Avalanche Statistics

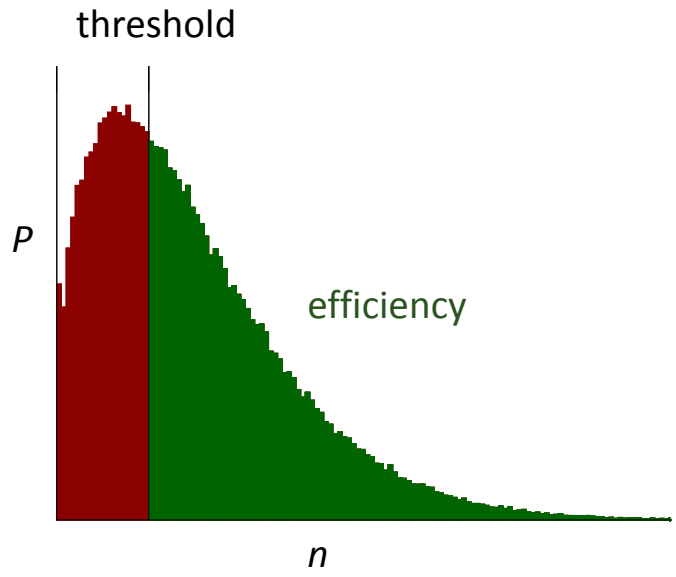
H. Schindler, R. Veenhof

Basics

- Since charge amplification is a stochastic process, the size of an electron avalanche is subject to fluctuations → probability $P(n, x)$ that an avalanche initiated by 1 electron comprises n electrons after a distance x.

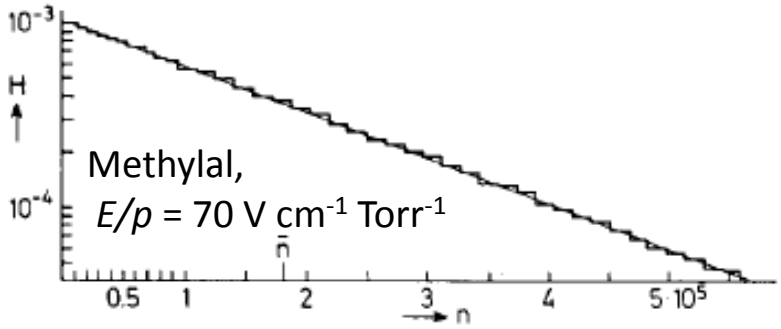
- The shape of the avalanche size distribution can be important for the detection performance
  - contribution to energy resolution
  - detection efficiency

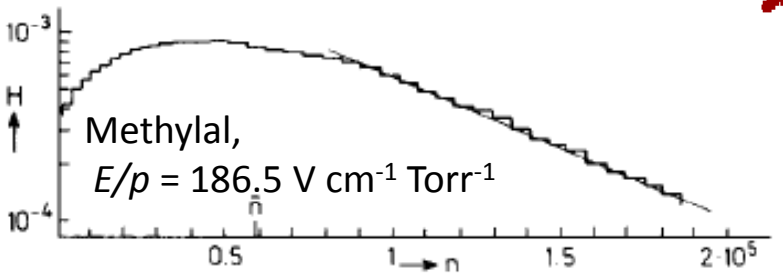
- In this talk we discuss avalanches in uniform fields initiated by single electrons.



Overview

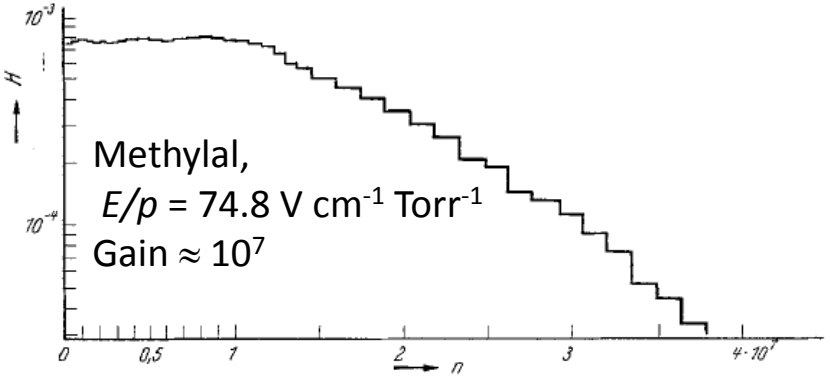
- At low fields *E/p* and moderate gain electron avalanches exhibit
  - exponential growth with distance
  - exponentially distributed fluctuations





- With increasing field the distribution departs from the exponential shape becoming more and more rounded (while the growth remains exponential). The relative width decreases, the maximum is shifted towards the mean.

- At high gain space charge leads to a departure from the exponential growth, additional deviations from the exponential shape are observed.



H. Schlumbohm, *Zur Statistik der Elektronenlawinen im Ebenen Feld. III*, Z. Phys. **151** (1958) 563-576

Low Fields: Yule - Furry Model (I)

- Key assumption: probability $\alpha\Delta x$ of ionising collision within a step Δx common to all electrons in the avalanche

- $P(n, x + \Delta x) = [1 - n\alpha(x)\,\Delta x]P(n, x) + (n-1)\alpha(x)\Delta x P(n-1, x) + O(\Delta x^2)$
  no multiplication — 1 electron ionises

- Letting $\Delta x \to 0$ yields the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}P(n,x) = \alpha(x)(n-1)P(n-1,x) - \alpha(x)nP(n,x), \quad P(n,0) = \delta_{n,1}$$

- Substituting $u = \int_0^x \alpha(s)\mathrm{d}s \;\rightarrow\; \frac{\mathrm{d}}{\mathrm{d}u}P(n,u) = (n-1)P(n-1,u) - nP(n,u)$

- Solution: geometric distribution $P(n,u) = e^{-u}\left(1-e^{-u}\right)^{n-1}$

W. Blum, W. Riegler, L. Rolandi, *Particle Detection with Drift Chambers,* Springer 2008

W. Legler, *Zur Statistik der Elektronenlawinen*, Z. Phys. **140** (1955) 221-240

Low Fields: Yule - Furry Model (II)

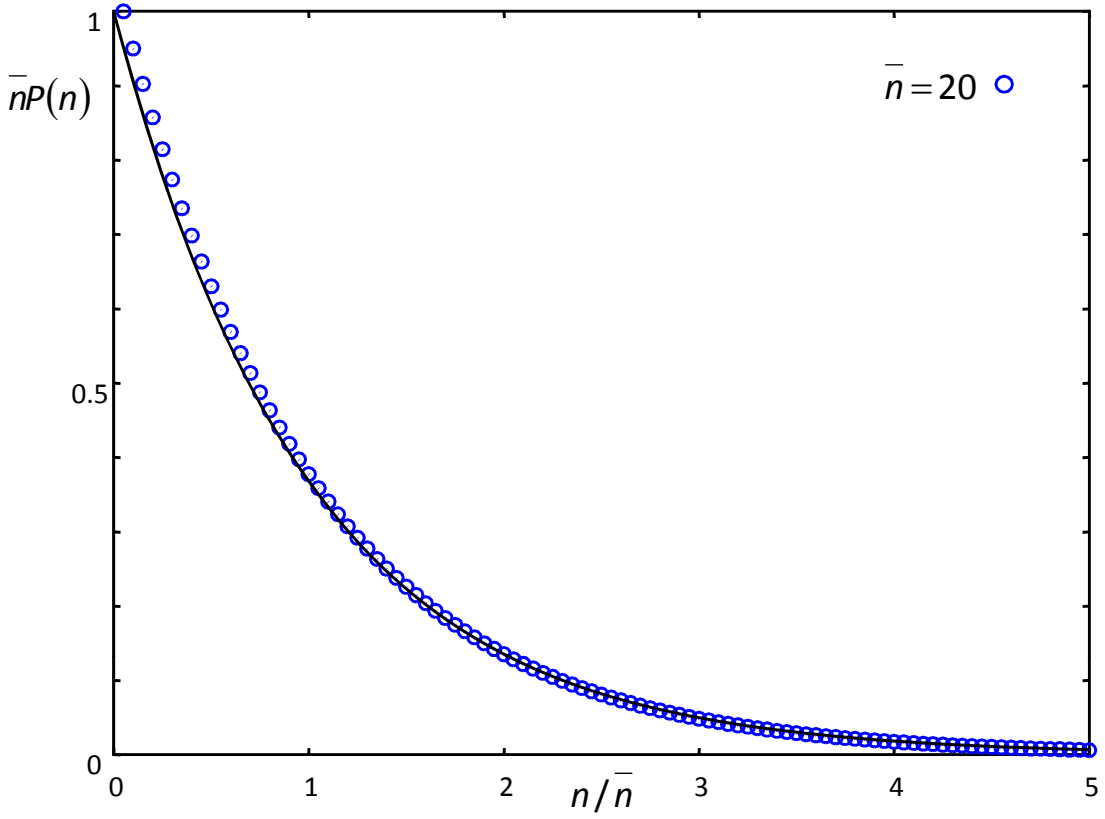
- Mean and variance are given by $\bar{n} = \exp\left(\int_0^x \alpha(s)\mathrm{d}s\right), \quad \sigma^2 = \bar{n}\left(\bar{n}-1\right)$

- Write $P(n, x)$ in terms of $\bar{n}$ → no explicit dependence on x

$$P(n,x) = \frac{1}{\bar{n}}\left(1-\frac{1}{\bar{n}}\right)^{n-1}$$

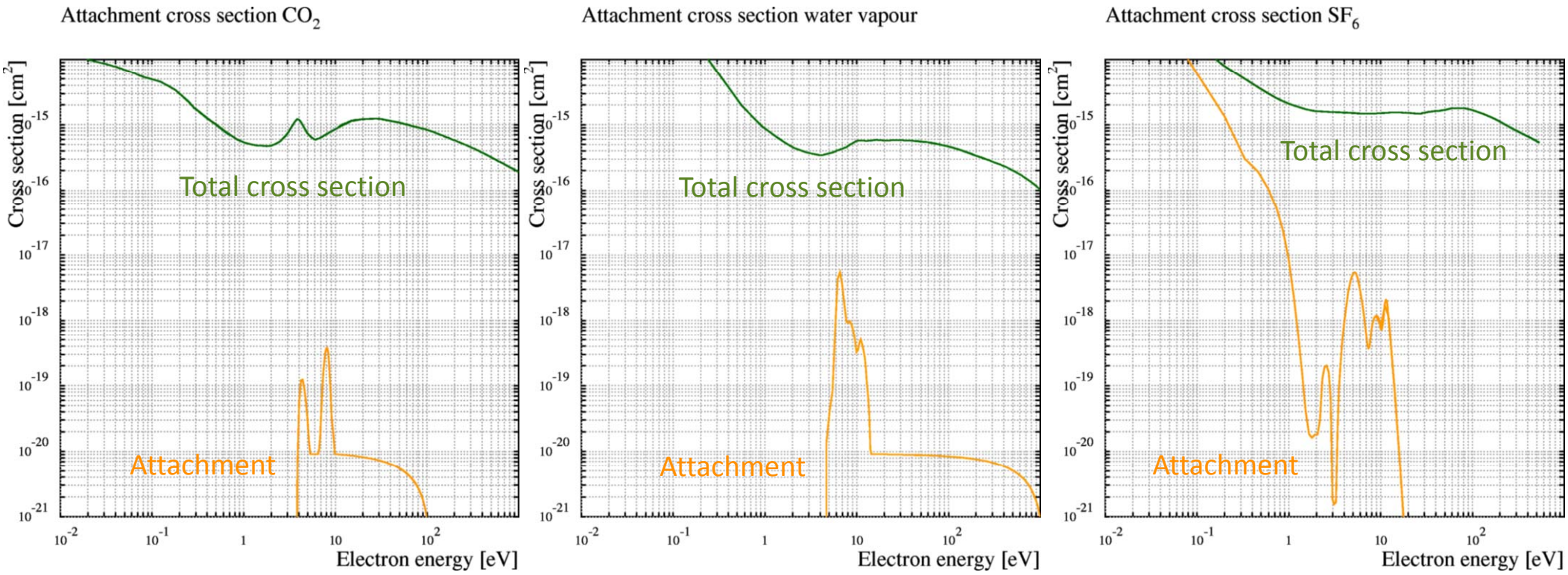
- For $\bar{n} >> 1$, $P(n, x)$ is well approximated by an **exponential** distribution (variance equal to mean).

$$P(n,x) \approx \frac{1}{\bar{n}} e^{-n/\bar{n}}, \quad \frac{\sigma^2}{\bar{n}^2} \approx 1$$



Attachment (I)

- Many non-noble gases attach electrons at somewhat lower than ionisation energies (typically > 5 eV). Some RPC gases massively attach at all energies (SF6).
- Main effect of attachment: electrons can be lost within the first steps prior to any multiplication.
- To overcome attachment, the initial multiplication must be large.



