$

from $\mathrm{d}f(0)$ to f and therefore the local indices are the same as well as the degrees of the corresponding maps g. Now $\mathrm{d}f(0) : D_i \to \mathbb{R}^m$ is a restriction of a linear isomorphism and therefore homotopic through such to an orthogonal map (a "movie" version of Gram-Schidt orthogonalization process). This homotopy applied to both f and g reduces via homotopy invariance to the case when f itself is orthogonal and thus $g = f|_{\partial D_i}$. In this case $\deg_0 f = \det f$ and as f preserves the outward pointing vectors $\deg g = \det f$ too[3].

Step III. "putting everything together": We have in a coordinate chart U a disc D containing all the zeros $\{x_1, \ldots, x_k\}$ of a vector field Y and around each x_i a disc D_i. We may assume that they are disjoint and contained in D. Putting $W = D - \bigcup_{i=1}^k D_i$ the above map g can be extended to W

$$g : W \to S^{m-1} \qquad y \mapsto \frac{f(y)}{|f(y)|}$$

As g is smooth we know that $\deg g|_{\partial W} = 0$. By step II. we have

$$\sum_{i=1}^k \deg g|_{\partial D_i} = \sum_{i=1}^k \operatorname{Ind}_{x_i} Y = \chi(M) = 0$$

[3] A basis $(e_2, \ldots, e_m)$ in $T(\partial D_i)$ is positive iff $(e_1, e_2, \ldots, e_m)$ is positive in TD_i where e_1 is outward pointing. Applying the differential of g we obtain $(f(e_2), \ldots, f(e_m))$. As $f(e_1)$ is still outward pointing this new basis has the same orientation in TS^{m-1} as $(f(e_1), f(e_2), \ldots, f(e_m))$ has in $T\mathbb{R}^m$. In particular it is oriented iff f preserves orientation.

and this implies $\deg g|_{\partial D} = 0$. But we know (either from algebraic topology or *preferably* from the homework) that

$$[\partial D, S^{m-1}] \xrightarrow[\cong]{\deg} \mathbb{Z}$$

and hence $g|_{\partial D} \sim *$. Consequently g extends to a map

$$\tilde{g} : D \to S^{m-1}$$

As $\tilde{g}|_{\partial D} = g|_{\partial D} \sim f|_{\partial D} : \partial D \longrightarrow \mathbb{R}^m - \{0\}$ there is also an extension

$$\tilde{f} : D \to \mathbb{R}^m - \{0\}$$

of $f|_{\partial D}$ (by the homotopy extension property of $\partial D \hookrightarrow D$) defining a new vector field $\tilde{Y}$ on D with no zeros. We can extend $\tilde{Y}$ to the whole M by the original Y as they agree on ∂D. Then $\tilde{Y}$ is nowhere vanishing. $\square$

8.A. Morse theory

Now we will identify $\chi(M)$ with the Euler characteristics. We will consider special vector fields arising from functions that will provide a cell decomposition.

A function $f : M \to \mathbb{R}$ is called *Morse* if for all critical points $x \in M$ (points where $\mathrm{d}f(x)$) the second derivative at x

$$T_xM \otimes T_xM \xrightarrow{\mathrm{d}^2 f(x)} \mathbb{R}$$

is a nondegenerate summetric bilinear form. Such critical points are called *nondegenerate*. The *index* of f at a nondegenerate critical point x is the index of $\mathrm{d}^2 f(x)$, i.e. the number of negative eigenvalues.

ASIDE. To define the second derivative we take the analytic approach. We express $\mathrm{d}^2 f(x)$ in a chart. If $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is a local diffeomorphism sending y to x then

$$\mathrm{d}^2(f \circ \varphi)(y) = \cdots = (\mathrm{d}^2 f(x)) \cdot (\mathrm{d}\varphi(y) \otimes \mathrm{d}\varphi(y)) + \mathrm{d}f(x) \cdot \mathrm{d}^2\varphi(y)$$

In the case that $\mathrm{d}f(x) = 0$ the second term vanishes and the second derivative is independent of the chart chosen to define it.

Geometrically let $X, Y \in T_xM$ and extend then to local vector fields. Then define

$$\mathrm{d}^2 f(x)(X, Y) = X(Y(f))(x)$$

but one has to show independence of the extensions.

In general for maps $f : M \to \mathbb{R}^n$ the second derivative is defined on

$$T_xM \otimes T_xM \to \operatorname{coker} \mathrm{d}f(x)$$

and to make $\mathrm{d}^2 f$ independent even of the charts on the target manifold (which is not what we were after with Morse functions) we need to further restrict to

$$(\ker \mathrm{d}f(x)) \otimes (\ker \mathrm{d}f(x)) \to \operatorname{coker} \mathrm{d}f(x)$$

An equivalent condition for f to be Morse is that the differential $\mathrm{d}f : M \to T^*M$ is transverse to the zero section M: locally

$$\begin{aligned} \mathrm{d}f : U &\longrightarrow U \times \mathbb{R}^m \\ x &\longmapsto (x, \partial_{x_1} f, \ldots, \partial_{x_m} f) \end{aligned}$$

and the nondegeneracy of x translates to $\mathrm{d}(pr_2 \circ \mathrm{d}f)(x)$ being an isomorphism, which is just the transversality. We will prove later that such functions exist and even such that $f(x_i) \neq f(x_j)$ for different critical points $x_i \neq x_j$.

a picture of a torus with the height function

From the Taylor expansion near a nondegenerate critical point 0

$$f(y) = f(0) + 1/2 \cdot \mathrm{d}^2 f(0)(y \otimes y) + \text{the remainder}$$

In fact we can ignore the remainder completely by choosing a good coordinate chart around x, the so-called Morse chart.

Theorem 8.5 (Morse's Lemma). *Let $x \in M$ be a nondegenerate critical point of index i then there is a chart φ around x in which*

$$(f \circ \varphi^{-1})(y_1, \ldots, y_m) = f(x) - y_1^2 - \cdots - y_i^2 + y_{i+1}^2 + \cdots + y_m^2$$

Remark. This means that f is 2-determined at x: for any function agreeing with f up to derivatives of order 2 at x there is a diffeomorphism relating the two functions

$$\begin{tikzcd} U \arrow[dr, "f"] \arrow[dd, dashed, "\exists g"', "\cong"] & \\ & \mathbb{R} \\ \tilde{U} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

This is because $\tilde{f}$ must be Morse too and the Morse charts for f and $\tilde{f}$ combine to produce the diffeomorphism g. Another point of view is that r-determined functions are polynomial (but not vice versa), for they agree with their Taylor polynomial and hence they only differ by a coordinate change.

