E2011: Theoretical fundamentals of computer science: Basic notions of graph theory

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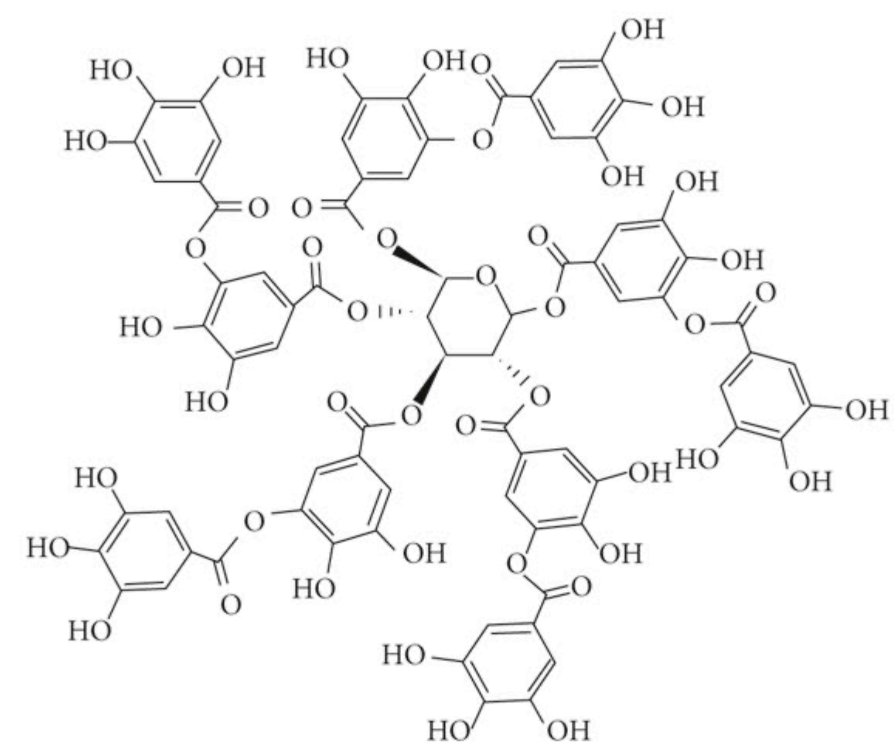
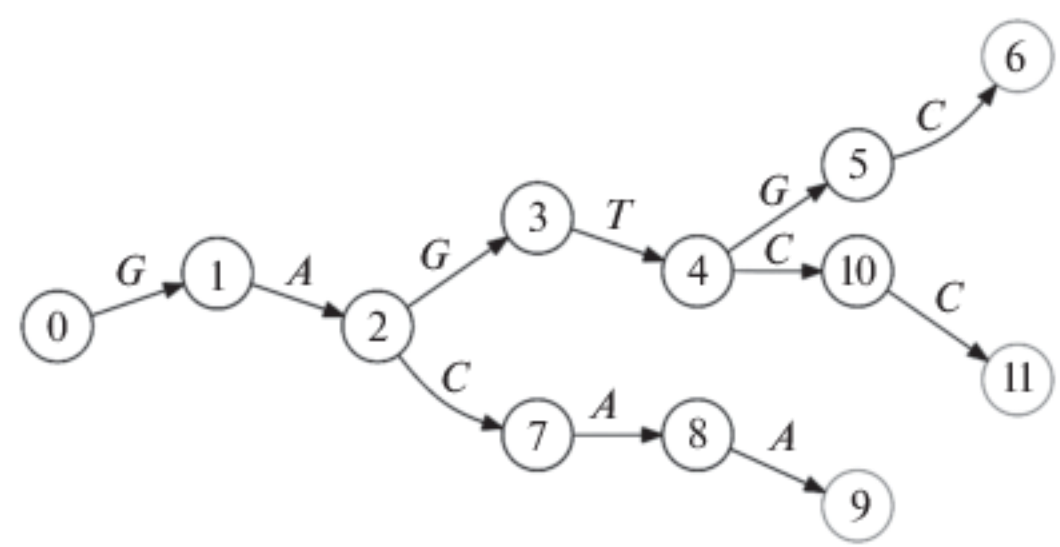
Applications

- Molecular models
- Computer networks
- Planning and scheduling
- Solve shortest path problems between cities
- Electrical circuits
- …and many, many, others…