Attachment (II)

- Introduce attachment coefficient η [1/cm]

- $P(n, x+\Delta x) = [1 - n(\alpha+\eta)\Delta x]P(n, x) + (n-1)\alpha\Delta x P(n-1, x) + (n+1)\eta\Delta x P(n+1, x) + \ldots$
  "nothing " happens 1 electron ionises 1 electron attaches

- Solution for $\bar{n} = e^{(\alpha-\eta)x} >> 1$: $P(n,x) = \begin{cases} \eta/\alpha, & n = 0 \\ \dfrac{(1-\eta/\alpha)^2}{\bar{n}} \exp\left(-(1-\eta/\alpha)\dfrac{n}{\bar{n}}\right), & n > 0 \end{cases}$
  (W. Legler)

- Considering only avalanches with $n > 0$, mean and variance are given by

$$\bar{n}^* = \frac{e^{(\alpha-\eta)x}}{1-\eta/\alpha}, \qquad \sigma^2 = \bar{n}^{*2}$$

- Distribution remains exponential.

W. Legler, *Die Statistik der Elektronenlawinen in elektronegativen Gasen, bei hohen Feldstärken und bei großer Gasverstärkung*, Z. Naturforschg. **16a** (1961), 253-261

Beyond the Yule-Furry Model

- To understand/reproduce rounded spectra as observed at higher fields, more elaborate models are required.

- In particular, the assumption that **all electrons** take part in the multiplication process with **equal probability** must be abandoned.

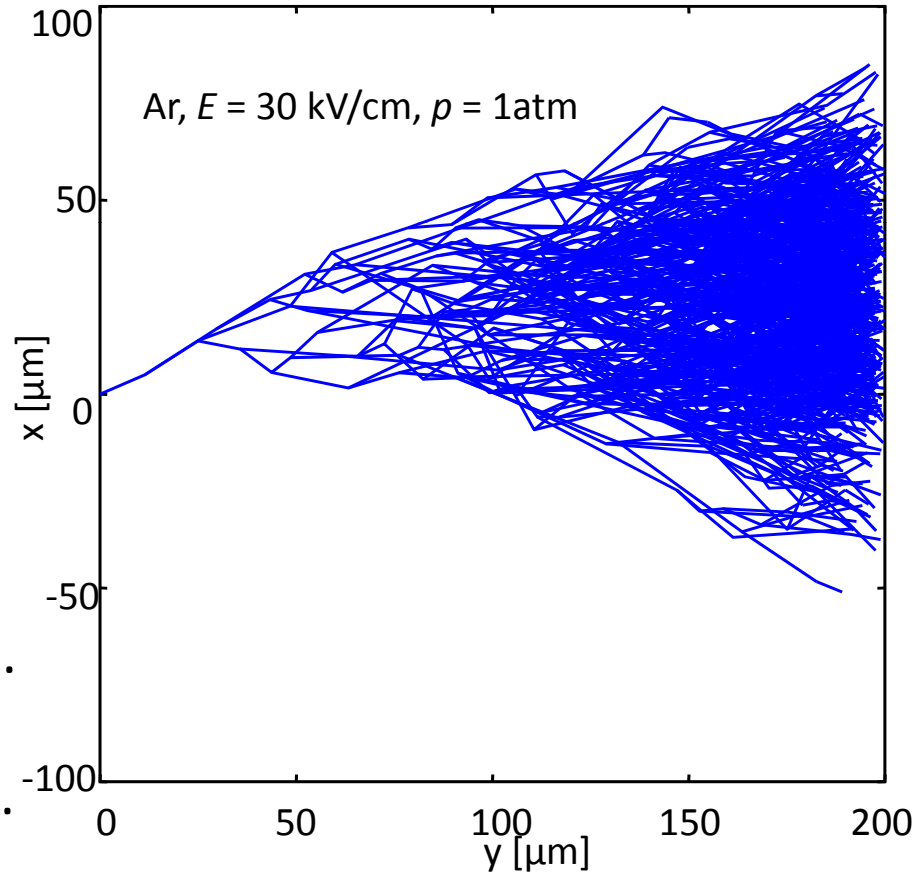
- Analytical models incorporating the ionisation threshold and inelastic interactions exist (more later) → useful for qualitative understanding

- For the quantitative prediction of avalanche spectra, Monte Carlo simulation represents a more suitable approach.

Avalanche Simulation in Garfield

- Since mid 2008 Garfield includes routines for microscopic Monte Carlo simulation of electron transport using Magboltz cross-sections
  - `MICROSCOPIC_AVALANCHE`
  - `DRIFT_MICROSCOPIC_ELECTRON`

- Output
  - numbers of produced electrons and ions
  - electron trajectories
  - electron energy distribution
  - interaction rates



- Here we discuss spectra for uniform fields. The microscopic tracking procedures can be used with arbitrary geometries though.

- The modification of the field due to space charge is not taken into account→ size dependent effects cannot be reproduced.

Experimental Data (I)

- Validation of simulation results by comparison with pulse height spectra measured in parallel-plate chambers

- We use data from
  - H. Schlumbohm, *Zur Statistik der Elektronenlawinen im Ebenen Feld. III*, Z. Phys. **151** (1958) 563-576
  - A. H. Cookson and T. J. Lewis, *Variations in the Townsend first ionization coefficient for gases*, Brit. J. Appl. Phys. **17** (1966) 1473-1481
  - P. Fonte et al., *Single-electron pulse-height spectra in thin-gap parallel-plate chambers*, Nucl. Instr. Meth. A **433** (1999) 513-517

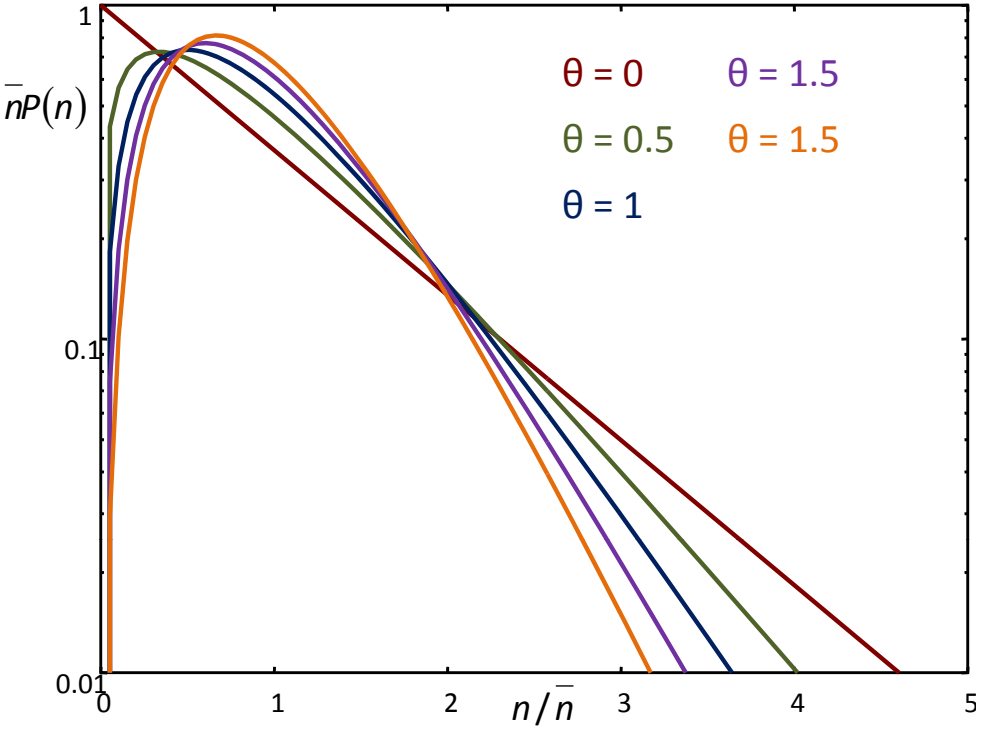
- Avalanches are initiated by single electrons emitted from the cathode due to illumination with UV light.

Excursion: Pólya Distribution

- $P(n) = \frac{1}{\bar{n}} \frac{(\theta+1)^{\theta+1}}{\Gamma(\theta+1)} \left(\frac{n}{\bar{n}}\right)^{\theta} e^{-(\theta+1)n/\bar{n}}$

- Relative variance $\sigma^2/\bar{n}^2 = 1/(\theta+1)$ falls with increasing shape parameter θ

- By tuning θ good agreement with measured spectra can be achieved.



- Physical significance of θ is dubious.

- "Derivation": introduce size dependent multiplication probability $\alpha(n, x) = \alpha(x)[1 + \theta/n]$

J. Byrne, *Statistics of the electron multiplication process in proportional counters*, Proc. R. Soc. Edinburgh A **66** (1962) 33-41

A. Lansiart and J.-P. Morucci, *Amplication gazeuse dans un compteur proportionnel* J. Phys. Radium (Supplement) **23/S6** (1962) 102A-104A

Comparing Spectra At Different Gain (I)

- Experimental data are given for gain values $\approx 10^4 - 10^6$, while reasonable avalanche sizes for simulation are of order $10^2 - 10^3$.

- Convenient figures of merit to characterise the "roundness" are
  - **shape parameter θ** of fitted Pólya function
  - **relative width** of the distribution $f = \sigma^2 / \bar{n}^2$
- To what extent are these parameters size-dependent?

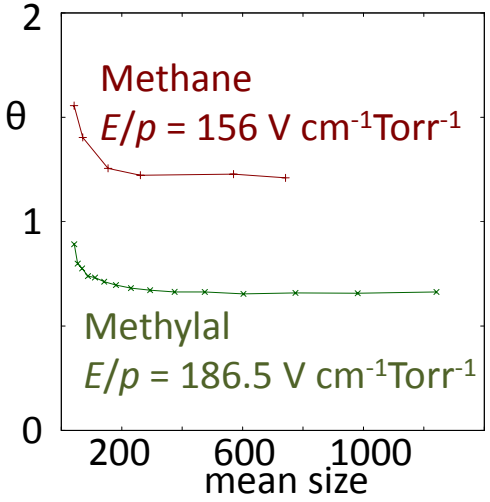
- **Hypothesis:** for $\bar{n} >> 1$ the shape of the distribution is independent of the mean size (provided that ***E/p*** is kept **constant**!)

$$P(n,x) = \frac{1}{\bar{n}} \varphi\left(n / \bar{n}\right)$$

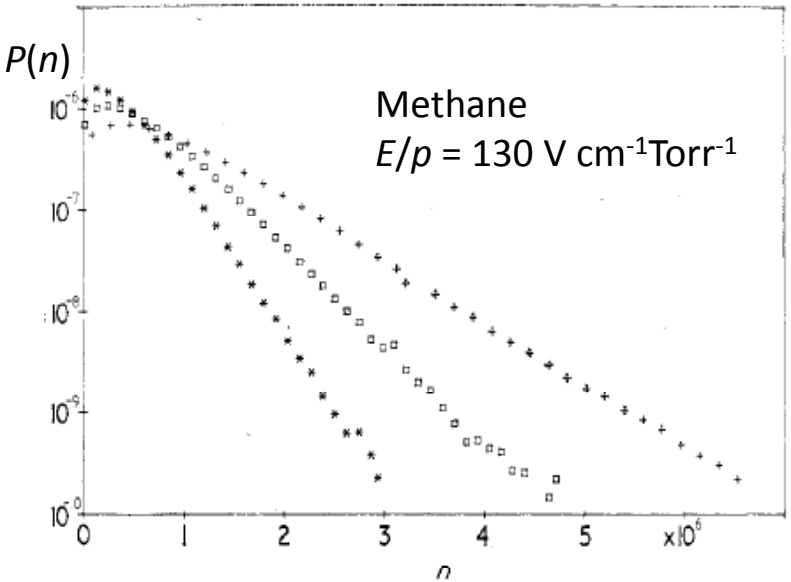
- However: If the field is altered during the avalanche growth (space charge), the shape of the distribution does depend on the gap!