Proof. Take any chart $\psi : M \to \mathbb{R}^m$ around x with $\psi(x) = 0$ and write

$$(f \circ \psi^{-1})(y) = f(x) + T_0^2(f \circ \psi^{-1})(y)(y \otimes y)$$

The map $T_0^2(f \circ \psi^{-1}) : U \to \hom_{\mathrm{sym}}(\mathbb{R}^m \otimes \mathbb{R}^m, \mathbb{R})$ is smooth and at 0

$$T_0^2(f \circ \psi^{-1})(0) = 1/2 \cdot \mathrm{d}^2(f \circ \psi^{-1})(0)$$

is nondegenerate, thus nondegenerate for all y near 0. For such y one can find a linear transformation Q_y such that

$$T_0^2(f \circ \psi^{-1})(y) \circ (Q_y \otimes Q_y)$$

is $A = \begin{pmatrix} -E_i & 0 \\ 0 & E_{m-i} \end{pmatrix}$. Setting $y = Q_y z$ or $z = Q_y^{-1} y$

$$T_0^2(f \circ \psi^{-1})(y)(y \otimes y) = A(z \otimes z)$$

Then $\eta : y \mapsto Q_y^{-1} y$ transforms f into the wanted form

$$\begin{tikzcd}
\mathbb{R}^m \arrow[dd, "\eta"'] & & \\
& M \arrow[ul, "\psi"'] \arrow[dl] \arrow[dr, "f"] & \\
\mathbb{R}^m \arrow[rr, "z \mapsto f(x)+A(z\otimes z)"'] & & \mathbb{R}
\end{tikzcd}$$

Assuming for a bit that Q is smooth we compute

$$\mathrm{d}\eta(0)(v) = Q_0^{-1} v \quad \Longrightarrow \quad \mathrm{d}\eta(0) = Q_0^{-1} \quad \text{invertible}$$

Hence we are left with the smoothness of Q. Let us fix a definite choice of Q_0 in the form of a product of matrices corresponding to elementary column operations. The entries of each of these matrices is a rational function of the entries of $\mathrm{d}^2(f \circ \psi^{-1})(0)$. Therefore in a neighbourhood of 0 one can use the same functions to get Q_y. $\square$

Let us call a Riemannian metric g on M compatible with f if there exists a Morse chart around each critical point (a chart from the Morse's lemma) in which g is the standard metric.

Lemma 8.6. *Let f be a Morse function on M. Then there exists a Riemannian metric compatible with f.*

Proof. Glue local Riemannian metrics as usual and use compatible local metrics near every critical point. $\square$

A gradient vector field $\operatorname{grad} f$ of f is a vector field dual (with respect to the metric) to $\mathrm{d}f$, i.e.

$$\mathrm{d}f(X) = \langle \operatorname{grad} f, X\rangle \qquad \forall X \in TM$$

As $\operatorname{grad} f$ corresponds to $\mathrm{d}f$ under the isomorphism $TM \cong T^*M$ and $\mathrm{d}f \pitchfork M$ so is $\operatorname{grad} f \pitchfork M$. Thus we can use the gradient vector field for computing $\chi(M)$. Before doing so we first describe how M is built in steps, a process controlled by the Morse function f. On the complement of the critical set in addition to $\operatorname{grad} f$ we also consider

$$X = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$$

Its main property is that f increases linearly along its flowlines

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} f(\mathrm{Fl}_t^X(x)) = Xf(\mathrm{Fl}_{t_0}^X(x)) = \mathrm{d}f(X(\mathrm{Fl}_{t_0}^X(x))) = \langle \operatorname{grad} f, X(\mathrm{Fl}_{t_0}^X(x))\rangle = 1$$

Therefore $f(\mathrm{Fl}_t^X(x)) = f(x) + t$. This proves the following theorem

Theorem 8.7. *If $a < b$ are such that there is no critical value in $[a, b]$ then*

$$f^{-1}[a, b] \cong f^{-1}(a) \times [a, b]$$

in a strong sense. Namely the following diagram commutes

$$\begin{tikzcd} (x,t) \arrow[d, mapsto] & f^{-1}(a)\times[a,b] \arrow[d, "\cong"'] \arrow[dr, "pr"] & \\ & & {[a,b]} \\ \mathrm{Fl}^X_{t-a}(x) & f^{-1}[a,b] \arrow[ur, "f"'] & \end{tikzcd}$$

picture

Proof. The inverse is given by $y \mapsto (\mathrm{Fl}^X_{a-f(y)}(y), f(y))$. $\square$

The outstanding question is: what happens to the level surface $f^{-1}(c)$ when passing though a critical level? *picture of a cell and the deformation*

Theorem 8.8. *Let $a < b$ be two regular values of f such that there is precisely one critical point in $f^{-1}[a,b]$, say of index i. Then there is a i-cell*

$$D^i \overset{\varphi}{\hookrightarrow} f^{-1}[a,b]$$

with $\varphi(\partial D^i) \subseteq f^{-1}(a)$ such that $f^{-1}[a,b]$ deformation retracts onto $f^{-1}(a)\cup_\varphi D^i$.

Proof. The (idea of the) proof was summarized in a picture which I am unable to draw in TeX. $\square$

Now we will finish the determination of $\chi(M)$. For our Morse function f with all critical points in distinct levels we choose $a_0,\dots,a_k \in \mathbb{R}$ such that for the critical points $x_1,\dots,x_k$: $f(x_j)\in(a_{j-1},a_j)$. Let i_j be the index of x_j and abbreviate

$$M_j = f^{-1}(-\infty, a_j)$$

so that $M_0=\emptyset$ and $M_k = M$. *picture* We showed in the last theorem that

$$\begin{aligned} M = M_k &\simeq M_{k-1}\cup D^{i_k} \\ &\vdots \\ M_j &\simeq M_{j-1}\cup D^{i_j} \\ &\vdots \\ M_1 &\simeq D^{i_1} \end{aligned}$$

(and $i_1 = 0$, $i_k = m$).