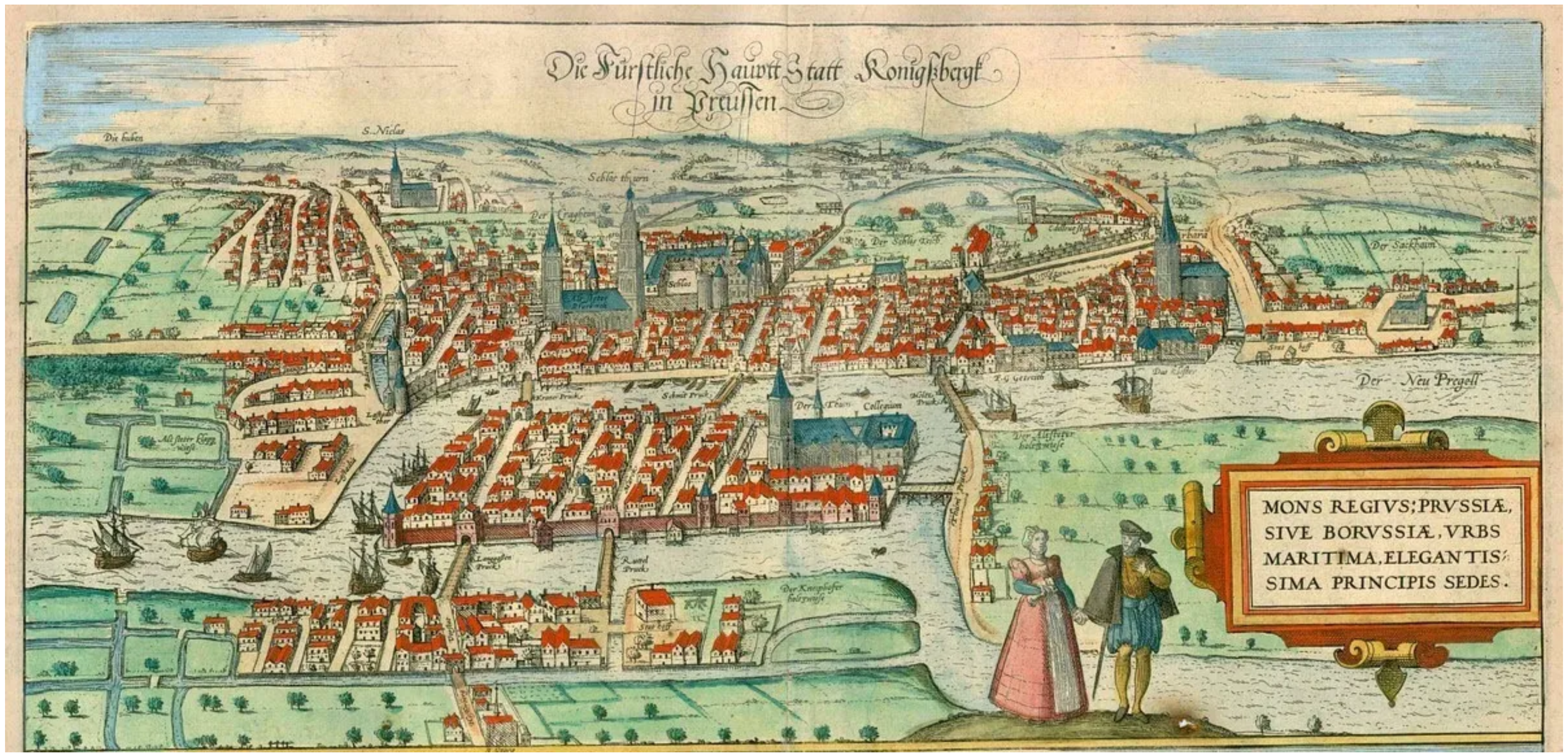


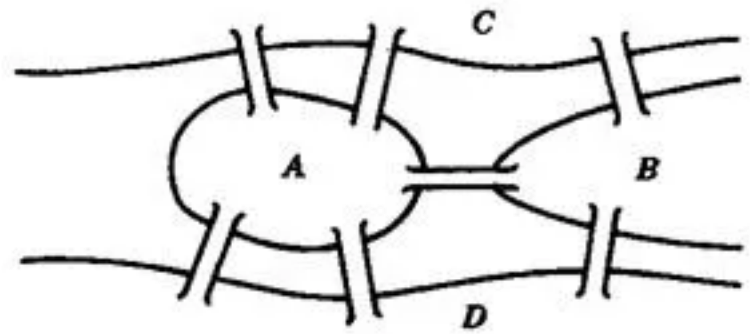


Some more examples

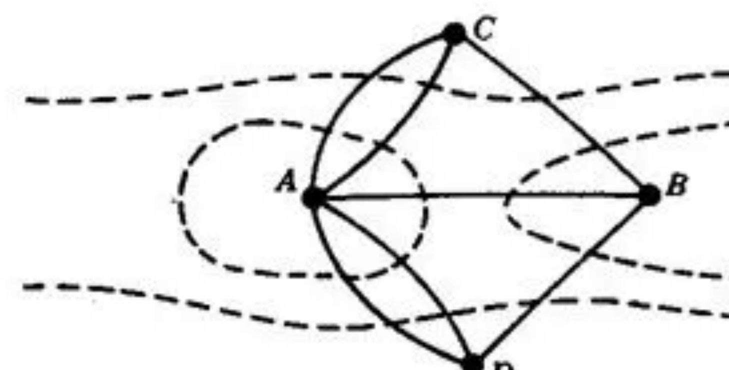
- Navigation/GPS: find the shortest path (using algorithm’s like Dijkstra)
- Games: e.g. chess - choices can be arrange in a tree-structure and best movement (within a given horizon) can be selected
- Computation distribution across machines of a cluster
- Neural networks

Euler and the 7 bridges of Königsberg





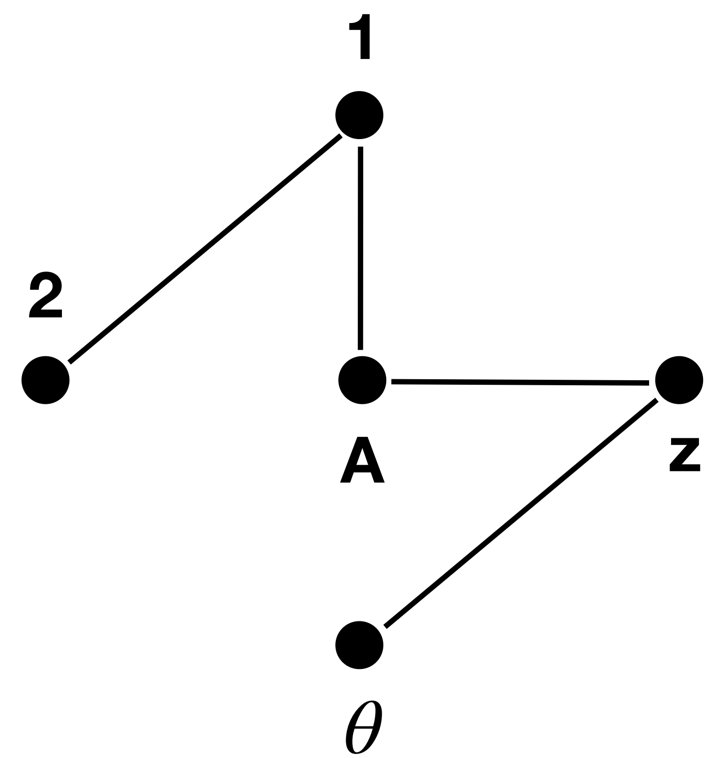
(a) Königsberg in 1736



(b) Euler's graphical representation

Definitions - graphs

- A **graph** $G = (V, E)$ is an ordered pairs of a set of vertices V and a set of edges E.
- If needed, the notation could be $V(G)$ to specify the set of vertices of graph G, and $E(G)$ for the set of edges of the same graph.

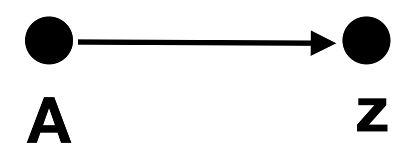


$$V = \{1, 2, A, z, \theta\}$$

$$E = \{\{1,2\}, \{1,A\}, \{A,z\}, \{\theta, z\}\}$$

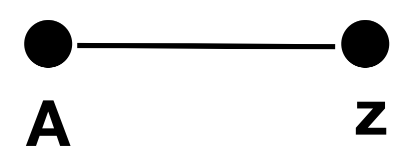
Definitions - edge types

- **Directed edge:** an ordered set of vertices, denoted as a tuple (u, v)



(A, z)

- **Undirected edge:** an unordered set of vertices, represented as a set $\{u, v\}$