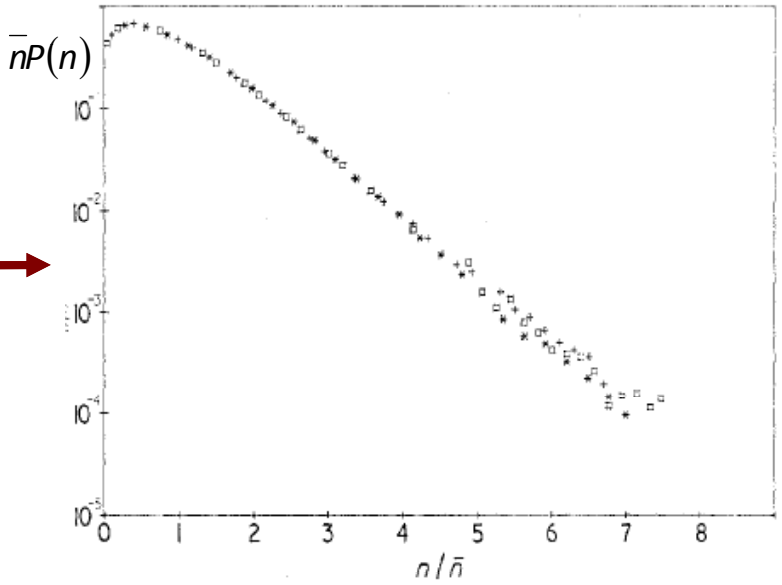
Comparing Spectra At Different Gain (II)

- Garfield: with increasing gap (and thus increasing size) the Pólya fit parameter approaches an asymptotic value



- Experimental evidence





G. Vidal, J. Lacaze and J. Maurel, *Microscopic evolution of the ionizing collision frequency in Townsend avalanches*, J. Phys. D: Appl. Phys. **7** (1974) 1684-1698

- Is this a universal feature? → use general statistical relations to calculate the evolution of the moments of the size distribution.

Excursion: Some Statistics ... (I)

- Recall: $P(n, x)$ denotes the size distribution after a **step *x*** for **one** primary electron

- Assuming that the individual avalanches evolve independently, the distribution for ***k* primary electrons** is given by k-fold **convolution** of $P(n, x)$ → for large k the size distribution tends to a normal distribution (central limit theorem)

- The single electron avalanche size distribution after a **step 2*x*** is given by

$$P(n,2x) = \sum_{n'=1}^{n} P(n',x) P^{*n'}(n-n',x)$$

where $P^{*n'}$ is the n' -fold convolution of P.

- "As one can easily show ": mean and variance after two steps are given by

$$\bar{n}(2x) = \bar{n}(x)^2 \qquad \sigma^2(2x) = \bar{n}(x)\left[1+\bar{n}(x)\right]\sigma^2(x)$$

Excursion: Some Statistics ... (II)

- Mean and variance after *k* steps are given by

$$\bar{n}(kx) = \bar{n}^k(x) \qquad \sigma^2(kx) = \sigma^2(x) \sum_{i=0}^{k-1} \bar{n}^{k+i-1}(x)$$

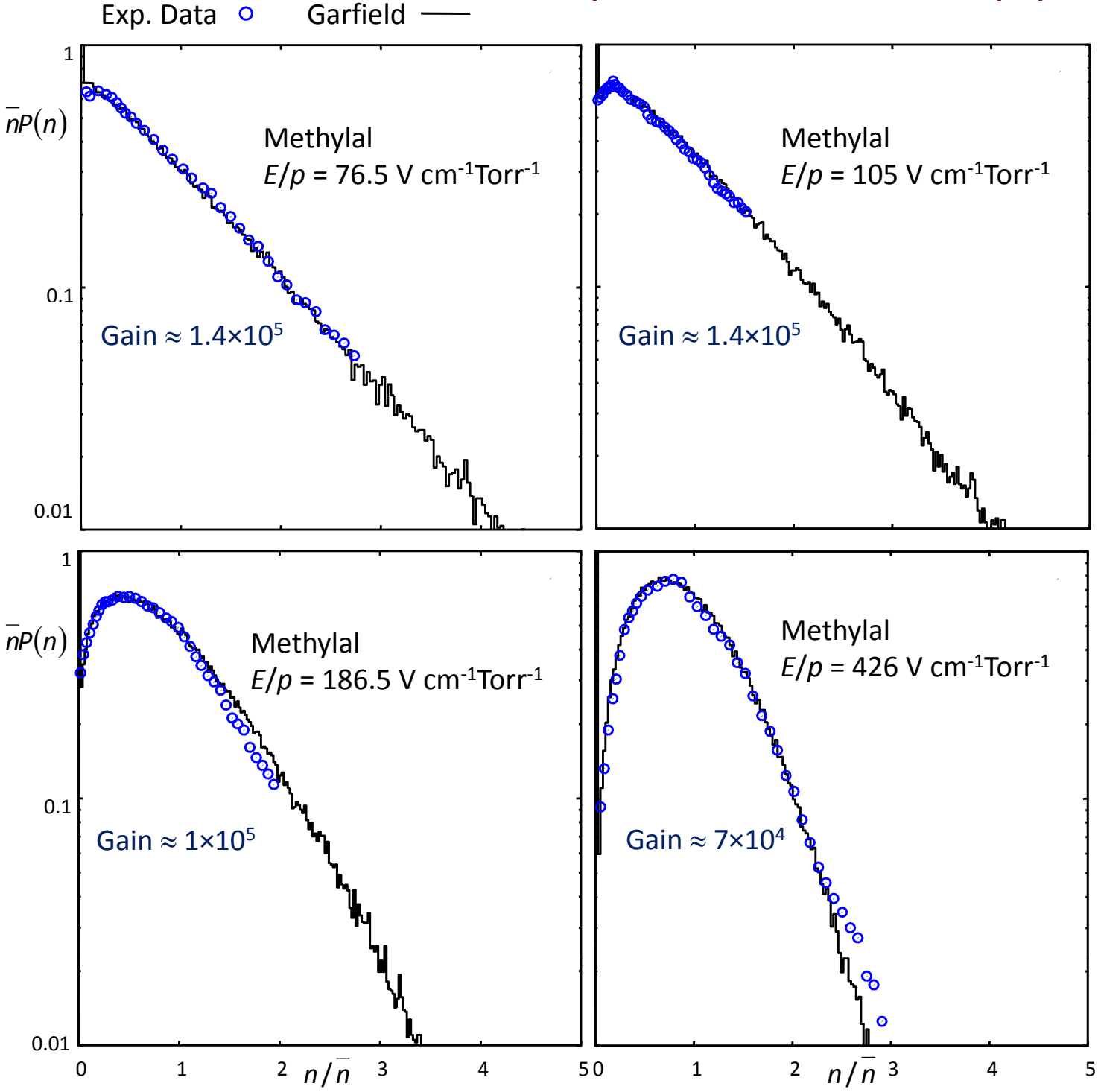
- The relative variance evolves as

$$\frac{\sigma^2(kx)}{\bar{n}^2(kx)} = \frac{\sigma^2(x)}{\bar{n}(x)} \left(\frac{1 - 1/\bar{n}^k(x)}{1 - 1/\bar{n}(x)} \right)$$

- The growth of the relative width is bounded (if $\bar{n}(x) > 1$)

- Once $\bar{n}$ is large, the relative width barely grows further.

Experimental Data (II)



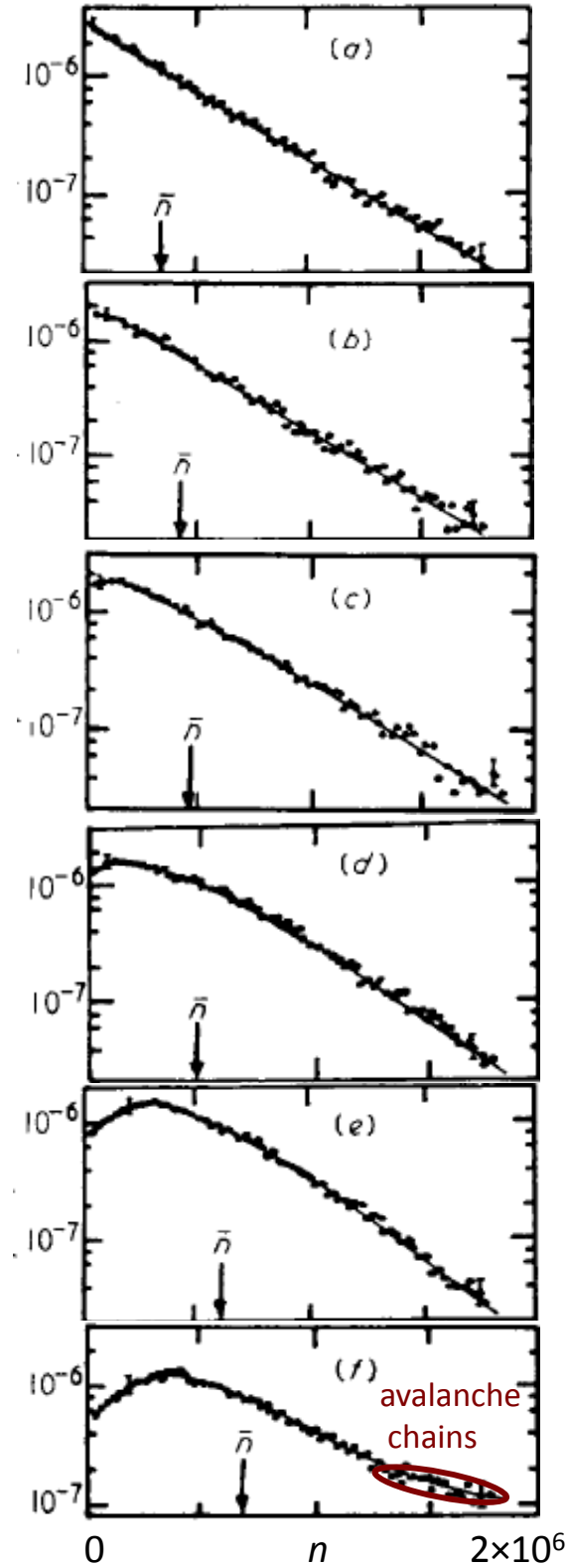
- Schlumbohm (1958): Observation of rounding-effect at high E/p in several gas mixtures. Plots are available for methylal, acetone and alcohol.
- For comparison with the simulation, the avalanche size is normalised to the mean obtained from a Pólya fit.