Remark. This implies that M is homotopy equivalent to a CW-complex with cells of dimensions $i_1,\dots,i_k$. For the above homotopy equivalences can be expressed in saying that

$$\begin{tikzcd} S^{i_j-1} \arrow[r, "f_j"] \arrow[d, hook] & M_{j-1} \arrow[d] \\ D^{i_j} \arrow[r] & M_j \end{tikzcd}$$

is a homotopy pushout. By induction M_{j-1} is homotopy equivalent to a CW-complex X_{j-1} with cells of dimensions $i_1, \dots, i_{j-1}$. The homotopy invariance of the homotopy pushout together with a homotopy of f_j to a cellular map $\tilde{f}_j$ gives $M_j \simeq X_{j-1} \cup_{\tilde{f}_j} D^{i_j} = X_j$. Now X_j is a CW-complex as the D^{i_j} is glued to the $(i_j - 1)$-skeleton.

In the proceeding we will only need a weaker version. We denote the usual (homological) Euler characteristics by

$$\chi_h(M) = \sum_{j=0}^{m} (-1)^j \dim H_j(M)$$

Proposition 8.9. *With the notation from above* $\chi_h(M) = \sum_{j=1}^{k} (-1)^j$.

Proof. Denoting the inclusion $S^{i_j - 1} \to M_{j-1}$ by f_j let us consider the mapping cylinder M_{f_j} of f_j. Obviously $M_{f_j} \simeq M_{j-1}$ and contains a copy of S^{i_j-1} and we form a long exact reduced homology sequence of the pair (M_{f_j}, S^{i_j-1}):

$$\cdots \longrightarrow \tilde{H}_p(S^{i_j-1}) \longrightarrow \tilde{H}_p(M_{f_j}) \longrightarrow \tilde{H}_p(M_{f_j}/S^{i_j-1}) \longrightarrow \cdots$$

As $M_{f_j} \simeq M_{j-1}$ and $M_{f_j}/S^{i_j-1} \simeq M_j$ for the Euler characteristics we get

$$(\chi_h(M_{j-1}) - 1) = (-1)^{i_1 - 1} + (\chi_h(M_j) - 1)$$

or in other words $\chi_h(M_j) = \chi_h(M_{j-1}) + (-1)^{i_j}$. By induction and $M = M_k$, $\chi_h(M_0) = 0$ we obtain the result. □

To identify $\chi(M)$ with $\chi_h(M)$ we only need

Proposition 8.10. *The index of the vector field* grad f *at* x_j *equals* $(-1)^{i_j}$.

Proof. Locally in the Morse chart (with $x_j = 0$)

$$f(y,z) = f(x_j) - |y|^2 + |z|^2 \qquad (y,z) \in \mathbb{R}^{i_j} \times \mathbb{R}^{m-i_j}$$

Then $(\operatorname{grad} f)(y,z) = (-2y, 2z)$ and $\operatorname{Ind}_0 \operatorname{grad} f = (-1)^{i_j}$ as the sign of the determinant of grad f. □

CHAPTER 9

Function spaces

To proceed further we need to topologize the set $C^\infty(M,N)$ of smooth maps between manifolds M, N. This topology should reflect all the differentiable qualities of maps: then maps transverse to a closed submanifold $A \subseteq N$ will form an open subset as transversality is an open condition on the first derivative. It would not be open if only the values of maps were considered. Similarly Morse functions are defined by conditions on the first and second derivative.

To get hold of smooth maps and in particular to their derivatives, we need to describe these geometrically. Namely we define $J^r(M,N)$ to be the set (of r-jets of maps $M \to N$) of equivalence classes of pairs (x,f) where $x \in M$ and $f : U \to N$ a smooth map defined on a neighbourhood U of x. The equivalence relation is: $(x,f) \sim (x',f')$ iff $x = x'$ and f agrees with f' at $x = x'$ up to derivatives of order r (expressed in but independent of charts around x and $f(x)$). We denote $[(x,f)] = j^r_x f$. The point x is called the source and $f(x)$ the target of the jet $j^r_x f$. In this way we obtain the map

$$J^r(M,N) \xrightarrow{(\sigma,\tau)} M \times N$$

and denote $J^r_x(M,N) = \sigma^{-1}(x)$, the jets with source x. Locally for open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$

$$J^r(U,V) \cong U \times V \times \prod_{k=1}^{r} \mathrm{hom}_{\mathrm{sym}}\big((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n\big)$$

The factor U corresponds to the source, V to the target and $\mathrm{hom}_{\mathrm{sym}}\big((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n\big)$ to the k-th derivative. The canonical identification is therefore

$$j^r_x f \mapsto (x, f(x), \mathrm{d}f(x), \ldots, \mathrm{d}^r f(x))$$

(and the inverse map could be realized by polynomials as there is exactly one polynomial or degree at most r with prescribed value and derivatives up to order r at a given point x). Varying these over charts on M and N we obtain an atlas on $J^r(M,N)$ and in fact a bundle structure[1] on

$$J^r(M,N) \xrightarrow{(\sigma,\tau)} M \times N$$

We will need later the composition of jets: we define $j^r_y f \circ j^r_x g := j^r_x(f \circ g)$ provided that $y = g(x)$. One needs to check that this is well-defined and therefore provides a canonical map

$$J^r(N,P) \times_N J^r(M,N) \to J^r(M,P)$$

[1]Moreover the chain $J^r(M,N) \to \cdots \to J^1(M,N) \to J^0(M,N) = M \times N$ consists of affine bundles.

which is moreover smooth[2]. Another important tool is the jet prolongation

$$\begin{aligned} j^r : C^r(M,N) &\longrightarrow C^0(M, J^r(M,N)) \\ f &\longmapsto (x \mapsto j^r_x f) =: j^r f \end{aligned}$$

In fact $j^r f$ is a section of the bundle $\sigma : J^r(M,N) \to M$ and captures all the derivatives of f. It is good for inducing topology on $C^r(M,N)$ once we say what the right topology on $C^0(M,N)$ is. In fact it turns out to be equally easy to topologize the set of continuous sections, the situation we would like to consider anyway.