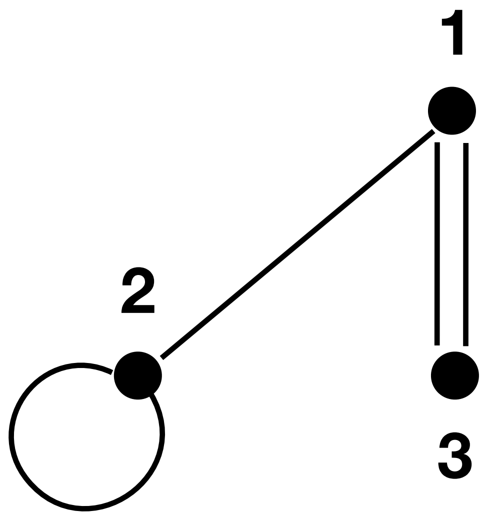


$\{A, z\}$

Definitions - edge type

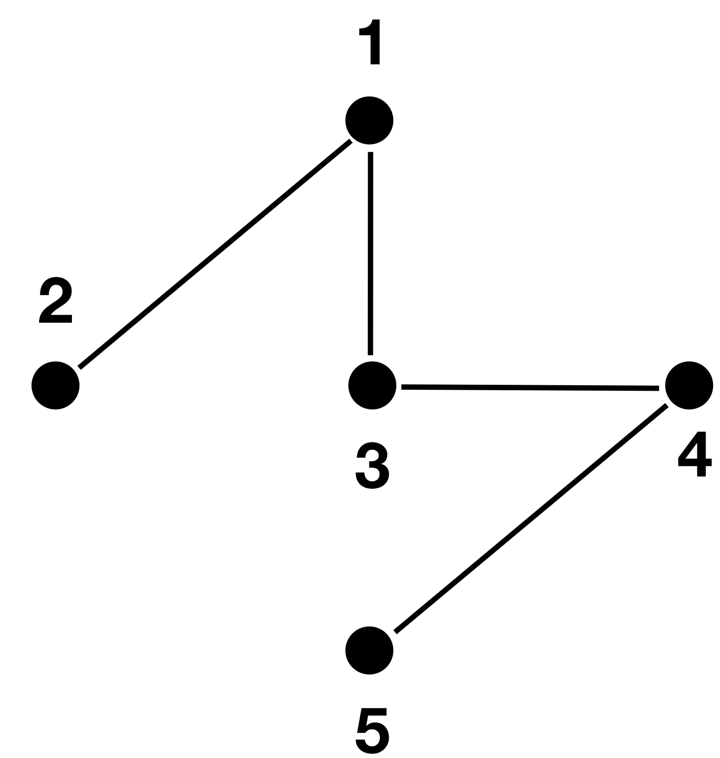
- **Loop:** an edge $\{u, v\}$(or (u, v)) with $u = v$

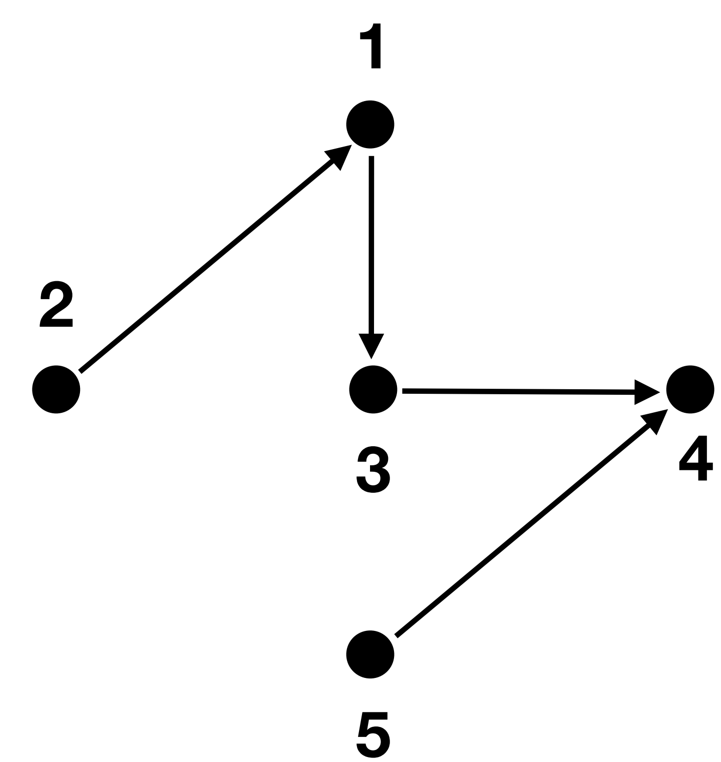
- **Multiple edges:** two or more edges connecting the same two vertices



Definitions - graph type

- **Undirected (simple) graph:** a graph $G(V, E)$, $V \neq \varnothing$, and E a set of undirected edges
- **Directed graph:** the set of edges contains oriented edges
- **Multigraph:** multiple edges are allowed
- **Pseudograph**: a multigraph with loops
- and combinations…





Terminology - undirected graphs

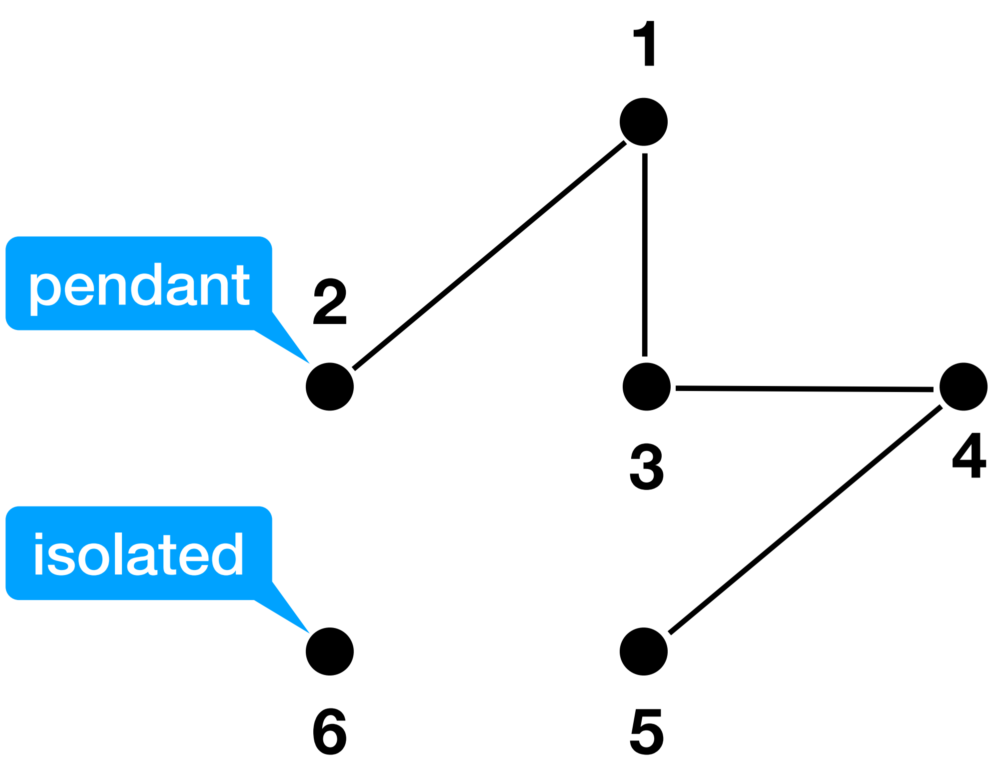
For an undirected graph $G(V, E)$:

