CHAPTER 12

Impossible Triangles

Is Euclid's geometry the only one?

Calculus was based on geometric principles, but the geometry was reduced to symbolic calculations, which were then formalized as analysis. However, the role of visual thinking in mathematics was also developing, in a new and initially rather shocking direction. For more than 2000 years the name Euclid had been synonymous with geometry. His successors developed his ideas, especially in their work on conic sections, but they did not make radical changes to the concept of geometry itself. Essentially, it was assumed that there can be only one geometry, Euclid's, and that this is an exact mathematical description of the true geometry of physical space. People found it difficult even to conceive of alternatives.

It couldn't last.

Spherical and projective geometry

The first significant departure from Euclidean geometry arose from the very practical issue of navigation. Over short distances, the Earth is almost flat, and its geographical features can be mapped on a plane. But as ships made ever longer voyages, the true shape of the planet had to be taken into account. Several ancient civilizations

knew that the Earth is round – there is ample evidence, from the way ships seem to disappear over the horizon to the shadow of the planet on the Moon during lunar eclipses. It was generally assumed that the Earth is a perfect sphere.

In reality, the sphere is slightly flattened: the diameter at the equator is 12,756 km, whereas that at the poles is 12,714 km. The difference is relatively small – one part in 300. In times when navigators routinely made errors of several hundred kilometres, a spherical Earth provided an entirely acceptable mathematical model. At that time, however, the emphasis was on spherical *trigonometry* rather than geometry – the nuts and bolts of navigational calculations, not the logical analysis of the sphere as a kind of space. Because the sphere sits naturally within three-dimensional Euclidean space, no one considered spherical geometry to be different from Euclidean. Any differences were the result of the Earth's curvature. The geometry of space *itself* remained Euclidean.

A more significant departure from Euclid was the introduction, from the early 17th century onwards, of *projective geometry*. This topic emerged not from science but from art: theoretical and practical investigations of perspective by the Renaissance artists of Italy. The aim was to make paintings look realistic; the outcome was a new way to think about geometry. But, again, this development could be seen as an innovation within the classical Euclidean frame. It was about how we *view* space, not about space.

The discovery that Euclid was not alone, that there can exist logically consistent types of geometry in which many of Euclid's theorems fail to hold, emerged from a renewed interest in the logical foundations of geometry, debated and developed from the middle of the 18th century to the middle of the 19th. The big issue was Euclid's Fifth Postulate, which – clumsily – asserted the existence of parallel lines. Attempts to deduce the Fifth Postulate from the remainder of Euclid's axioms eventually led to the realization that no such deduction is possible. There are consistent types of geometry other than Euclidean. Today, these non-Euclidean geometries have become indispensable tools in pure mathematics and mathematical physics.

Geometry and art

As far as Europe was concerned, geometry was becalmed in the doldrums between the years 300 and 1600. The revival of geometry as a living subject came from the question of perspective in art: how to render a three-dimensional world realistically on two-dimensional canvas.

The artists of the Renaissance did not just create paintings. Many were employed to carry out engineering works, for peaceful or warlike purposes. Their art had a practical side, and the geometry of perspective was a practical quest, applying to architecture as well as to the visual arts. There was also a growing interest in optics, the mathematics of light, which blossomed once the telescope and microscope were invented. The first major artist to think about the mathematics of perspective was Filippo Brunelleschi. In fact, his art was mostly a vehicle for his mathematics. A seminal book is Leone Battista Alberti's *Della Pittura*, written in 1435 and printed in 1511. Alberti began by making some important, and relatively harmless, simplifications – the standard reflex of a true mathematician. Human vision is a complex subject. For example, we use two slightly separated eyes to generate stereoscopic images, providing a feeling of depth. Alberti simplified reality by assuming a single eye with a pinprick pupil, which worked like a pinhole camera. He imagined an artist painting a scene, setting up his easel and trying to make the image on the canvas match the one perceived by his (single) eye. Both canvas and reality project their images on to the retina, at the back of the eye. The simplest (conceptual) way to ensure a perfect match is to make the canvas transparent, look through it from a fixed location and draw upon the canvas exactly what the eye sees. So the three-dimensional scene is *projected* on to

the canvas. Join each feature of the scene to the eye by a straight line, note where this line meets the plane of the canvas: that's where you paint that feature.