For this part let X be a topological space and $p : Y \to X$ a space over X, one may think of it as a bundle. We would like to describe a convenient topology on the set $\Gamma(Y)$ of continuous sections of $p : Y \to X$. If X is compact Hausdorff and Y metric (in fact one only needs metric on each fibre) then $\Gamma(Y)$ could be equipped with a metric

$$d(f,g) = \max_{x \in X} d(f(x), g(x))$$

In the case of the product projection $X \times Z \to X$ we use the notation $C(X,Z)$ instead of $\Gamma(X \times Z)$ (which does not tell what the map p is).

Lemma 9.1. *The topology induced by the above metric on* $\Gamma(Y)$ *is generated by the open sets*

$$O(W) = \{f \in \Gamma(Y) \mid \textit{the image of } f \textit{ lies in } W\}$$

with $W \subseteq Y$ *varying over all open subsets*[3].

Proof. Let $g \in \Gamma(Y)$ and $\varepsilon > 0$. Defining W as $\{y \mid d(y, g(p(y))) < \varepsilon\}$ we easily verify $O(W) = B_\varepsilon(g)$. On the other hand for $g \in O(W)$ we take $\varepsilon = \inf_{y \notin W} d(y, g(p(y)))$ and verify that $\varepsilon > 0$: this is because $\varepsilon \geq d(Y - W, g(X)) > 0$. Clearly $B_\varepsilon(g) \subseteq O(W)$. $\square$

This topology behaves nicely (with respect to composition, evaluation, etc.). It extends to noncompact spaces in two ways

- the strong topology $\Gamma_S(Y)$ is generated again by sets $O(W)$ as $W \subseteq Y$ varies over all open subsets. Alternatively for X paracompact and Y metric we can exhaust a neighbourhood basis of a fixed g via sets of the form

  $$N(g, \varepsilon) = \{f \in \Gamma(Y) \mid d(f(x), g(x)) < \varepsilon(x)\}$$

  with ε ranging over all positive functions $X \to (0, \infty)$. To verify this we are required to find for every open neighbourhood W of the image of g a positive function ε such that

  $$\{y \in Y \mid d(y, g(p(y))) < \varepsilon(p(y))\} \subseteq W$$

[2]Locally one takes two polynomials of degree r and then substitutes one into the other, truncating the part of degree bigger then r. The coefficients of the result are smooth functions of the coefficients of the original polynomials.

[3]Typically neighbourhoods of the image of a fixed section $g : X \to Y$. In this case the set of all such $O(W)$ provide a neighbourhood basis of g.

Choose for each $x \in X$ some $\varepsilon_x > 0$ such that $B_{2\varepsilon_x}(g(x)) \subseteq W$. Then for $z \in U_x := g^{-1}B_{\varepsilon_x}(g(x))$ we have $B_{\varepsilon_x}(g(z)) \subseteq W$. Let λ_x be a partition of unity subordinate to (U_x). Then the required ε is $\varepsilon = \sum_{x\in X} \lambda_x \varepsilon_x$.

- the weak topology $\Gamma_W(Y)$ is generated by $\operatorname{res}_K^{-1} O(W)$ where K is a compact subset of X and W an open subset of $Y|_K = p^{-1}(K)$. Here
$$\operatorname{res}_K : \Gamma(Y) \to \Gamma(Y|_K)$$
is the restriction map. The weak topology is usually referred to as the compact-open topology.

Homework: show that a sequence (f_i) in $C_S(M, N)$ converges iff f_i are eventually constant outside of some compact subset of M where it converges uniformly.

Remark. Another point of view on the weak topology is as the limit topology
$$C(X, Y) = \lim_{K\subseteq X} C(K, Y)$$
which is necessary (if one wants to have a categorically well-behaved internal hom functor) at least on the category of compactly generated Hausdorff spaces:
$$X \text{ compactly generated} \iff X = \operatorname*{colim}_{K\subseteq X} K$$
What we are describing here is an extension of $C(-, Y)$ from the category of compact Hausdorff spaces to its closure CGHaus under colimits in the category of all Hausdorff spaces. Again $C_W(-,-)$ turns CGHaus into a cartesian closed category, a very useful property.

For the purpouses of differential topology the strong topology is better, we will therefore restrict ourselves just to that and drop the index "S". We can now use the map
$$j^r : C^r(M, N) \longrightarrow \Gamma(J^r(M, N))$$
to induce topology from the right hand side, i.e. endow $C^r(M, N)$ with the subspace topology. The topology on $C^\infty(M, N)$ is defined as a limit (intersection)
$$C^\infty(M, N) = \lim_r C^r(M, N)$$
In concrete terms, the neighbourhood basis of $g \in C^\infty(M, N)$ consists of (the intersections with $C^\infty(M, N)$ of) the neighbourhoods of g in $C^r(M, N)$, $r \geq 0$. Alternatively one can define
$$J^\infty(M, N) = \lim_r J^r(M, N)$$
together with $j^\infty : C^\infty(M, N) \to \Gamma(J^\infty(M, N))$ and induce the topology from there (which is the best way). The most useful property of $C^\infty(M, N)$ happens to be that it satisfies the Baire property.

Definition 9.2. A subset of a topological space X is called *residual* if it contains a countable intersection of open dense subsets of X. A topological space X is called *Baire* if every residual subset is dense.

Recall that every complete metric space is Baire. We will now generalize this significantly. Let $Y \to X$ be a space over X with Y equipped with a metric. A subset $Q \subseteq \Gamma(Y)$ is called *uniformly closed* if it contains the limit of any uniformly convergent sequence in Q.

Proposition 9.3. *If X is paracompact and Y complete metric then any uniformly closed $Q \subseteq \Gamma(Y)$ is Baire (in the strong topology).*

Remark. By taking $X = *$ and $Q = C(*, Y) = Y$ we deduce that every complete metric space is Baire.