Experimental Data (III)

- Cookson and Lewis (1966):
  Measurements in methane, fitted with Pólya curves

Plot	E/p	Pólya fit parameter θ	
	[V cm^{-1}Torr^{-1}]	Cookson/Lewis	Garfield
(a)	48.2	0.0	0.13
(b)	51.3	0.1	0.15
(c)	78.9	0.3	0.36
(d)	120.0	0.4	0.76
(e)	156.0	1.0	1.21
(f)	218.0	1.2	1.99

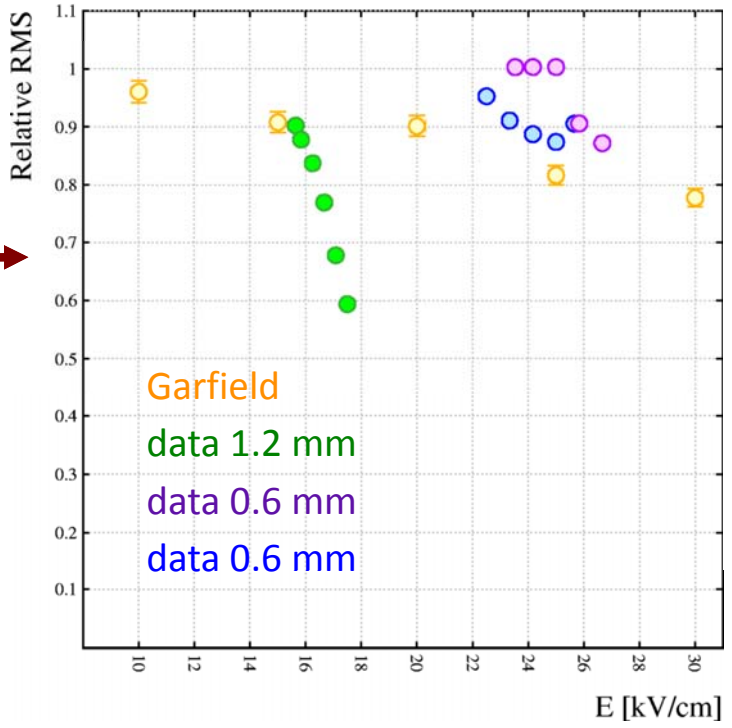
- Garfield simulation: Tendency of rounding is reproduced, shape deviates somewhat from experimental spectra

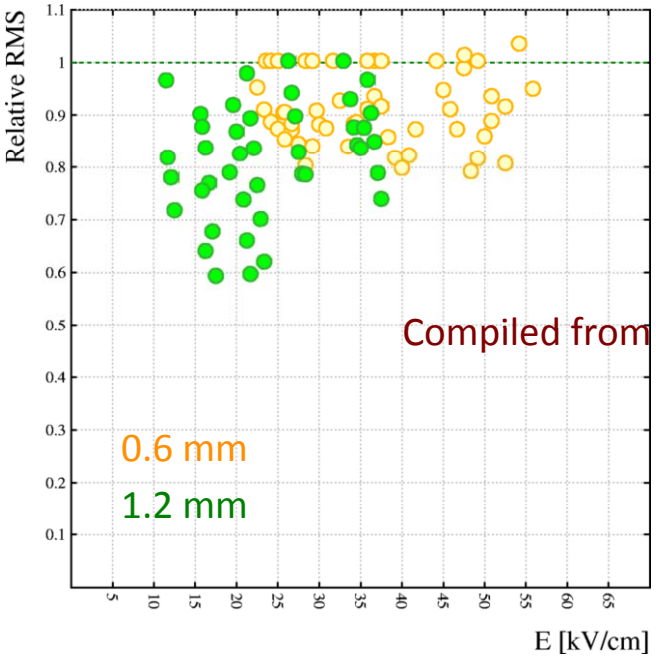


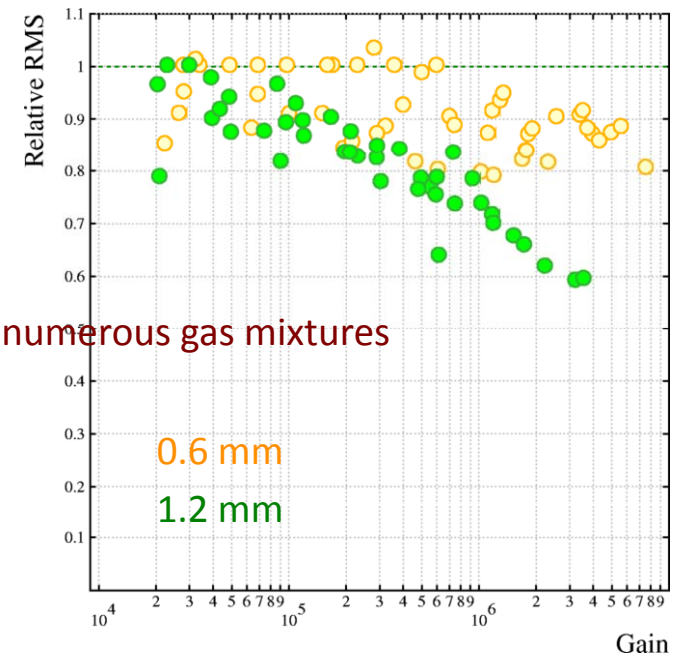
Experimental Data (IV)

- P. Fonte et al. (1999): Measurements with a range of gas mixtures and gaps of 0.6 mm and 1.2 mm. Data files with parametrisations of the gain spectra (unpublished) were provided by Paulo Fonte. The data for 1.2 mm gaps looks cleaner.

- The data show a steep fall of relative RMS with increasing *E* field, while Garfield predicts a more gentle decrease. Example: 90% Ar + 10 % C_2H_6 (similar disagreement for all other mixtures)





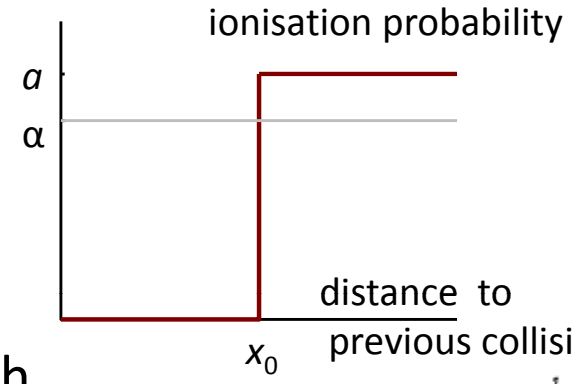


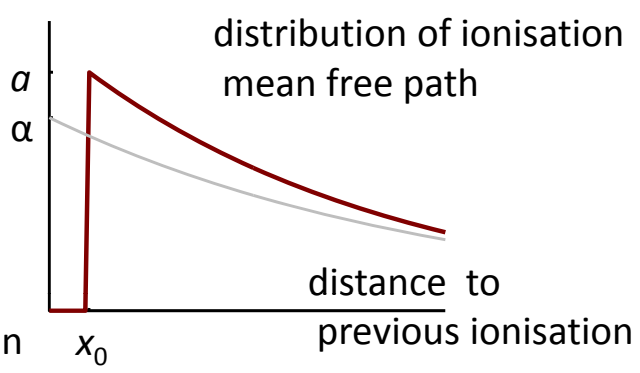
Compiled from numerous gas mixtures

- **Gain** seems to be the driving factor for the shape of these spectra.

Analytical Models (I)

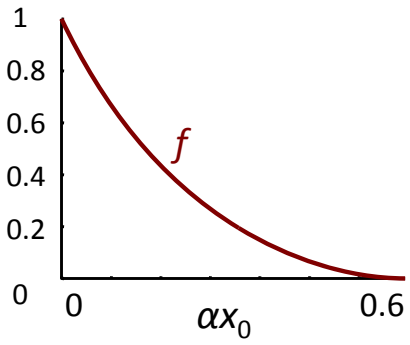
- **Legler's model**: threshold energy for ionisation → ionisation probability depends on electron energy or the distance travelled since the last collision, respectively.
- If the electron starts with zero kinetic energy, the minimum step length can be estimated as $x_0 \approx U_i/E$. Exponential growth requires that $a = \alpha / (2\exp(-\alpha x_0) - 1)$.

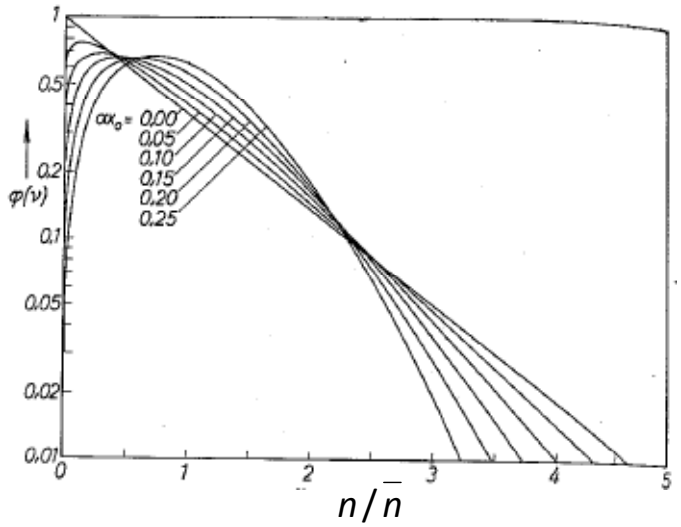




- relative width:

$$f = \frac{\sigma^2}{\bar{n}^2} = \frac{\left(2e^{-\alpha x_0} - 1\right)^2}{4e^{-\alpha x_0} - 2e^{-2\alpha x_0} - 1}$$

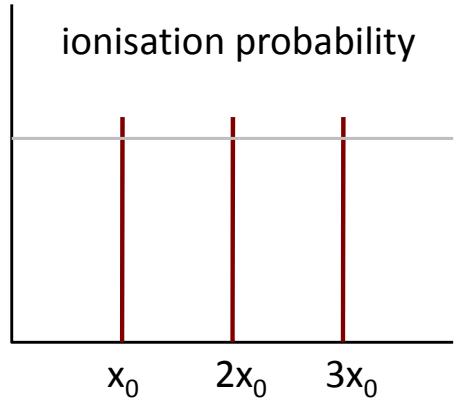




W. Legler, *Die Statistik der Elektronenlawinen in elektronegativen Gasen, bei hohen Feldstärken und bei großer Gasverstärkung,* Z. Naturforschg. **16a** (1961) 253-261

G. D. Alkhazov, *Statistics of Electron Avalanches and Ultimate Resolution of Proportional Counters*, Nucl. Instr Meth. **89** (1970) 155-165

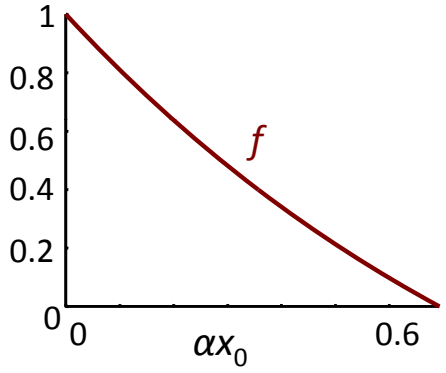
Analytical Models (II)

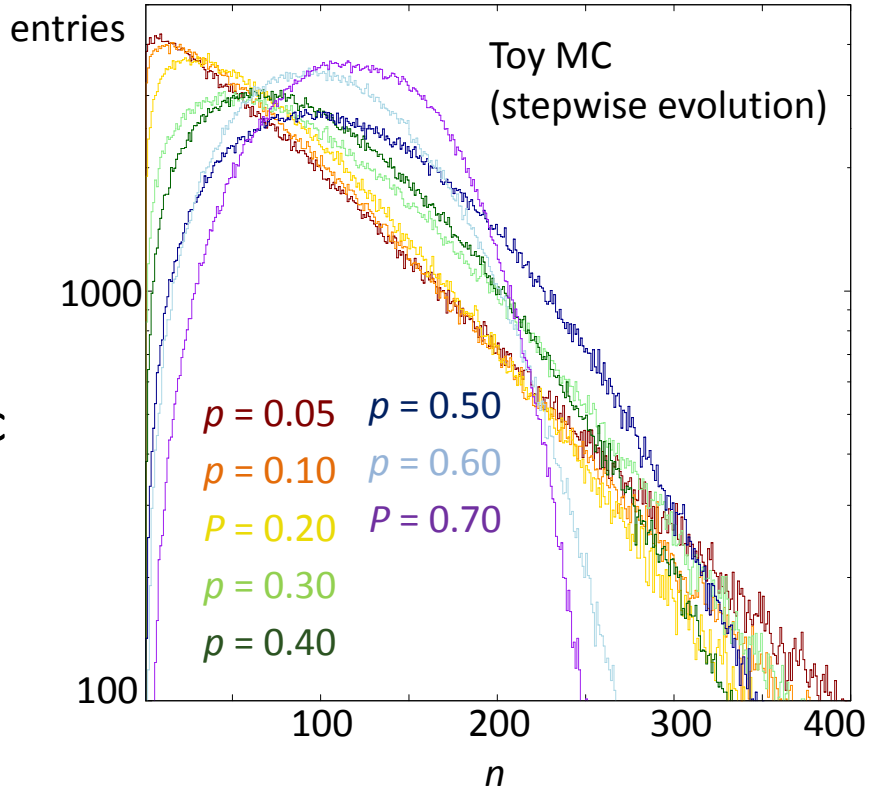


- **Stepwise evolution:** multiplication occurs in steps of fixed length x_0. After each step an electron ionises with probability p or loses its energy through another process with probability $1 - p$.