This idea is not terribly practical if you take it literally, although some artists did just that, using translucent materials, or glass, in place of a canvas. They often did this as a preliminary step, transferring the resulting outline to canvas for proper painting. A more practical approach is to use this conceptual formulation to relate the geometry of the three-dimensional scene to that of the two-dimensional image. Ordinary Euclidean geometry is about features that remain unchanged by rigid motions – lengths, angles. Euclid did not formulate it that way, but his use of congruent triangles as a basic tool has the same effect. (These are triangles of the same size and shape, but in different locations.) Similarly, the geometry of perspective boils down to features that remain unchanged by projection. It is easy to see that lengths and angles do not behave like that. You can cover the Moon with your thumb, so lengths can change. Angles fare no better – when you look at the corner of a building, a right angle, it only *looks* like a right angle if you view it square on.

What properties of geometrical figures, then, are preserved by projection? The most important ones are so simple that it is easy to miss their significance. Points remain points. Straight lines remain straight. The image of a point lying on a straight lines lies on the image of that line. Therefore, if two lines meet at a point, their images meet at the corresponding point. Incidence relations of points and lines are preserved by projection.

An important feature that is not *quite* preserved is the relation 'parallel'. Imagine standing in the middle of a long, straight road and look ahead. The two sides of the road, which in three-dimensional reality are parallel – so never meet – do not appear parallel. Instead, they converge towards a single point on the distant horizon. They behave like this on an ideal infinite plane, not just on a slightly rounded Earth. In fact, they only behave exactly like this on a plane. On a sphere, there would be a tiny gap, too small to see, where the lines cross the horizon. And the whole issue of parallel lines on a sphere is tricky anyway.

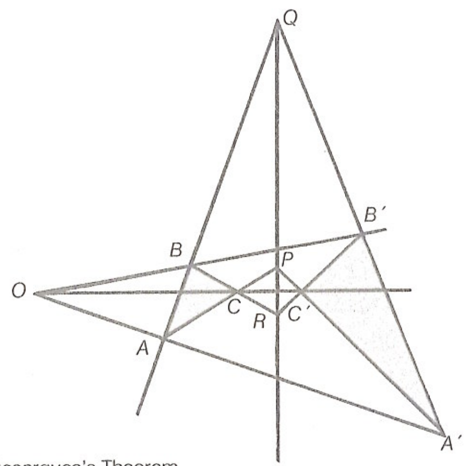
This feature of parallel lines is very useful in perspective drawing. It lies behind the usual way of drawing right-angled boxes in perspective, using a horizon line and two vanishing points, which are where parallel edges of the box cross the horizon in perspective. Piero della Francesca's *De Prospettiva Pingendi* (1482–87) developed Alberti's methods into practical techniques for artists, and he used them to great effect in his dramatic and very realistic paintings.

The writings of the Renaissance painters solved many problems in the geometry of perspective, but they were semi-empirical, lacking the kind of logical foundation that Euclid had supplied for ordinary geometry. These foundational issues were finally resolved by Brook Taylor and Johann Heinrich Lambert in the 18th century. But by then, more exciting things were going on in geometry.

Desargues

The first non-trivial theorem in projective geometry was found by the engineer/architect Girard Desargues and published in 1648 in a book by Abraham Bosse. Desargues proved the following remarkable theorem. Suppose that triangles ABC and $A'B'C'$ are in perspective, which means that the three lines AA', BB' and CC' all pass through the same point O. Then the three points P, Q and R at which corresponding sides of the two triangles meet all lie on the same line. This result is called Desargues's Theorem to this day. It mentions no lengths, no angles – it is purely about incidence relations among lines and points. So it is a projective theorem.

There is a trick which makes the theorem obvious: imagine it as a drawing of a three-dimensional figure, in which the two triangles lie in two planes. Then the line along which those planes intersect is the line containing Desargues's three points P, Q and R. With a little care,



Desargues's Theorem

the theorem can even be proved this way, by constructing a suitable three-dimensional figure whose projection looks like the two triangles. So we can use Euclidean methods to prove projective theorems.