Proof. Let $U \subseteq Q$ be a nonempty open subset and (U_i) a sequence of open dense subsets of Q. We must show that $U \cap \bigcap_{i=0}^{\infty} U_i \neq \emptyset$. To achieve this we inductively construct a sequence of sections $f_i \in Q$ together with positive functions $\varepsilon_i : M \to (0, \infty)$ in such a way that $\varepsilon_0 \leq 1$, $\varepsilon_i \leq 1/2 \cdot \varepsilon_{i-1}$ and

$$Q \cap N(f_i, 2\varepsilon_i) \subseteq U_i \cap N(f_{i-1}, \varepsilon_{i-1})$$

This is possible because U_i being open dense must intersect $N(f_{i-1}, \varepsilon_{i-1})$ in a nonempty open subset. Choosing any f_i from this intersection ε_i is guaranteed by the fact that $N(f_i, \varepsilon)$ form a neighbourhood basis of f_i (as X is paracompact). By our choice the distance $d(f_i(x), f_j(x))$, $i < j$, is uniformly bounded by $\varepsilon_i \leq 2^{-i}$. Hence (f_i) converges uniformly (as Y is complete) and the limit $f = \lim f_i \in Q$. Finally, for $j > i$, $f_j \in N(f_i, \varepsilon_i)$ and thus

$$f \in Q \cap \overline{N(f_i, \varepsilon_i)} \subseteq Q \cap N(f_i, 2\varepsilon_i) \subseteq U_i$$

□

We are trying to complete the proof of the space $C^r(M, N)$ being Baire. First we find a complete metric on every smooth manifold, in particular on $J^r(M, N)$.

Lemma 9.4. *Let M be a manifold. Then there exists a proper function $f : M \to \mathbb{R}_+$.*

Proof. By definition f is proper if $f^{-1}(K)$ is compact for all K compact. Recall the decomposition of M into a union

$$M = \bigcup_{i=0}^{\infty} K_i$$

of compact subsets K_i with K_i lying in the interior of K_{i+1}. Take $L_i = \mathring{K}_i - K_{i-2}$ and consider its corresponding partition of unity λ_i. Then $f = \sum_{i=1}^{\infty} \lambda_i \cdot i$ is the required proper function - outside of K_i, $f(x) > i$, thus $f^{-1}[0, i] \subseteq K_i$. □

Theorem 9.5. *Every smooth manifold admits a complete metric.*

Proof. Let $d : M \times M \to \mathbb{R}_+$ be any metric on M. Define

$$d'(x, y) = d(x, y) + |f(x) - f(y)|$$

where f is any proper function on M. Easily d' is a metric and for the corresponing balls around x, $B'_\varepsilon(x) \subseteq B_\varepsilon(x)$. On the other hand $B'_\varepsilon(x) = \{y \mid d'(x, y) < \varepsilon\}$ is open (with respect to the usual topology) and so the topologies induced by d and d' coincide. It remains to show that the metric d' is complete. Hence let (x_i) be a sequence which is Cauchy with respect to d'. Then $f(x_i)$ is necessarily bounded, say by L, and therefore $\{x_i\} \subseteq f^{-1}[0, L]$, a compact set. Consequently (x_i) has a limit point and converges. □

REMARK. This metric satisfies an additional property: the closed balls in this metric are all compact. This is because $B_L(x) \subseteq f^{-1}[0, f(x) + L]$.

The last piece of the puzzle is the following proposition

PROPOSITION 9.6. *The image of*

$$j^r : C^r(M, N) \to \Gamma(J^r(M, N))$$

is uniformly closed. In particular it is a Baire space.

PROOF. Let $(j^r f_i)$ be a sequence uniformly converging to $g : M \to J^r(M, N)$. Then we claim that $g = j^r(\tau \circ g)$. This is a local problem where this means that when all the derivatives $\mathrm{d}^k f_i$, $k = 0, \ldots, r$, converge uniformly to g^k then $g^k = \mathrm{d}^k g^0$, a well-known fact. □

Observe that the proof also works for $r = \infty$. We will state this as a theorem

THEOREM 9.7. *The space $C^\infty(M, N)$ is Baire.*

PROOF. As noted before the statement it only remains to show that $J^\infty(M, N)$ admits a complete metric. By definition as a limit there is an inclusion

$$J^\infty(M, N) \subseteq \prod_{r=0}^{\infty} J^r(M, N)$$

as a susbet of compatible jets. This subset is clearly closed and thus the proof is finished by refering to the standard fact that a countable product of complete metric spaces has a complete metric. If the complete metrics on the factors are d_i with $d_i \leq 2^{-i}$ (which can be achieved by replacing d_i by $\max\{2^{-i}, d_i\}$) then the metric on the product is e.g.

$$d((x_i), (y_i)) = \sum_{i=0}^{\infty} d_i(x_i, y_i)$$

□

Now we discuss certain naturality properties. From the diagram

$$\begin{tikzcd} C^\infty(M, N) \arrow[r, hook] \arrow[d, dashed, "f_*"'] & \Gamma(J^\infty(M, N)) \arrow[d, "f_*"] \\ C^\infty(M, P) \arrow[r, hook] & \Gamma(J^\infty(M, P)) \end{tikzcd}$$

the map f_* on the left is continuous. It is not the case that g^* is continuous (unless g is proper).

THEOREM 9.8. *The canonical map*

$$j^r : C^\infty(M, N) \to C^\infty(M, J^r(M, N))$$

is an embedding of a subspace.

PROOF. There is a retraction $\tau_* : C^\infty(M, J^r(M, N)) \to C^\infty(M, N)$, induced by the target map. What we are left with is the continuity of

$$C^\infty(M, N) \xrightarrow{j^r} C^\infty(M, J^r(M, N)) \xrightarrow{j^s} \Gamma(J^s(M, J^r(M, N)))$$

This composition can be also written as

$$C^\infty(M, N) \xrightarrow{j^{r+s}} \Gamma(J^{r+s}(M, N)) \xrightarrow{\alpha_*} \Gamma(J^s(M, J^r(M, N)))$$

where α_* is induced by the canonical map $\alpha : J^{r+s}(M, N) \to J^s(M, J^r(M, N))$ sending $j^{r+s}_x f$ to $j^s_x(j^r f)$ once one checks this to be independent of the representing map f (within the maps with the same $(r + s)$-jet at x) and also smooth. Both facts are easily verified using polynomials. $\square$

We can now make certain statements involving “close” maps and “small” homotopies precise. Let $f : M \to N$ be a smooth map and $A \subseteq N$ a submanifold. We say that f is transverse to A on $K \subseteq A$ if f is transverse to A at each point $x \in f^{-1}(K)$.