- u and v are called adjacent if $e = \{u, v\} \in E$; e is called incident with u and v; u and v are called endpoints of e

- the degree of a vertex v ($\deg(v)$) is the number of edges incident on v

- pendant vertex: $\deg(v) = 1$

- isolated vertex: $\deg(v) = 0$

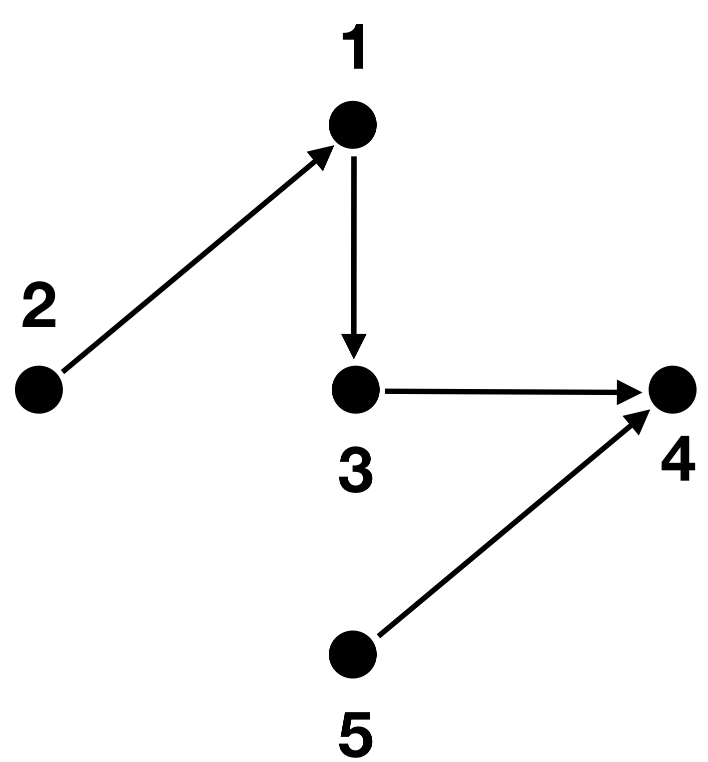


$\deg(u) = 2, \forall u \in \{1,3,4\}$

Terminology - directed graphs

For a directed graph $G(V, E)$:

- for $e = (u, v) \in E$; u is adjacent to v; v is adjacent from u; u is initial vertex and v is terminal vertex
- the in-degree of a vertex v ($\deg^-(v)$) is the number of edges incident with v terminal vertex
- the out-degree of a vertex v ($\deg^+(v)$) is the number of edges incident with v initial vertex



$\deg^+(4) = 0$
$\deg^-(4) = 2$

$\deg^+(1) = 1$
$\deg^-(1) = 1$

Some basic results

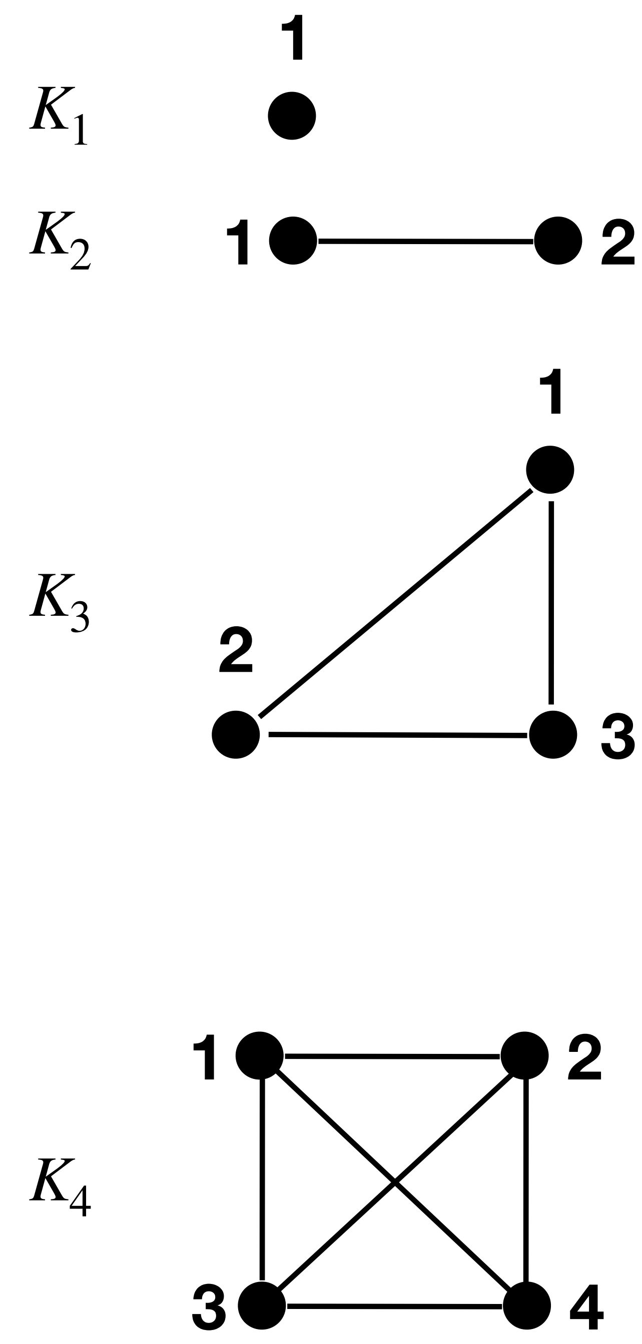
- Theorem (Euler): in an undirected graph, $\sum_{v \in V} \deg(v) = 2|E|$ (handshaking theorem)
- Theorem (Euler): an undirected graph has an even number of vertices with odd degree
- Theorem: in a directed graph, $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$

Special cases

- **Complete graph:** K_n is a simple graph where any two vertices are connected by an edge
- **Cycle:** C_n, $n \geq 3$ consists of n vertices such that $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\} \in E$