Euclid's axioms

Projective geometry differs from Euclidean geometry as far as its viewpoint goes (pun intended), but it is still related to Euclidean geometry. It is the study of new kinds of transformation, projections, but the underlying model of the space that is being transformed is Euclidean. Nevertheless, projective geometry made mathematicians more receptive to the possibility of new kinds of geometric thinking. And an old question, one that had lain dormant for centuries, once more came to the fore.

Nearly all of Euclid's axioms for geometry were so obvious that no sane person could seriously question them. All right angles are equal, for instance. If that axiom failed, there had to be something wrong with the definition of a right angle. But the Fifth Postulate, the one that was really about parallel lines, had a distinctly different flavour. It was complicated. Euclid states it this way: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

It sounded more like a theorem than an axiom. *Was* it a theorem? Might there be some way to prove it, perhaps starting from something simpler, more intuitive?

One improvement was introduced by John Playfair in 1795. He substituted the statement that for any given line, and any point not on that line, there exists one and only one line through the point that is parallel to the given line. This statement is logically equivalent to Euclid's Fifth Postulate – that is, each is a consequence of the other, given the remaining axioms.

Legendre

In 1794 Adrien-Marie Legendre discovered another equivalent statement, the existence of *similar triangles* – triangles having the same angles, but with edges of different sizes. But he, and most other mathematicians, wanted something even more intuitive. In fact, there was a feeling that the Fifth Postulate was simply superfluous – a consequence of the other axioms. All that was missing was a proof. So Legendre tried all sorts of things. Using only the other axioms, he proved – to his own satisfaction, at any rate – that the angles of a triangle either add up to 180° or less. (He must have known that in spherical geometry the sum is greater, but that is the geometry of the sphere, not the plane.) If the sum is always 180°, the Fifth Postulate follows. So he assumed that the sum could be less than 180°, and developed the implications of that assumption.

A striking consequence was a relation between the triangle's area and the sum of its angles. Specifically, the area is proportional to the amount by which the angle sum falls short of 180°. This seemed promising: if he could construct a triangle whose sides were twice

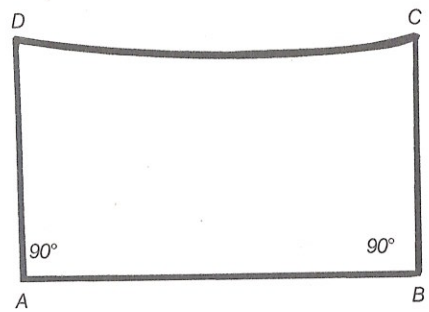
those of a given triangle, but with the same angles, then he would obtain a contradiction, because the larger triangle would not have the same area as the smaller one. But however he tried to construct the larger triangle, he found himself appealing to the Fifth Postulate.

He did manage to salvage one positive result from the work. Without assuming the Fifth Postulate, he proved that it is impossible for some triangles to have angle-sums greater than 180°, while others have angle-sums less than 180°. If one triangle has angles that summed to more than 180°, so does every triangle; similarly if the sum is less than 180°. So there are three possible cases:

- The angles of every triangle add up to 180° exactly (Euclidean geometry)
- The angles of every triangle add up to less than 180°
- The angles of every triangle add up to more than 180° (a case that Legendre thought he had excluded; it later turned out that he had made other unstated assumptions to do so).

Saccheri

In 1733 Gerolamo Saccheri, a Jesuit priest at Pavia, published a heroic effort, *Euclides ab Omni Naevo Vindicatus* (*Euclid Vindicated from All Flaws*). He also considered three cases, of which the first was Euclidean geometry, but he used a quadrilateral to make the



Saccheri's quadrilateral: the line CD has been drawn curved to avoid Euclidean assumptions about angles C and D distinction. Suppose the quadrilateral is *ABCD*, with *A* and B right angles and *AC* = *BD*. Then, said Saccheri, Euclidean geometry implies that angles C and D are right angles. Less obviously, if C and D are right angles in any one quadrilateral of this kind, then the Fifth Postulate follows.