PROPOSITION 9.9. *Let M, N be smooth manifolds and $A \subseteq N$ a submanifold. Let $K \subseteq A$ be a compact subset (or more generally K closed in N). Then the set*

$$\mathfrak{X} = \{f : M \to N \mid f \pitchfork A \text{ on } K\}$$

is open in $C^\infty(M, N)$.

PROOF. Clearly transversality is a condition on a 1-jet of f (i.e. condition on the values and the first derivatives). Therefore we consider $J^1(M, N)$ and its subset of jets fulfilling our condition

$$W = \{j^1_x f \mid f(x) \notin K \text{ or } f_*(T_x M) + T_{f(x)}A = T_{f(x)}N\}$$

Then the set $\mathfrak{X}$ from the statement can be also described as $\mathfrak{X} = (j^1)^{-1}(O(W))$ with

$$j^1 : C^\infty(M, N) \to \Gamma(J^1(M, N))$$

It suffices to show that W is open as then $\mathfrak{X}$ will be one of the generating open sets in $C^\infty(M, N)$. As openness is a local property we can work in one of the charts on $J^1(M, N)$, i.e. reduce to the case

$$W \subseteq \mathbb{R}^m \times \mathbb{R}^n \times \hom(\mathbb{R}^m, \mathbb{R}^n)$$

with A corresponding to a linear subspace $\mathbb{R}^k \subseteq \mathbb{R}^n$ and $K \subseteq \mathbb{R}^k$ a closed subset. Then W is defined by the requirement: either the component in $\mathbb{R}^n$ lies in the complement of K or the component in $\hom(\mathbb{R}^m, \mathbb{R}^n)$ when composed with the projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^k$ is surjective, a union of two open subsets. $\square$

Corollary 9.10. *Let M, N be smooth manifolds and $A \subseteq J^r(M,N)$ a submanifold. Let $K \subseteq A$ be a compact subset (or more generally K closed in $J^r(M,N)$). Then the set*

$$\mathfrak{X} = \{f : M \to N \mid j^r f \pitchfork A \text{ on } K\}$$

is open in $C^\infty(M,N)$.

Proof. One reduces to the previous proposition via the canonical *continuous* map $j^r : C^\infty(M,N) \to C^\infty(M, J^r(M,N))$. □

The situation of the previous corollary is called a jet transversality. A lot of situations we were studying can be described by jet transversality.

Example 9.11. We saw that a smooth function $f : M \to \mathbb{R}$ is Morse iff $\mathrm{d}f : M \to T^*M$ is transverse to the zero section. It is not hard to identify $J^1(M,\mathbb{R})$ with $\mathbb{R} \times T^*M$ and under this identification $j^1 f : M \to J^1(M,\mathbb{R})$ becomes $(f, \mathrm{d}f)$. Thus a function f is Morse iff $j^1 f \pitchfork \mathbb{R} \times M$ with $\mathbb{R} \times M \subseteq J^1(M,\mathbb{R})$ as the subset of all jets (at various points) of all constant functions. In particular Morse functions form an open subset of $C^\infty(M,\mathbb{R})$.

Example 9.12. A map $f : M \to N$ between two smooth manifolds is an immersion iff $j^1 f$ does not meet the subset of $J^1(M,N)$ of the non-immersive jets. Locally

$$J^1(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^m \times \mathbb{R}^n \times \hom(\mathbb{R}^m, \mathbb{R}^n$$

and f is an immersion iff $j^1 f$ misses all $\mathbb{R}^m \times \mathbb{R}^n \times \hom_r(\mathbb{R}^m, \mathbb{R}^)$ for the rank $r < m$. We computed earlier that each $\hom_r(\mathbb{R}^m, \mathbb{R}^n)$ is a submanifold of $\hom(\mathbb{R}^m, \mathbb{R}^n)$ with the lowest codimension $n - m + 1$ for $r = m - 1$. Therefore when $n \geq 2m$ this condition is equivalent to jet transversality (in this case with respect to a finite number of submanifolds).

The main theorem on jet transversality is the following.

Theorem 9.13 ((Thom's) Jet Transversality Theorem). *Let M, N be smooth manifolds, $\partial N = \emptyset$, and $A \subseteq J^r(M,N)$ a submanifold. Then the set*

$$\mathfrak{X} = \{f : M \to N \mid j^r f \pitchfork A\}$$

is residual in $C^\infty(M,N)$. It is moreover open provided that A is closed (as a subset).

Proof. We cover A by a countable number of compact subsets A_i with the following properties.

- $\sigma(A_i)$ lies in a coordinate chart $\mathbb{R}^m \cong U_i$ on M.
- $\tau(A_i)$ lies in a coordinate chart $\mathbb{R}^n \cong V_i$ on N.

We will show that each of the sets

$$\mathfrak{X}_i = \{f : M \to N \mid j^r f \pitchfork A \text{ on } A_i\}$$

is open dense. The openness part was proved in Corollary 9.10 so we are left with the density.

We identify U_i with $\mathbb{R}^m$ and V_i with $\mathbb{R}^n$ using the charts (and thus reduce to a local problem). Let $f_0 : M \to N$ be a smooth map, we will find an arbitrarily close map which is transverse to A on A_i. First we denote $W_i = U_i \cap f_0^{-1}(V_i) \subseteq \mathbb{R}^m$ the open subset where

f_0 can be written in the charts U_i and V_i. Let $\lambda : W_i \to \mathbb{R}_+$ be a function with compact support and which equals 1 on a neighbourhood Z_i of $\sigma(A_i) \cap f_0^{-1}(\tau(A_i))$. Consider the following family of smooth maps $W_i \to \mathbb{R}^n$

$$\begin{aligned} F : J_0^r(\mathbb{R}^m, \mathbb{R}^n) \times W_i &\longrightarrow \mathbb{R}^n \\ (j_0^r p, x) &\longmapsto f_0(x) + \lambda(x) \cdot p(x) \end{aligned}$$