  $C_3 = K_3$

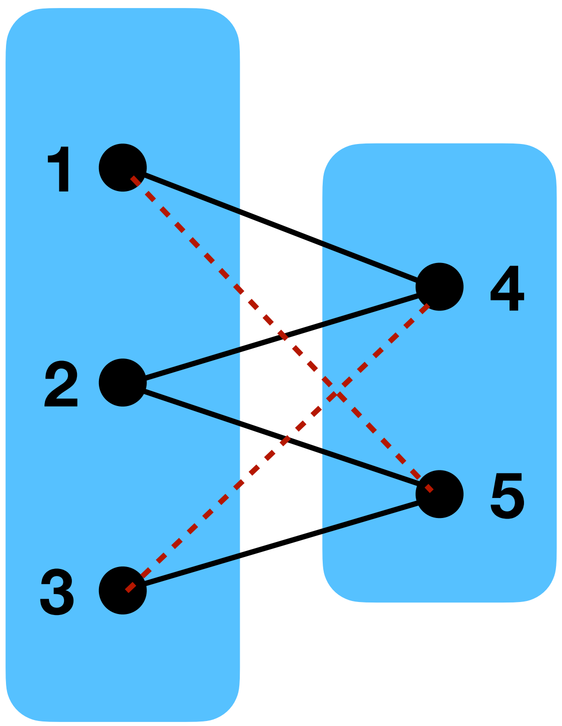
  $C_4 = \{1,2,3,4\}$



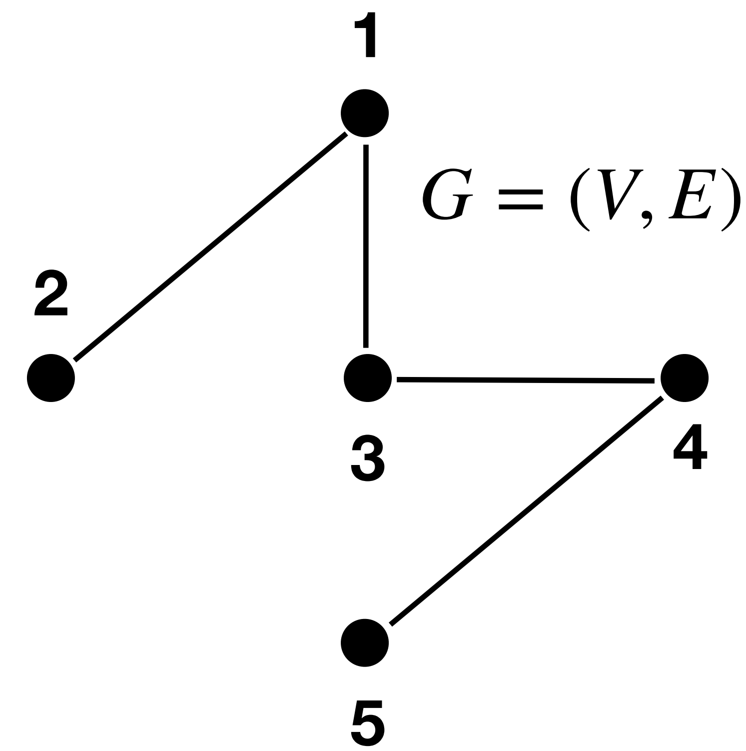
Special cases - Bipartite graph

- A simple graph G for which the vertices V set can be partitioned $V = V_1 \cup V_2,\ V_1 \cap V_2 = \emptyset$, such that all edges have one end in V_1 and the other one in V_2

- Complete bipartite graph $K_{m,n}$ is a bipartite graph $(|V_1| = m,\ |V_2| = n)$ in which all the vertices in one partition are connected to all the vertices in the second partition

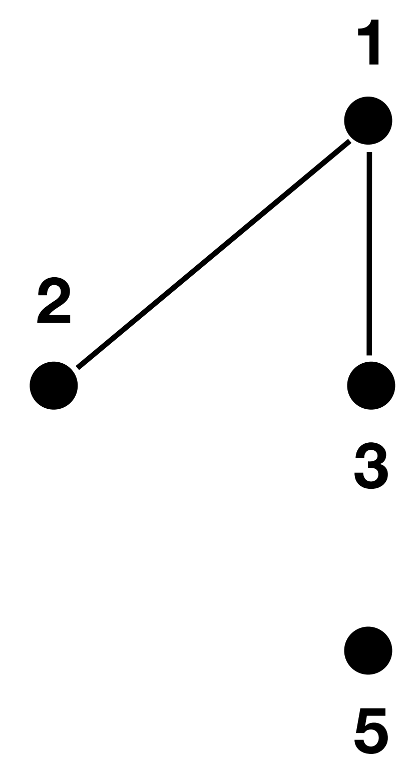


Subgraphs

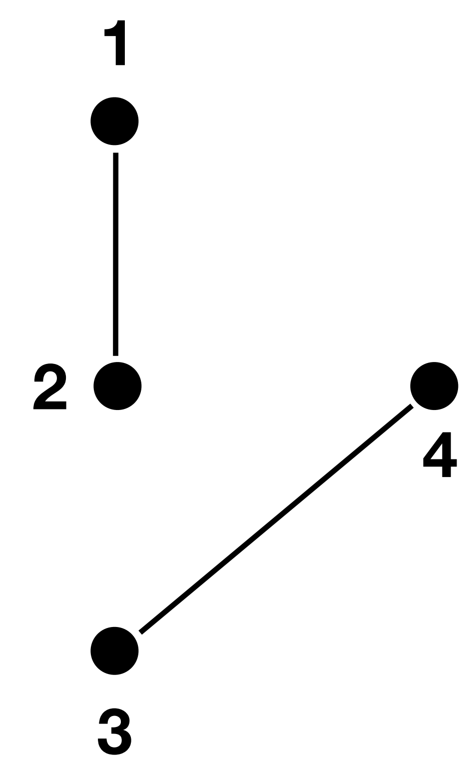


- Let $G = (V, E)$ be a graph.