Without using the Fifth Postulate, Saccheri proved that angles C and D are equal. So that left two distinct possibilities:

- *Hypothesis of the obtuse angle*: both C and D are greater than a right angle.
- *Hypothesis of the acute angle*: both C and D are less than a right angle.

Saccheri's idea was to assume each of these hypotheses in turn, and deduce a logical contradiction. That would then leave Euclidean geometry as the only logical possibility.

He began with the hypothesis of the obtuse angle, and in a series of theorems deduced – so he thought – that angles C and D must in fact be right angles after all. This was a contradiction, so the hypothesis of the obtuse angle had to be false. Next, he assumed the hypothesis of the acute angle, which led to another series of theorems, all correct, and fairly interesting in their own right. Eventually he proved a rather complicated theorem about a family of lines all passing through one point, which implied that two of these lines would have a common perpendicular at infinity. This is not actually a contradiction, but Saccheri thought it was, and declared the hypothesis of the acute angle to be disproved as well.

That left only Euclidean geometry, so Saccheri felt that his programme was vindicated, along with Euclid. But others noticed that he had not really obtained a contradiction from the hypothesis of the acute angle; just a rather surprising theorem. By 1759 d'Alembert declared the status of the Fifth Postulate to be 'the scandal of the elements of geometry'.

What Non-Euclidean geometry did for them

By 1813 Gauss was becoming ever more convinced that what he first called anti-Euclidean, then astral and finally non-Euclidean geometry was a logical possibility. He began to wonder what the true geometry of space was, and measured the angles of a triangle formed by three mountains near Göttingen – the Brocken, the Hohehagen and the Inselberg. He used line-of-sight measurement so the curvature of the Earth did not come into play. The sum of the angles that he measured was 15 seconds of arc greater than 180°. If anything, this was the obtuse-angle case, but the likelihood of observational errors made the whole exercise moot. Gauss needed a much bigger triangle, and much more accurate instruments to measure its angles.

Lambert

A German mathematician, Georg Klügel, read Saccheri's book, and offered the unorthodox and rather shocking opinion that belief in the truth of the Fifth Postulate was a matter of experience rather than logic. Basically, he was saying that something in the way we think about space makes us believe in the existence of parallel lines of the kind envisaged by Euclid.

In 1766 Johann Henrich Lambert, following up on Klügel's suggestion, embarked on an investigation that was similar to Saccheri's, but he started from a quadrilateral with three right angles. The remaining angle must either be a right angle (Euclidean geometry), acute or obtuse. Like Saccheri, he thought that the obtuse angle case led to a contradiction. More precisely, he decided that it led to spherical geometry, where it had long been known that the angles of a quadrilateral add up to more than 360°, because the angles of a triangle add up to more than 180°. Since the sphere is not the plane, the obtuse case is ruled out.

However, he did not claim the same for the acute angle case. Instead, he proved some curious theorems, the most striking being a formula for the area of a polygon with n sides. Add all the angles, and subtract this from $2n - 4$ right angles: the result is proportional to the polygon's area. This formula reminded Lambert of a similar formula for spherical geometry: add all the angles, and subtract $2n - 4$ right angles from this: again the result is proportional to the polygon's area. The difference is minor: the subtraction is performed in the opposite order. He was led to a remarkably prescient but obscure prediction: the geometry of the acute-angle case is the same as that on a sphere with *imaginary radius*.

He then wrote a short article about trigonometric functions of imaginary angles, obtaining some beautiful and perfectly consistent formulas. We now recognize these functions as the so-called hyperbolic functions, which can be defined without using imaginary numbers, and they satisfy all of Lambert's formulas. Clearly something interesting must lie behind his curious, enigmatic suggestion. But what?

Gauss's dilemma

By now the best-informed geometers were getting a definite feeling that Euclid's Fifth Postulate could not be proved from the remaining axioms. The acute-angle case seemed too self-consistent ever to lead to a contradiction. On the other hand, a sphere of imaginary radius was not the sort of object that could be proposed to justify that belief.