Here p is the unique polynomial representative of the jet in question. Since λ is compactly supported we can extend this family to a globally defined one, still denoted by F

$$F : J_0^r(\mathbb{R}^m, \mathbb{R}^n) \times M \longrightarrow N$$

For each $\alpha \in J_0^r(\mathbb{R}^m, \mathbb{R}^n)$ take the r-jet of F_α to obtain a map

$$G : J_0^r(\mathbb{R}^m, \mathbb{R}^n) \times M \longrightarrow J^r(M, N)$$

Our goal is to show that $G \pitchfork A$ on a neighbourhood of A_i so that we can apply the parametric transversality theorem to conclude that there is a sequence (α_k) in $J_0^r(M, N)$ converging to 0 so that each $G_{\alpha_k} \pitchfork A$ on A_i. This will finish the proof of density of $\mathfrak{X}_i$ as $\lim_{k\to\infty} F_{\alpha_k} = f_0$ in[4] $C^\infty(M, N)$. To achieve this observe that on Z_i we can express G in the charts and, as $\lambda \equiv 1$ there,

$$G(j_0^r p, x) = j_x^r(f_0 + p)$$

Thus G is in fact submersive on $J_0^r(\mathbb{R}^m, \mathbb{R}^n) \times Z_i$. Now $f_0(\sigma(A_i) - Z_i)$ is a compact subset disjoint with $\tau(A_i)$. Therefore even for $\alpha \in J_0^r(\mathbb{R}^m, \mathbb{R}^n)$ small the same will be true[5] for F_α. As for such α, $G_\alpha^{-1}(A_i)$ is a subset of $\sigma(A_i)$ disjoint with Z_i and the transversality holds on Z_i the restricted map G is transverse to A on A_i. If we start instead of A_i with its compact neighbourhood $B_i \subseteq A$ (still with $\sigma(B_i) \subseteq U_i$ and $\tau(B_i) \subseteq V_i$) we get that the (now even more) restricted map G is even transverse to A on B_i and in particular on the interior of B_i - a manifold. Therefore we can invoke the parametric transversality theorem to conclude the proof. □

Therefore Morse functions form a residual subset of $C^\infty(M, \mathbb{R})$ and, for $n \geq 2m$, so do the immersions in $C^\infty(M, N)$.

Next we discuss a relative version of the jet transversality theorem. Let $F \subseteq M$ be a closed subset and $s_0 : F \to J^r(M, N)$ a section of the jet bundle over F. Then clearly

$$\{s \in \Gamma(J^r(M, N)) \mid s|_F = s_0\}$$

is a uniformly closed subset (observe that this works even for $r = \infty$). Therefore the corresponding subspace

$$C^\infty(M, N)_{s_0} = \{f \in C^\infty(M, N) \mid j^r f|_F = s_0\}$$

is a Baire space. We then obtain the following generalization of Theorem 9.13:

[4]This is because the association $\alpha \mapsto F_\alpha$ is continuous. This on the other hand follows from the fact that the family F is constant outside of a compact set.

[5]As $\alpha \mapsto F_\alpha$ is continuous this follows from the openness of the corresponding subset $O\,(M \times (N - \tau(A_i)) \cup (M - (\sigma(A_i) - Z_i)) \times N)$ in $C^0(M, N) = \Gamma(M \times N \to M)$.

THEOREM 9.14. *Let M, N be smooth manifolds, $\partial N = \emptyset$, $A \subseteq J^r(M,N)$ a submanifold and $s_0 : B \to J^r(M,N)$ a smooth section over some fixed closed (as a subset) submanifold $B \subseteq M$ for which $s_0 \pitchfork A$. Then the set*

$$\mathfrak{X} = \{f \in C^\infty(M,N)_{s_0} \mid j^r f \pitchfork A\}$$

is residual in $C^\infty(M,N)_{s_0}$. It is moreover open provided that A is closed (as a subset).

PROOF. First note that when $\overline{\sigma(A)} \subseteq M - B$ this is proved by the same argument as Theorem 9.13. Then write $M - B$ as a countable union of compact subsets K_i and denote $A_i = A \cap \sigma^{-1}(K_i)$ and correspondingly

$$\mathfrak{X}_i = \{f \in C^\infty(M,N)_{s_0} \mid j^r f \pitchfork A \text{ on } A_i\}$$

Clearly $\mathfrak{X} = \bigcap \mathfrak{X}_i$ while each of the terms is residual (in fact open dense). □

COROLLARY 9.15. *Let M be a smooth manifold and $B \subseteq M$ a closed (as a subset) submanifold. Then every immersion $f : B \to \mathbb{R}^n$ extends to an immersion $M \to \mathbb{R}^n$ provided $n \geq 2m$.*

PROOF. Suppose first that f extends to a smooth map $\tilde{f} : M \to \mathbb{R}^n$ which is an immersion near B. Then the last theorem applies to $s_0 = (j^1\tilde{f})|_B$ to conclude that the set $\mathfrak{X}$ of immersions $M \to \mathbb{R}^n$ is residual in a nonempty space $C^\infty(M,\mathbb{R}^n)_{s_0}$ (it contains $\tilde{f}$). In particular $\mathfrak{X}$ itself is nonempty.

Thus it remains to find an extension $\tilde{f}$. We assume here that B has codimension 0, the only case we will need later[6]. Then f extends locally at each point of ∂B and hence to a neigbhbourhood of ∂B in M by the means of the partition of unity. The required extension to M, automatically an immersion near B, is guaranteed by the contractibility of $\mathbb{R}^n$. □

To describe injective immersions (and later embeddings) in a way similar to immersions we need a notion of a multijet. A multijet is a number of jets with different sources. More formally denote by $M^{(k)}$ the subset of the cartesian power M^k of pairwise distinct k-tuples. We define

$${J_k}^r(M,N) = (J^r(M,N))^k|_{M^{(k)}}$$

An element of ${J_k}^r(M,N)$ is called a *multijet*. Again we have a multijet prolongation map[7]

$${j_k}^r : C^\infty(M,N) \to C^\infty(M^{(k)}, {J_k}^r(M,N))$$

THEOREM 9.16 (Multijet Transversality Theorem). *Let M, N be smooth manifolds, $\partial N = \emptyset$, and $A \subseteq {J_k}^r(M,N)$ a submanifold. Then the set*

$$\mathfrak{X} = \{f : M \to N \mid {j_k}^r f \pitchfork A\}$$

is residual in $C^\infty(M,N)$.