- A graph $H = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$



$H_1 = (V_1', E_1')$

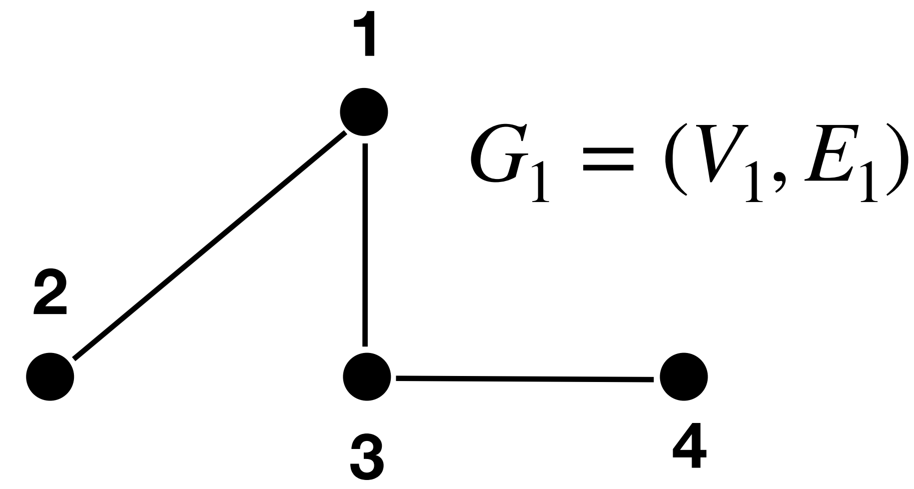


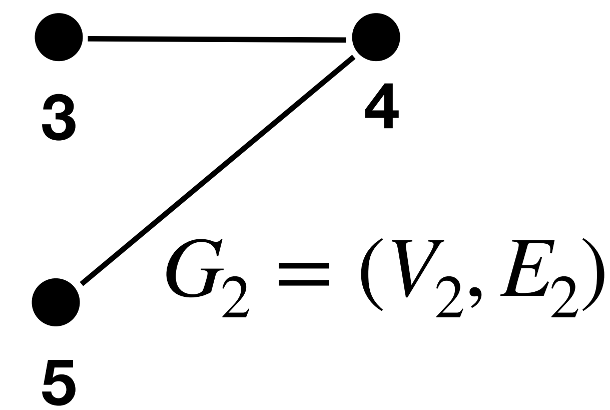
$H_2 = (V_2', E_2')$

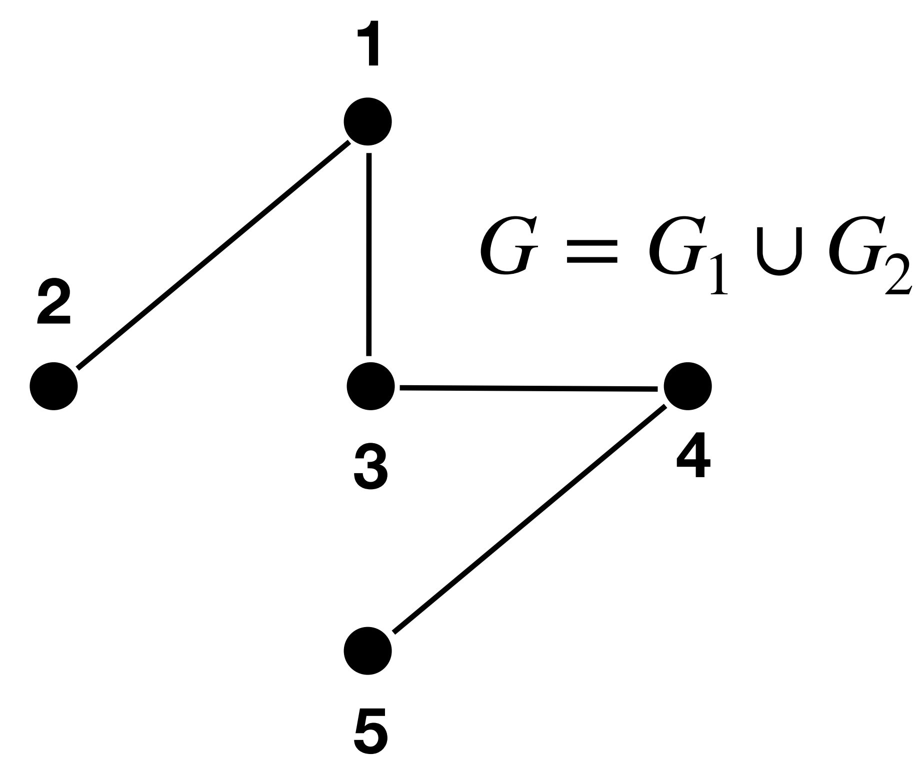
Graph union

- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs

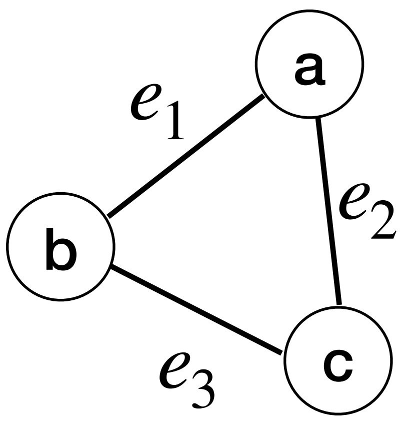
- Then their union is a graph $G = G_1 \cup G_2 = (V, E)$ such that $V_1 \cup V_2$ and $E = E_1 \cup E_2$







Graph representation



$G = (V, E); V = \{v_1, \ldots, v_n\}, E = \{e_1, \ldots, e_m\}$

- Incidence matrix ($n \times m$ matrix)

  $$M = [m_{ij}]; \quad m_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is adjacent with } v_i \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} & e_1 & e_2 & e_3 \\ a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 1 \end{bmatrix}$$

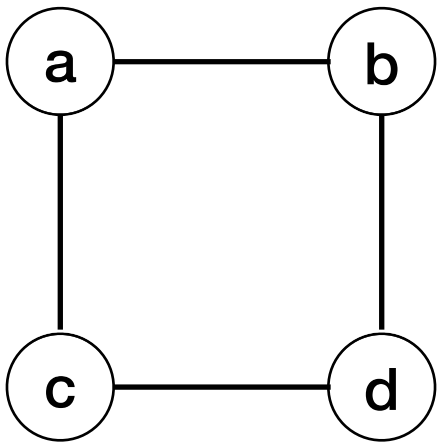
- Adjacency matrix ($n \times n$ matrix)

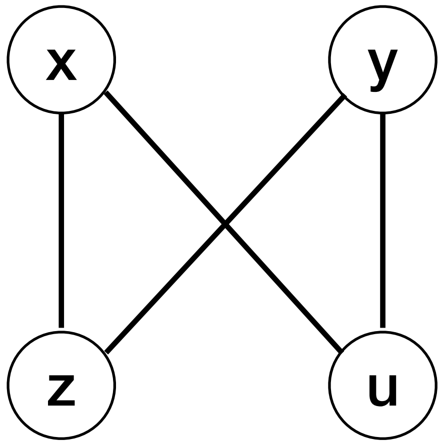
  $$A = [a_{ij}]; \; a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} & a & b & c \\ a & 0 & 1 & 1 \\ b & 1 & 0 & 1 \\ c & 1 & 1 & 0 \end{bmatrix}$$

Graph isomorphism

- Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$, $\forall u, v \in V_1$.
- The function f is called isomorphism.