- relative width:

$$f = \frac{\sigma^2}{\bar{n}^2} \approx \frac{1-p}{1+p} = 2e^{-\alpha x_0} - 1$$





- Alkhazov's model 3: distribution of inelastic collision probability given by step-function; probability of ionisation in each inelastic collision given by p

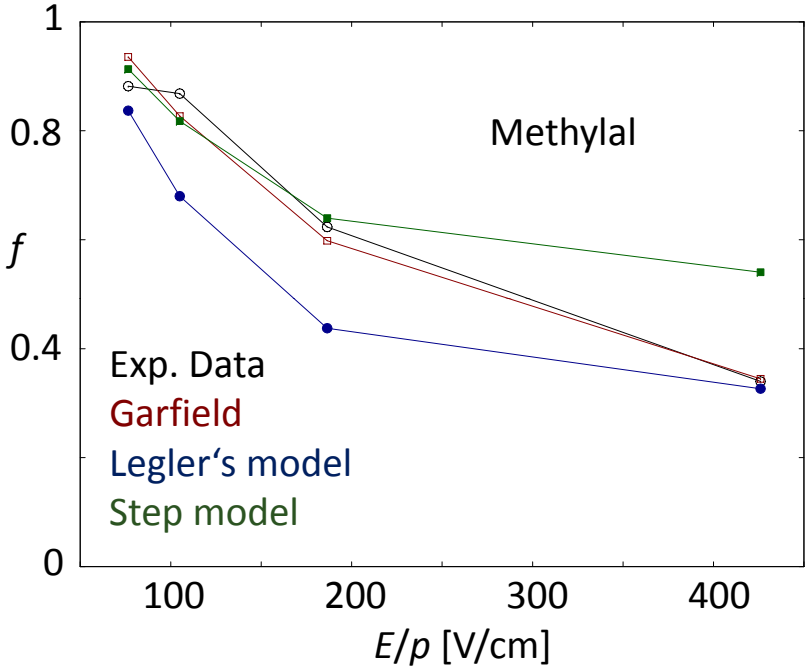
$$f \approx \frac{(1+3p)e^{-2\alpha x_0} - 2(1+p)e^{-2\alpha x_0} + 1}{2(1+p)e^{-\alpha x_0} - (1+p)e^{-2\alpha x_0} - 1}$$

G. D. Alkhazov, *Statistics of Electron Avalanches and Ultimate Resolution of Proportional Counters*, Nucl. Instr Meth. **89** (1970) 155-165

Analytical Models (III)

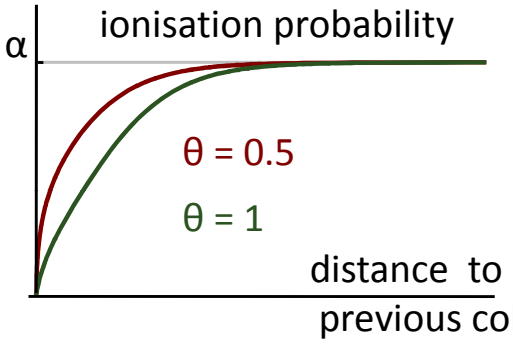
- Closed-form expressions for these distributions are not available, but their shape can be characterised in terms of the moments.

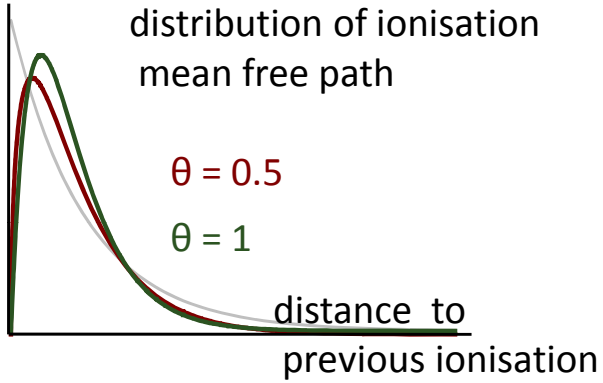
- The models reflect (in a simplified way)
  - **energy dependence** of electron-atom collision cross-sections
  - interplay between **ionisation** and other **inelastic** collisions



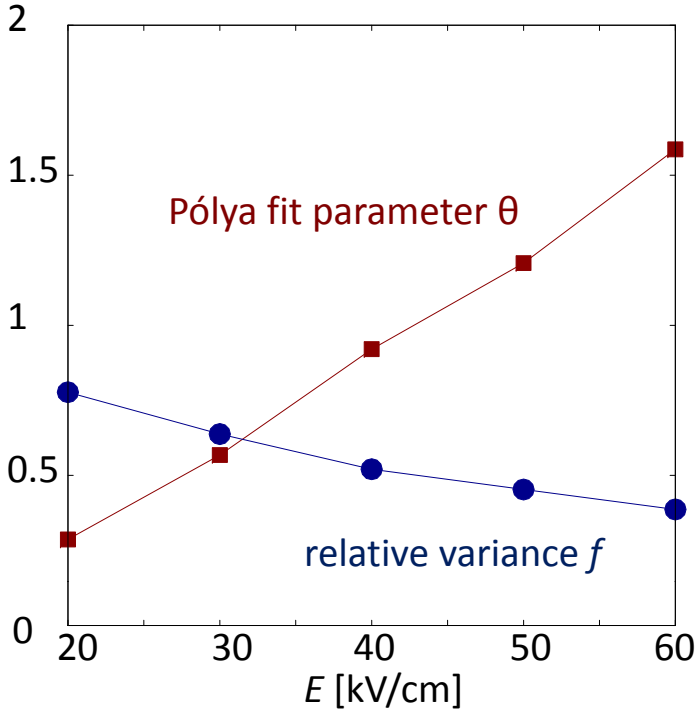
- The relative width falls with increasing parameter αx_0 ($x_0 \propto 1/E$) and/or increasing ratio ionisation/excitation.

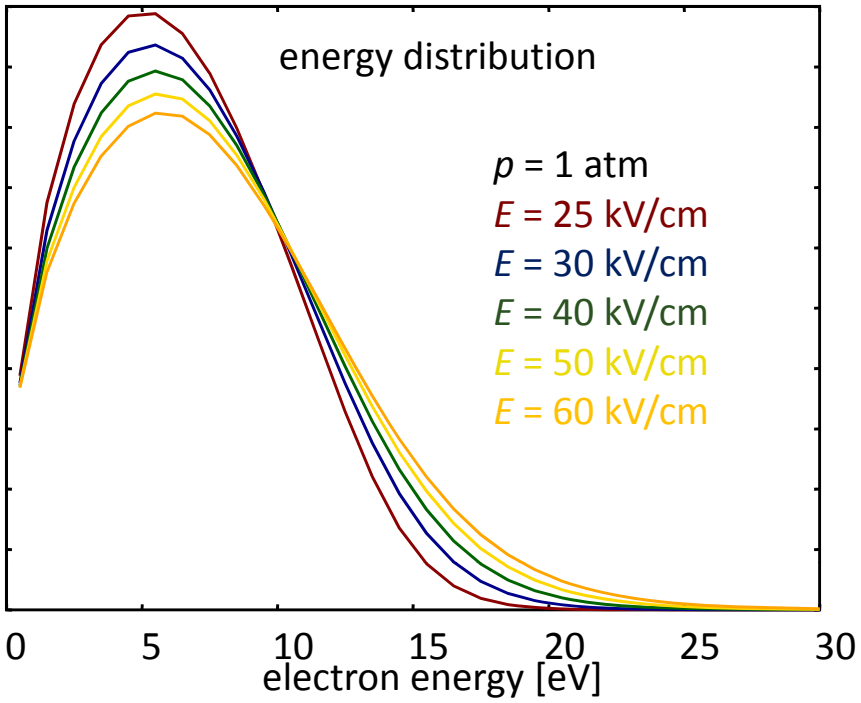
- Side note:
  Which distribution of the ionisation mean free path would lead to the Pólya function?