[6] In the general case one assumes some nice position of B in M. This means that either B lies in the interior of M or in the boundary or is a neat submanifold. In any case one extends f to a tubular neighbourhood by a vector bundle argument which is left as a HW.

[7] A serious homework: explain why ${j_k}^r$ is NOT continuous.

PROOF. The proof follows that of Theorem 9.13 with the following modifications: we require $\sigma(A_i)$ to be contained in a product $U_i^1 \times \cdots \times U_i^k$ of *disjoint* charts on M. Then in the density part we embed simlarly f_0 into a family of maps

$$F : (J_0^r(\mathbb{R}^m, \mathbb{R}^n))^k \times M \longrightarrow N$$

and proceed analogously.[8] □

EXAMPLE 9.17 (injective maps). Let $A \subseteq J_2{}^0(M,N) \cong M^{(2)} \times N^2$ consists exactly of those (x_1, x_2, y_1, y_2) with $y_1 = y_2$. Then f is injective iff $j_2{}^0 f$ misses A. The codimension of A is precisely n and thus for $n > 2m$, f is injective iff $j_2{}^0 f \pitchfork A$. In particular the set of injective maps is residual in these dimensions. Combining with the result for immersions, injective immersions are residual for $n > 2m$. To solve the problem for embeddings note that $f : M \to N$ is an embedding as a *closed* submanifold of N iff f is a *proper* injective immersion. We will prove that proper maps form an open subset. Then the existence of embeddings of any manifold M^m into $\mathbb{R}^{2m+1}$ follows from the previous.

Let N be equipped with a complete metric with compact balls (like the one constructed in Theorem 9.5) and $g : M \to N$ be a proper map. Then

$$\{f : M \to N \mid d(f(x), g(x)) < 1\}$$

is clearly an open neighbourhood of g and for any f in this set

$$f^{-1}(B_L(y)) \subseteq g^{-1}(B_{L+1}(y))$$

a compact subset.

EXAMPLE 9.18 (special Morse functions). Let $A \subseteq J_2{}^1(M, \mathbb{R})$ consist precisely of the pairs $(j^1_{x_1} f, j^1_{x_2} f)$ with both x_1 and x_2 critical points of f and with $f(x_1) = f(x_2)$. The codimension of A is $2m+1$ and therefore a function f satisfies $j_2{}^1 f \pitchfork A$ iff all the critical points are in different levels and the set of such is residual. Together with the residuality of the set of Morse functions, the same holds for these special Morse functions.

In fact special Morse functions are also stable: any function close to a special Morse function f in $C^\infty(M, \mathbb{R})$ is equivalent to f (i.e. there are diffeomorphisms of M and $\mathbb{R}$ relating the two functions).

Injective immersions are also stable. To get a stable version of immersions one needs to restrict only to immersions with normal crossings *picture*.

The relative version of the multijet transversality theorem is hard even to state so we content ourselves with the case of injective immersions. Let M be a compact manifold with boundary and $s_0 : \partial M \to N$ an injective smooth map.

THEOREM 9.19. *Assuming that* $n > 2m$ *the set*

$$\mathfrak{X} = \{f \in C^\infty(M, N)_{s_0} \mid f \textit{ is injective}\}$$

is residual in $C^\infty(M, N)_{s_0}$.

[8] A proof of openness should go as follows: the map

$$(j^r)^k : C^\infty(M, N) \to C^\infty(M^k, (J^r(M, N))^k)$$

is continuous and $\mathfrak{X}_i$ is a preimage of an open subset.

PROOF. Filling $M - \partial M$ by a countable number of compact subsets K_i we again form

$$\mathfrak{X}_i = \{f \in C^\infty(M, N)_{s_0} \mid f \text{ is injective on } \partial M \cup K_i\}$$

Obvously $\mathfrak{X} = \bigcap \mathfrak{X}_i$. By an easy modification of the proof of Theorem 9.16 we now prove that each $\mathfrak{X}_i$ is residual (or even open dense). We make sure that none of the charts used on M meets both ∂M and K_i. We then have three possibilities on the pair U_i^1, U_i^2: if both miss ∂M then we use the same argument, if both hit ∂M then f_0 does not need to be perturbed and in the mixed case we create a family only indexed by a single copy of $J^r(\mathbb{R}^m, \mathbb{R}^n)$ which is still transverse to the diagonal. $\square$

COROLLARY 9.20. *The forgetful map*

$$\left\{\begin{matrix}\textit{the cobordism classes of closed}\\ \textit{m-dimensional submanifolds of } S^{2m+2}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\textit{the cobordism classes of closed}\\ \textit{m-dimensional manifolds}\end{matrix}\right\}$$

is a bijection.

PROOF. The surjectivity was proved already in Theorem 2.12. To prove injectivity let W be a cobordism between two m-dimensional submanifolds M_0, M_1 of S^{2m+2} which we think of as an embedding $\iota : \partial W \hookrightarrow \partial(I \times S^{2m+2})$. Removing a point from S^{2m+2} not contained in $M_0 \cup M_1$ we can replace S^{2m+2} by $\mathbb{R}^{2m+2}$. We extend the embedding ι to a neat embedding, also denoted by ι, of a collar V of ∂W in an obvious way. Denoting by U a collar with $\overline{U} \subseteq V$ (e.g. the one corresponding to $[0, 1) \subseteq \mathbb{R}_+$) we are then asked to extend an embedding

$$\overline{U} - \partial W \hookrightarrow (0, 1) \times \mathbb{R}^{2m+2}$$

to $W - \partial W$. The extension is provided by Corollary 9.15 and Theorem 9.19. $\square$

COROLLARY 9.21. *There is a natural bijection*

$$\pi_{m+d}(\mathrm{Th}(\gamma_{k,d})) \xrightarrow{\cong} \left\{\begin{matrix}\textit{the cobordism classes of closed}\\ \textit{m-dimensional manifolds}\end{matrix}\right\}$$

provided that $d > m + 1$ and $k > m + d$.