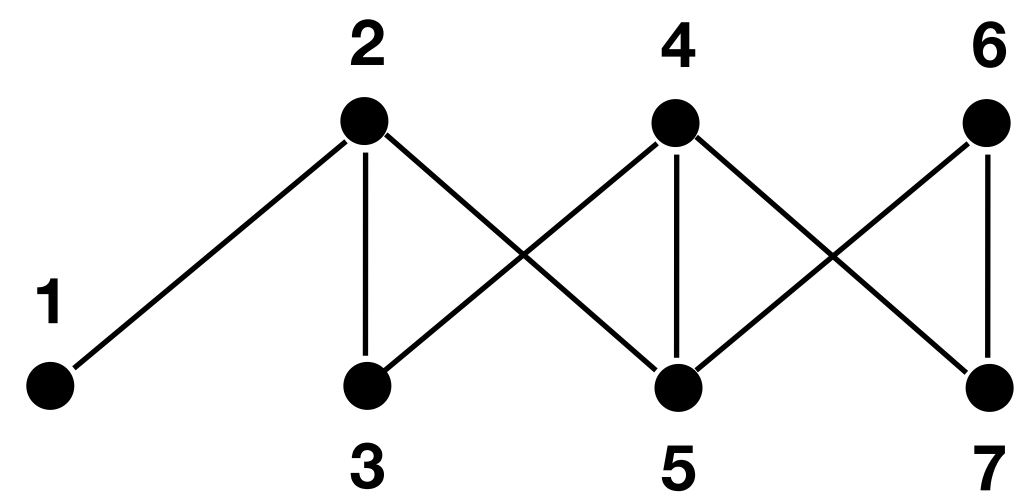




$f(a) = x; f(b) = u; f(c) = z; f(d) = y$

Connectivity

- **Path:** a sequence of edges of the form $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_i, v_j\}, \{v_j, v_k\}$
- Length of a path: number of edges
- A cycle: a path with first vertex identical to the last vertex
- A simple path: no edge is traversed more than once
- A graph is connected if there is a path between any pair of its vertices



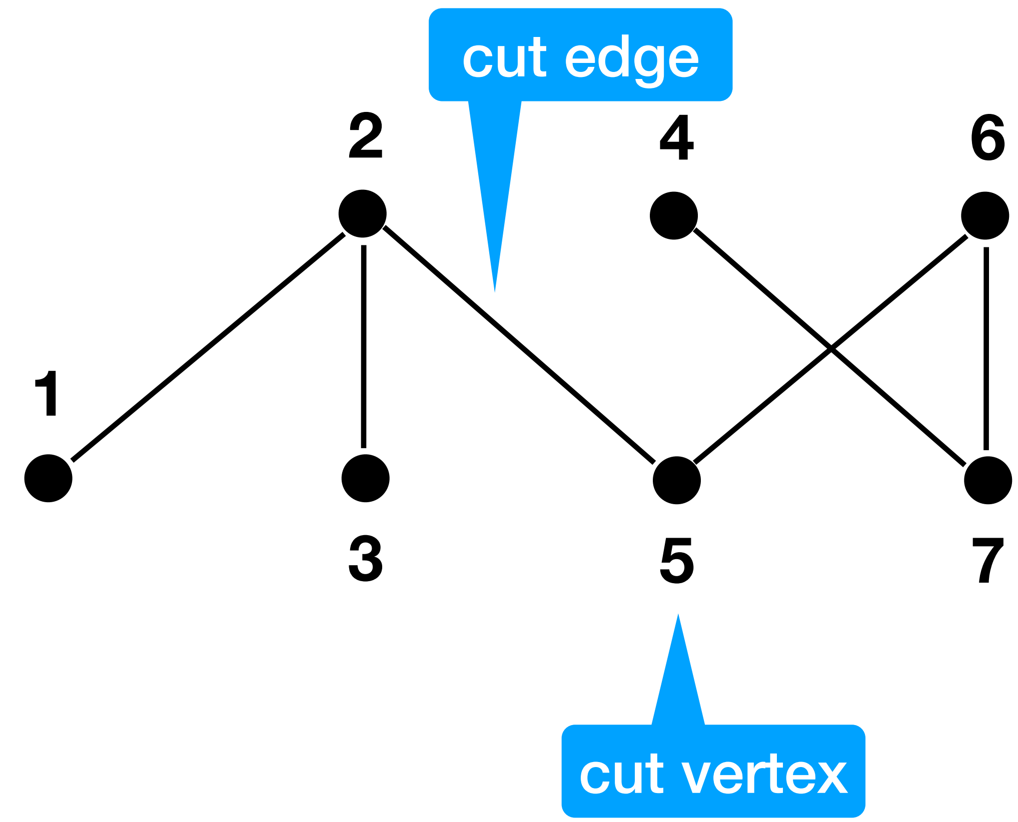
Example of a path:

$\{1,2\}, \{2,5\}, \{5,4\}, \{4,7\}$

Cuts

In an undirected graph,

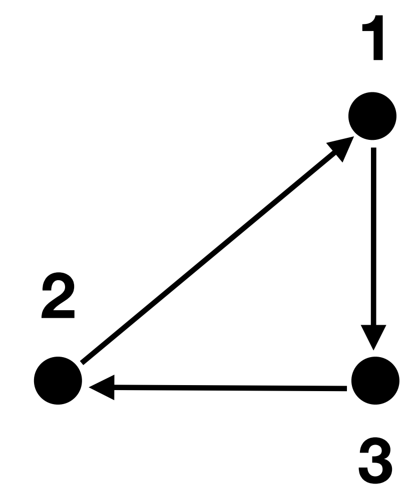
- an **articulation point** (cut vertex) is vertex whose removal would increase the number of connected components
- a **cut edge** is an edge whose removal would increase the number of connected components



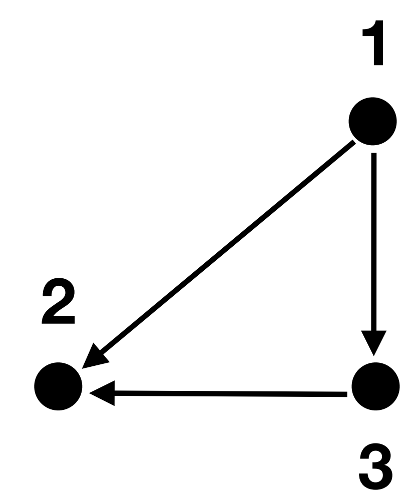
Connectivity of directed graphs

A directed graph is

- **strongly connected** if for any two vertices $u, v \in V$ there is a path from u to v and from v to u



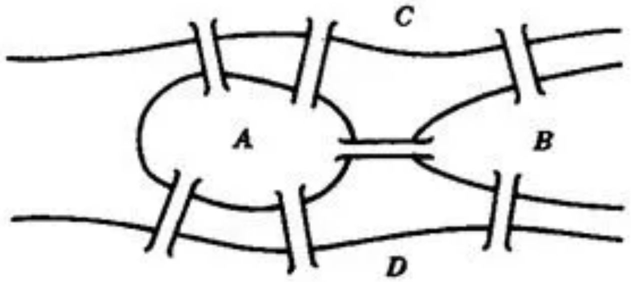
- **weakly connected** if, by disregarding the orientation of the edges the resulting (undirected) graph is connected