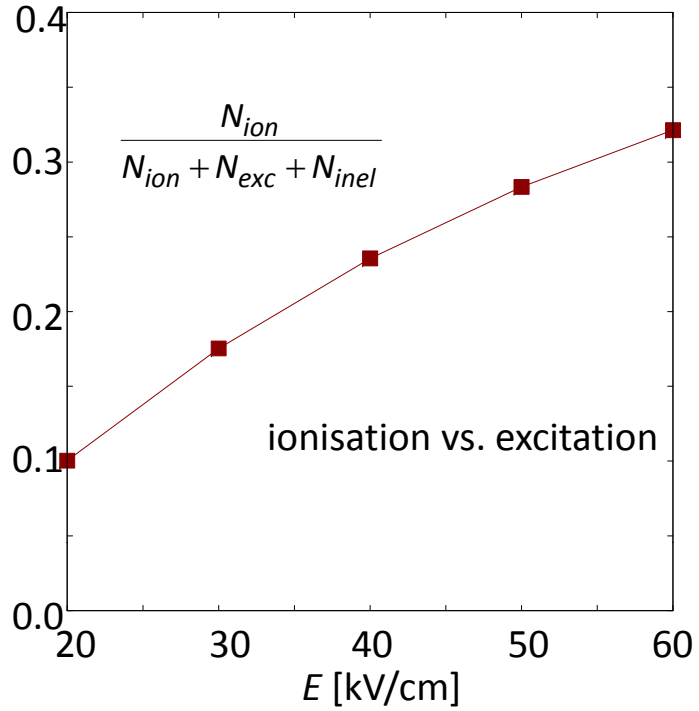


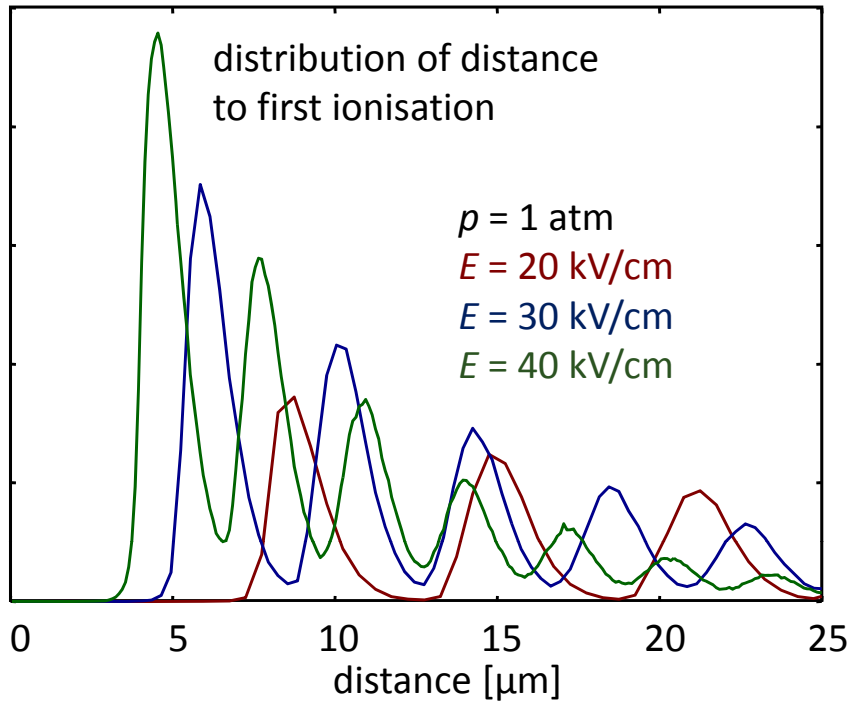


Case Study: Argon

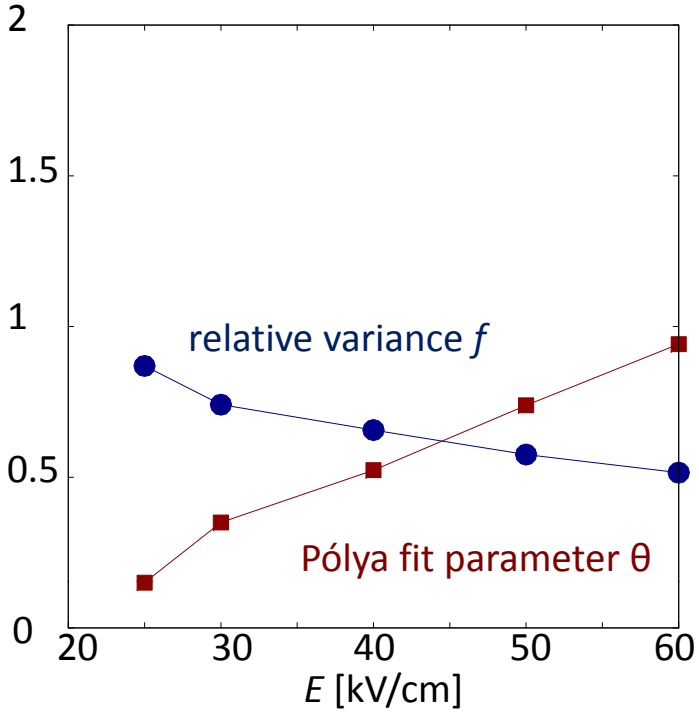


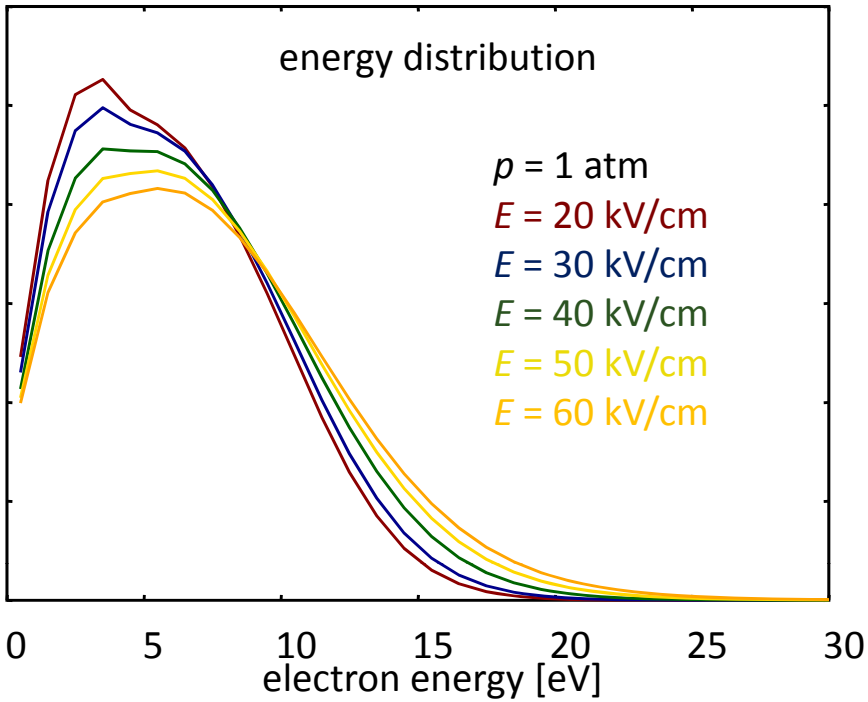


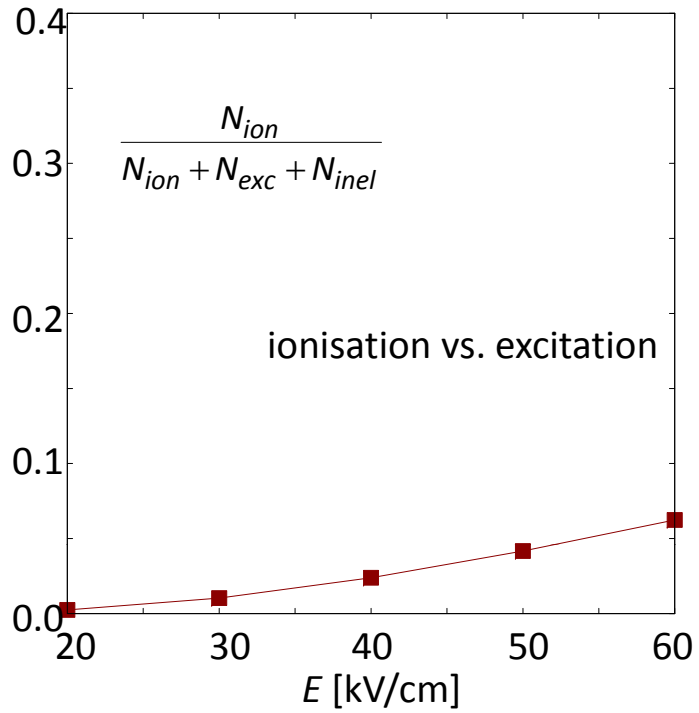


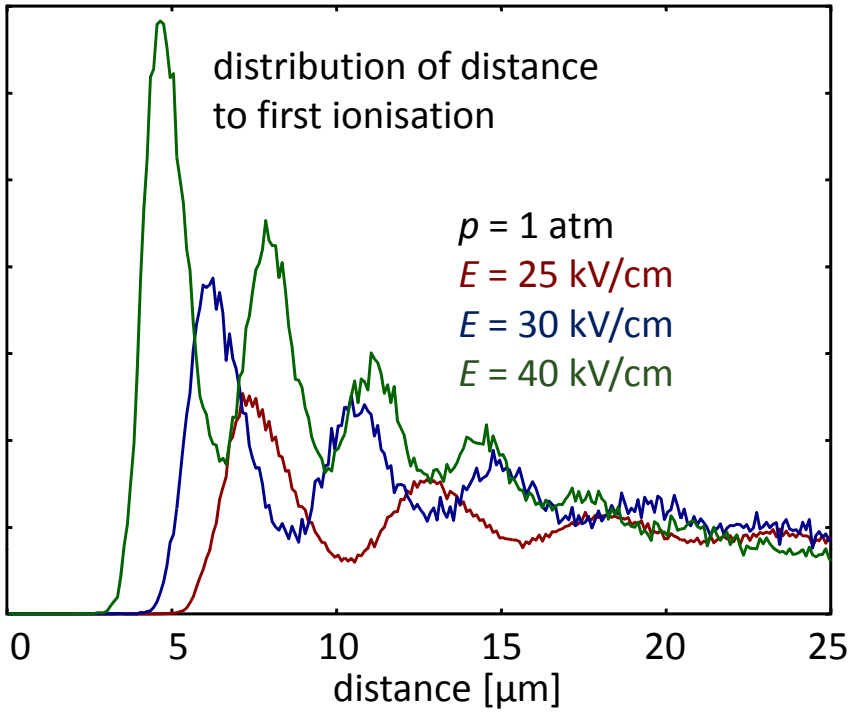


Case Study: 80% Ar + 20% CO_2



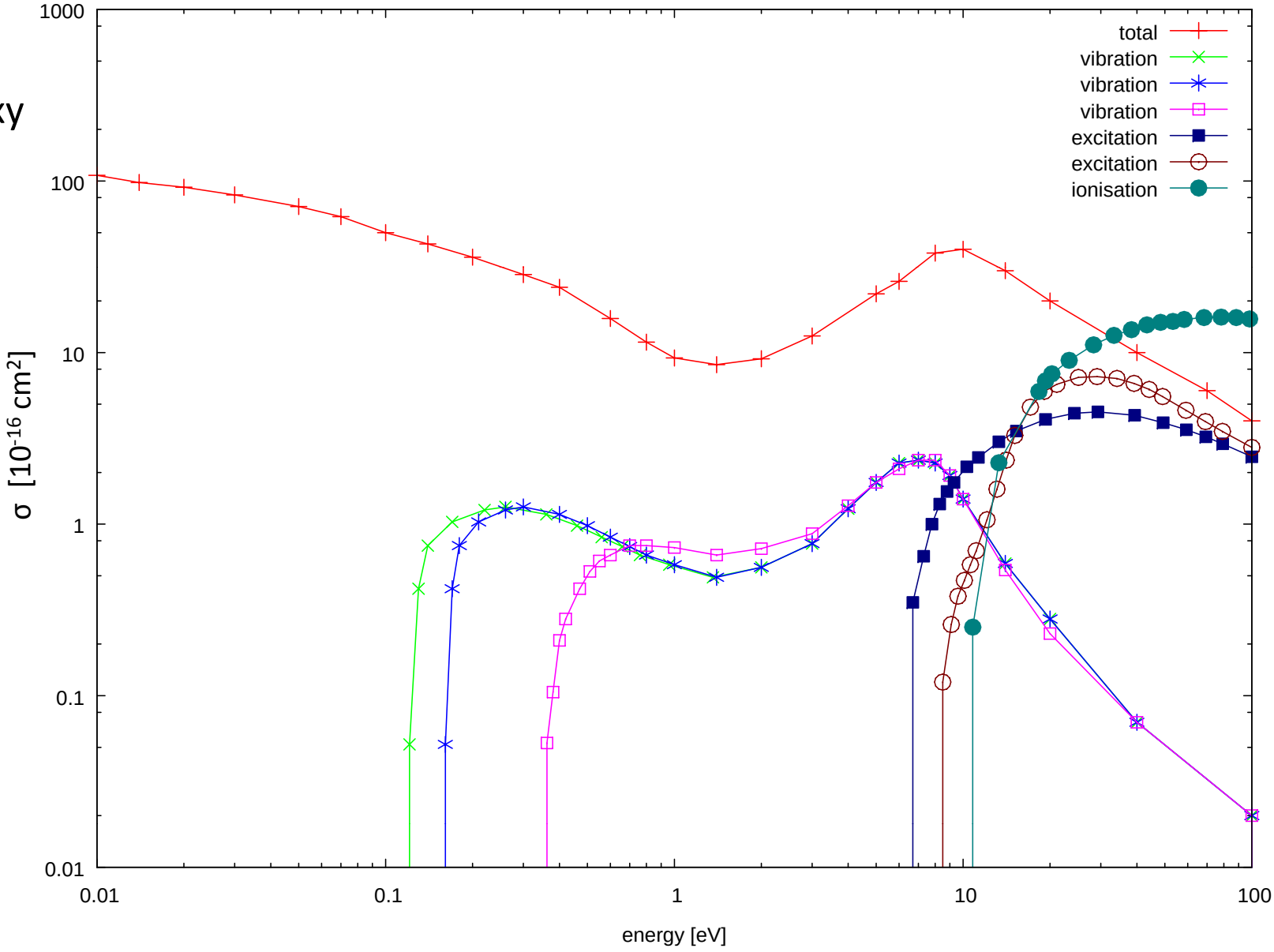






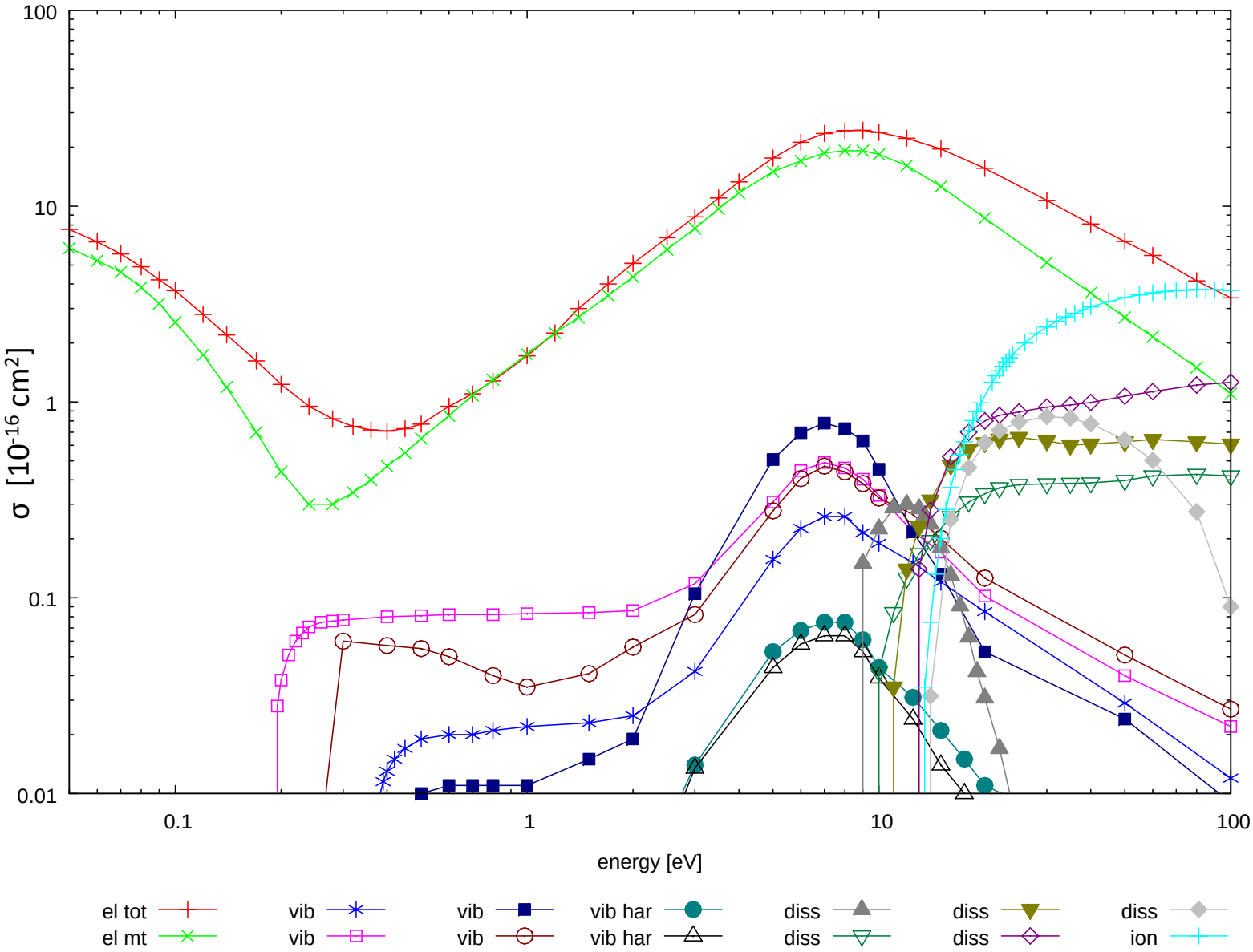
Methylal

- $CH_2(OCH_3)_2$ aka
  - DMM
  - dimethoxy methane
  - formal
- Excitations
  - 6.3 eV
  - 8.3 eV
- Ionisation
  - 10 eV



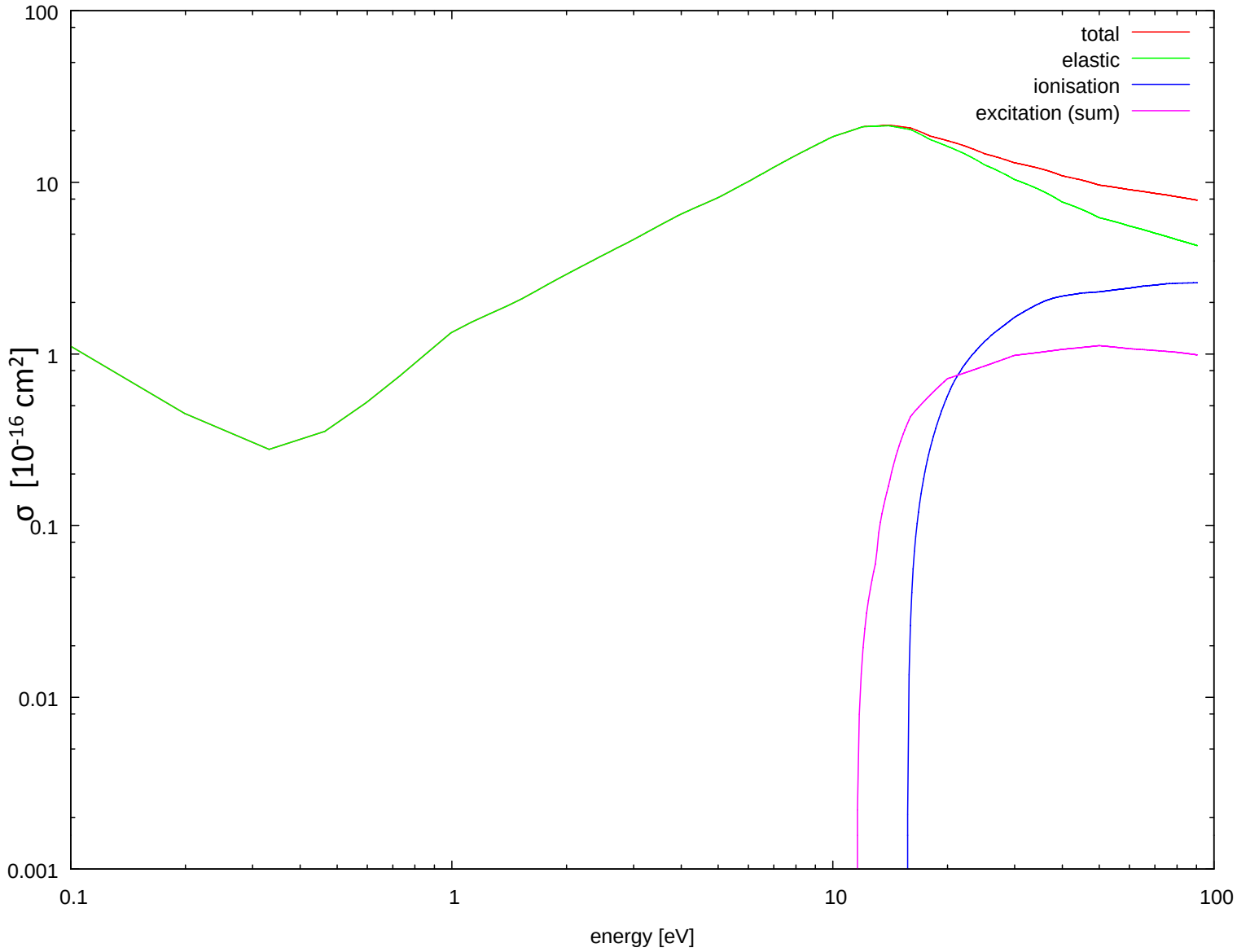
Methane

- CH_4
- Dissociation
  - 9 eV
  - 10 eV
  - 11 eV
  - 11.8 eV
- Ionisation
  - 12.99 eV