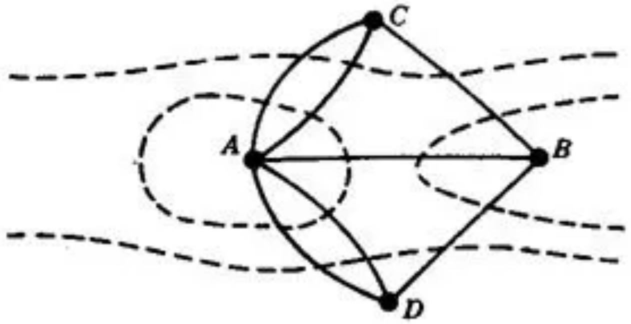
Number of paths between 2 vertices

- Theorem: Let G be a graph with adjacency matrix A (for a fixed permutation of vertices $v_1, \ldots, v_n$). Then, the number of different paths of length $r > 0$ between two vertices v_i and v_j is $[A^r]_{ij}$ (the (i, j)-th element of the matrix A^r).

Eulerian graphs



(a) Königsberg in 1736



(b) Euler's graphical representation

- An Eulerian path/cycle is a path/cycle that contains all the edges exactly once.
- A graph is called traversable if it contains an Eulerian path.
- A graph is called Eulerian if it contains an Eulerian cycle.

- Theorem 1: A connected graph G is Eulerian if and only if it has no vertices of odd degree.
- Theorem 2: A connected graph contains an Eulerian path from vertex u to vertex $v \neq u$ if and only if it is connected and u, v are the only two vertices of odd degree.

Conclusion: there is no solution to Königsberg 7 bridges problem.

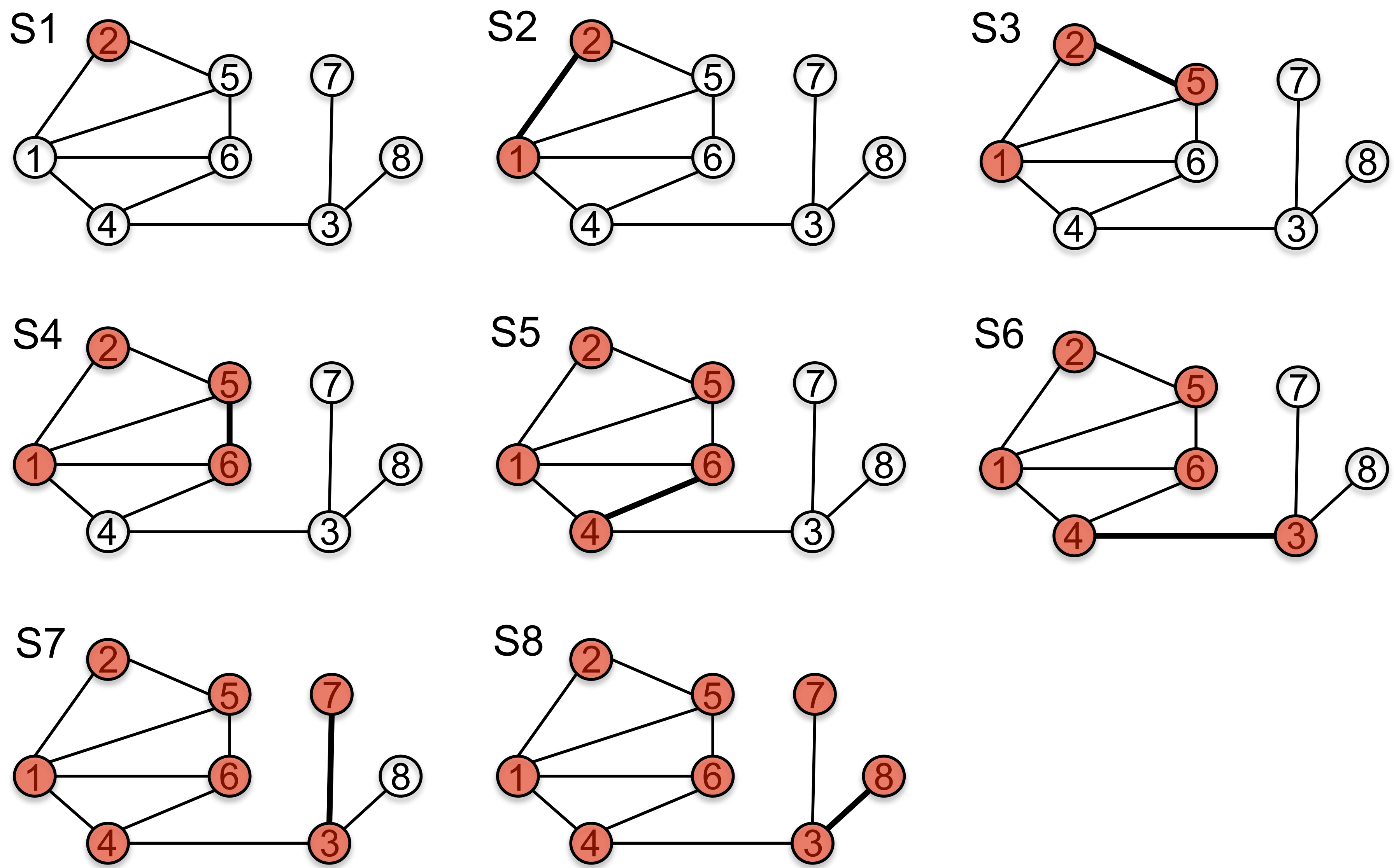
Graph connectivity test

Output : List M of marked vertices in the component
Input : Graph G (e.g., adjacency list)
Input : Starting vertex s

```
L := {s}; M := {s}; % Initialize exploration and marking lists
% Repeat while there are still nodes to explore
while L ≠ ∅ do
    choose u ∈ L; % Pick arbitrary vertex to explore
    if ∃ (u, v) ∈ E such that v ∉ M then
        choose (u, v) with v of smallest index;
        L := L ∪ {v}; M := M ∪ {v}; % Mark and augment
    else
        L := L \ {u}; % Prune
    end
end
```

Graph connectivity test - example

L	Mark
{2}	2
{2,1}	1
{2,1,5}	5
{2,1,5,6}	6
{1,5,6}	
{1,5,6,4}	4
{5,6,4}	
{5,4}	
{5,4,3}	3
{5,3}	
{5,3,7}	7
{5,3}	
{3}	
{3,8}	8
{3}	
{}	



Questions?