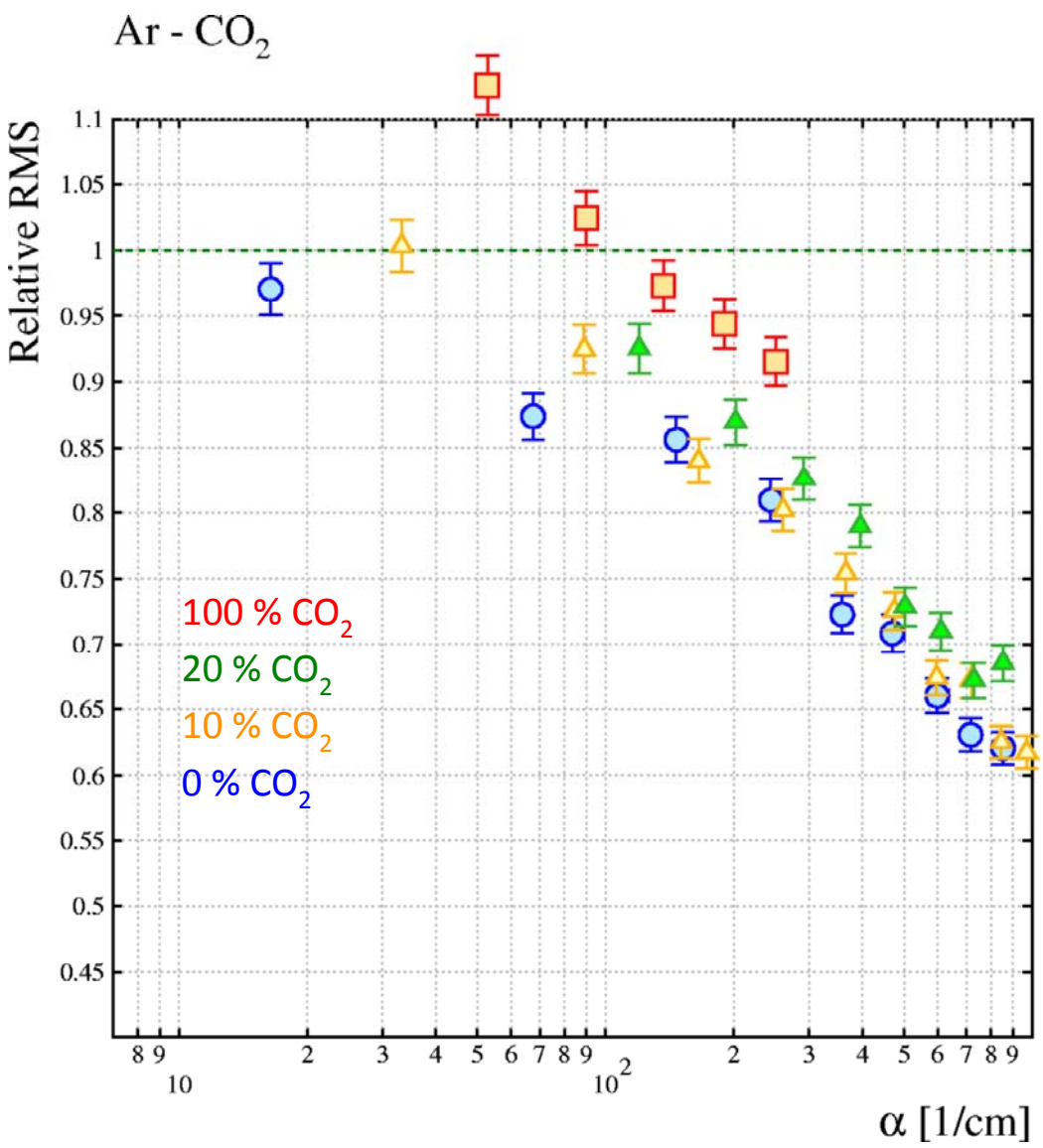


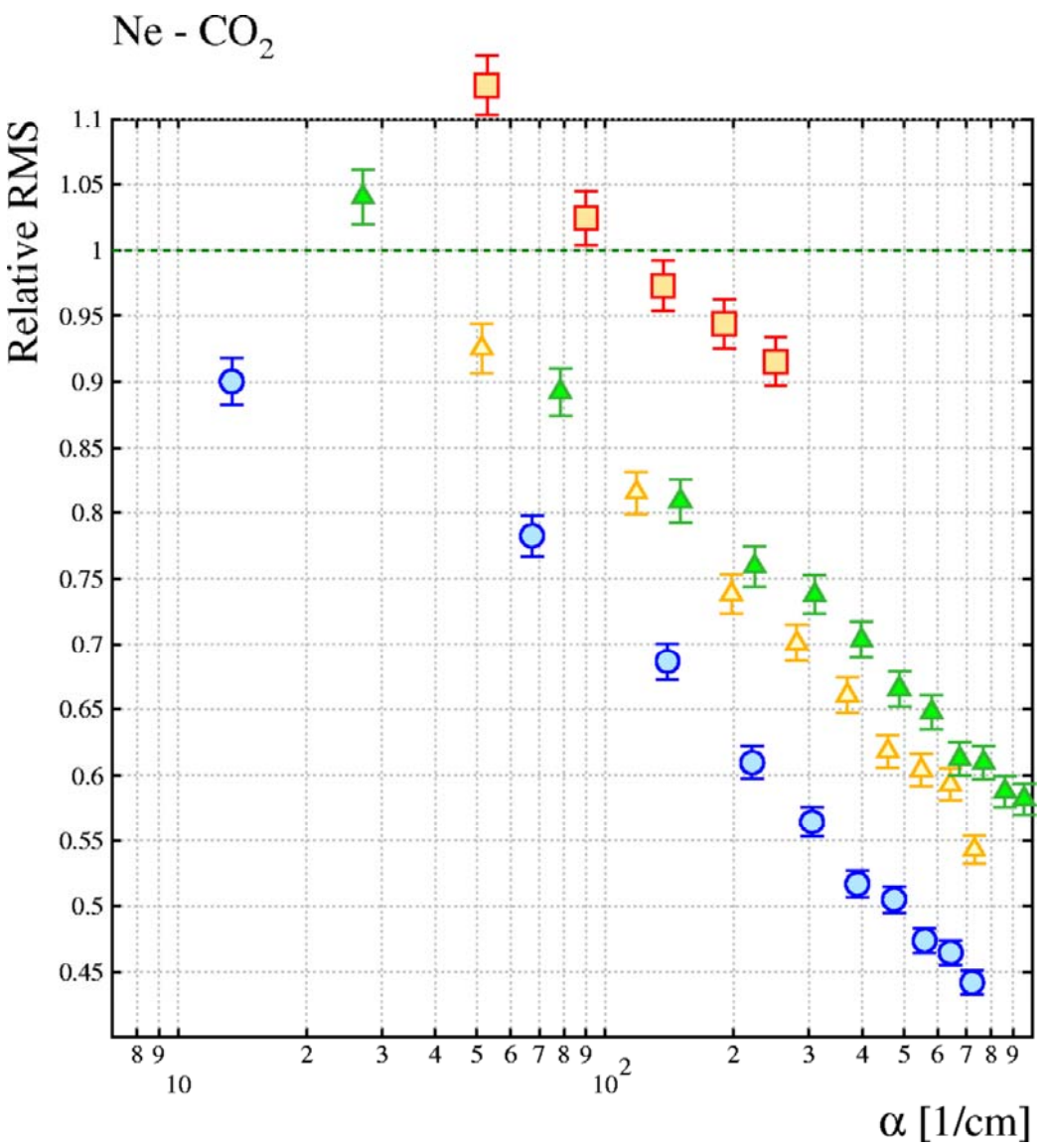
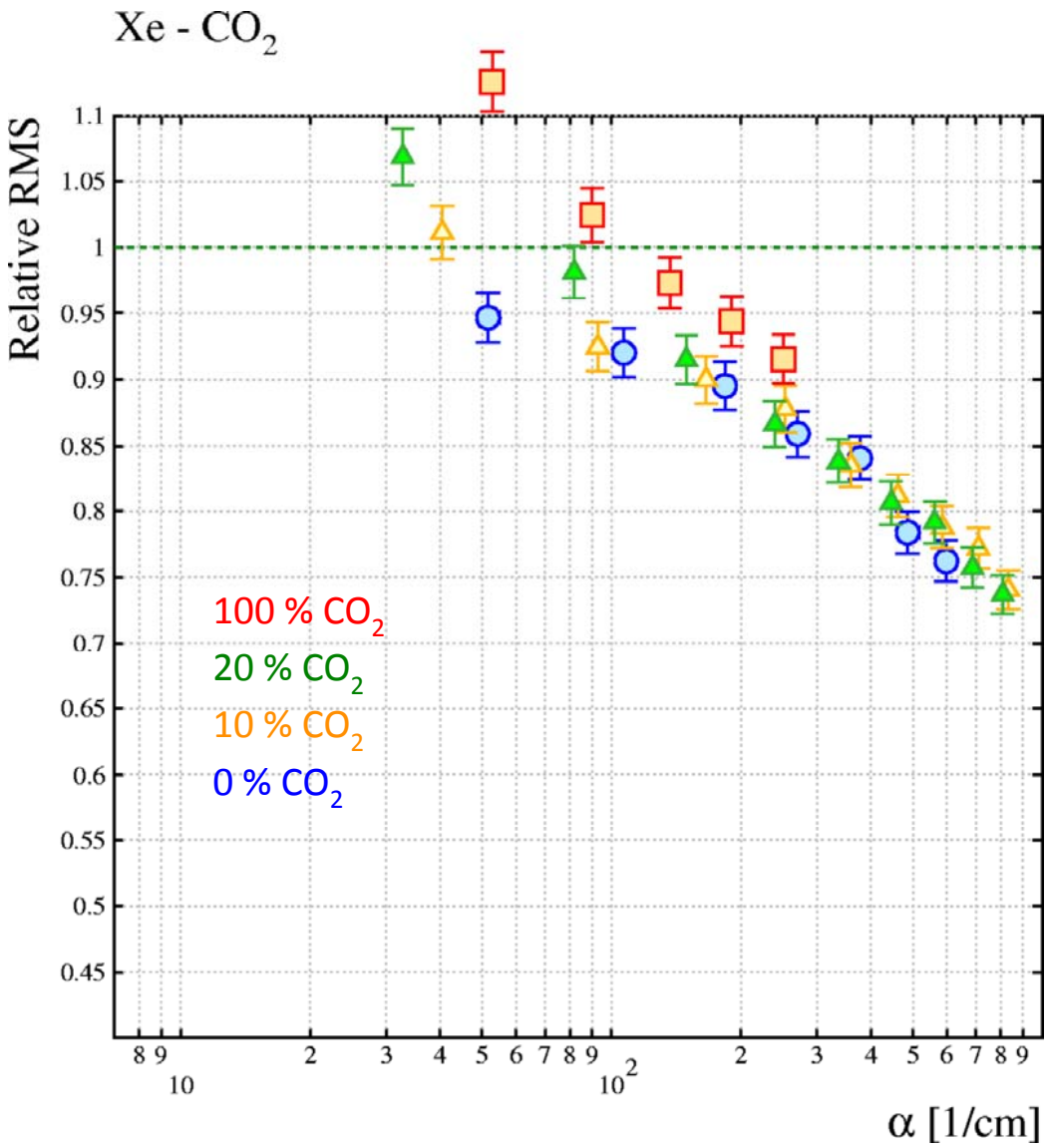
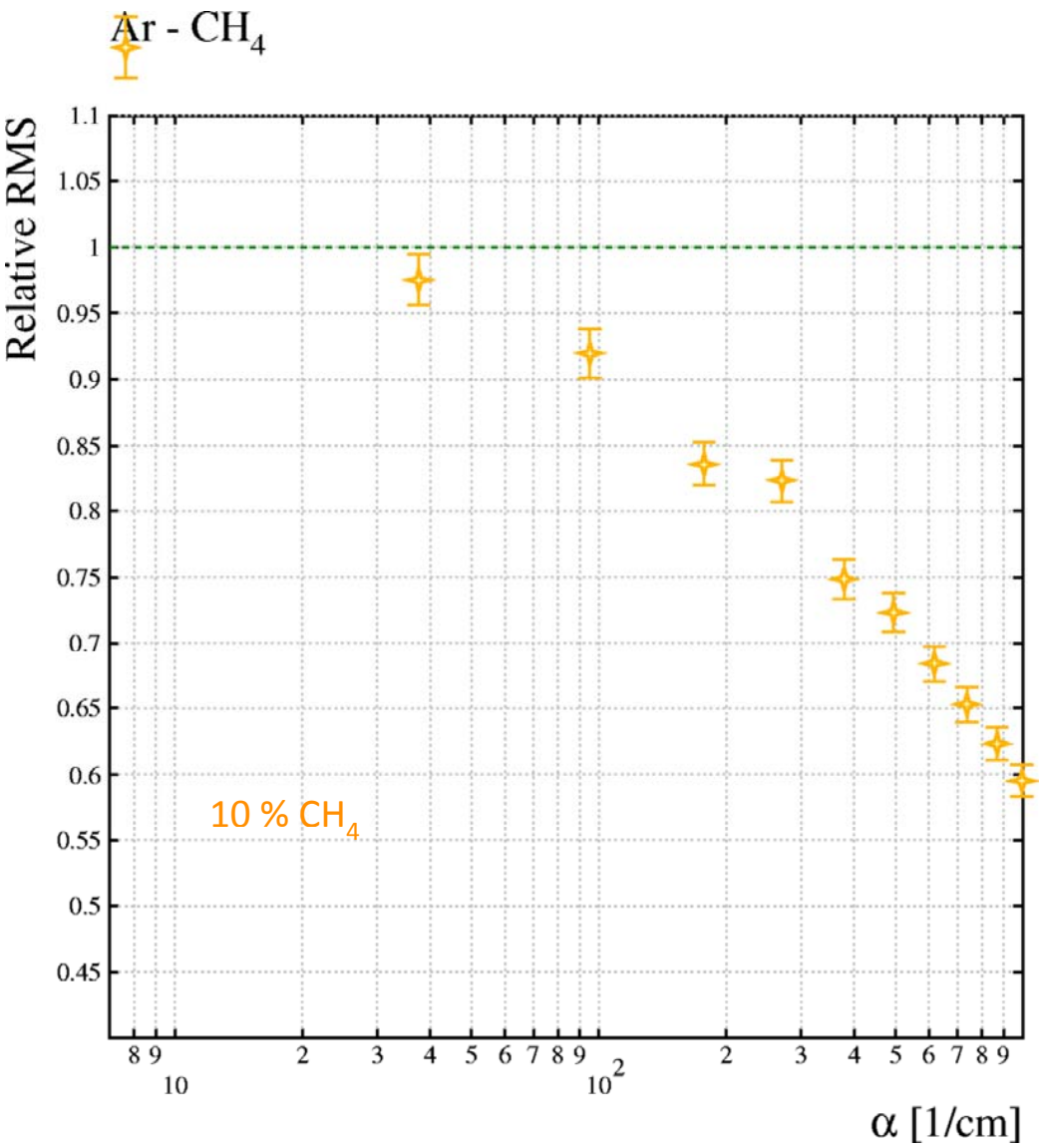
Argon

- Excitation
  - 11.55 eV
  - 13 eV
  - 14 eV
- Ionisation
  - 15.7 eV



Comparing Gases (I)





Comparing Gases (II)