One such geometer was Gauss, who convinced himself from an early age that a logically consistent non-Euclidean geometry was possible, and proved numerous theorems in such a geometry. But, as he made clear in an 1829 letter to Bessel, he had no intention of publishing any of this work, because he feared what he called the 'clamour of the Boeotians'. Unimaginative people would not understand, and in their ignorance and hidebound adherence to tradition, they would ridicule the work. In this he may have been

influenced by the overarching status of Kant's widely acclaimed work in philosophy; Kant had argued that the geometry of space must be Euclidean.

By 1799 Gauss was writing to the Hungarian Wolfgang Bolyai, telling him that the research 'seems rather to compel me to doubt the truth of geometry itself. It is true that I have come upon much which by most people would be held to constitute a proof [of the Fifth Postulate from the other axioms]; but in my eyes it is as good as nothing.'

Other mathematicians were less circumspect. In 1826 Nikolai Ivanovich Lobachevsky, at the University of Kazan in Russia, gave lectures on non-Euclidean geometry. He knew nothing of Gauss's work, but had proved similar theorems using his own methods. Two papers on the topic appeared in 1829 and 1835. Rather than starting riots, as Gauss had feared, these papers sank pretty much without trace. By 1840 Lobachevsky was publishing a book on the topic, in which he complained about the lack of interest. In 1855 he published a further book on the subject.

Independently, Wolfgang Bolyai's son János, an officer in the army, came up with similar ideas around 1825, writing them up in a 26-page paper that was published as an appendix to his father's geometry text *Tentamen Juventum Studiosam in Elementa Matheseos* (*Essay on the Elements of Mathematics for Studious Youths*) of 1832. 'I have made such wonderful discoveries that I am myself lost in astonishment,' he wrote to his father.

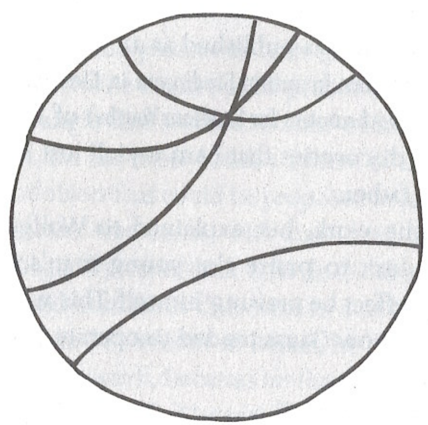
Gauss read the work, but explained to Wolfgang that it was impossible for him to praise the young man's efforts, because Gauss would in effect be praising himself. This was perhaps a little unfair, but it was how Gauss tended to operate.

Non-Euclidean geometry

The history of non-Euclidean geometry is too complicated to describe in any further detail, but we can summarize what followed these pioneering efforts. There is a deep unity behind the three cases noticed by Saccheri, by Lambert, and by Gauss, Bolyai and Lobachevsky. What unites them is the concept of *curvature*. Non-Euclidean geometry is really the natural geometry of a curved surface.

If the surface is positively curved like a sphere, then we have the case of the obtuse angle. This was rejected because spherical geometry differs from Euclidean in obvious ways – for example, any two lines, that is, great circles (circles whose centres are at the centre of the sphere), meet in two points, not the one that we expect of Euclidean straight lines.

Actually, we now realize that this objection is unfounded. If we identify diametrically opposite points of the sphere – that is, pretend they are identical – then lines (great circles) still make sense, because if a point lies on a great circle, so does the diametrically opposite point. With this identification, nearly all of the geometric properties remain unchanged, but now lines meet in *one* point. Topologically, the surface that results is the projective plane, although the geometry concerned is not orthodox projective



Poincaré's model of hyperbolic geometry makes it clear there are infinitely many parallel lines through a point that do not meet a given line

geometry. We now call it elliptic geometry, and it is considered just as sensible as Euclidean geometry.

If the surface is negatively curved, shaped like a saddle, then we have the case of the acute angle. The resulting geometry is called *hyperbolic*. It has numerous intriguing features, which distinguish it from Euclidean geometry.

If the surface has zero curvature, like a Euclidean plane, then it is the Euclidean plane, and we get Euclidean geometry.

All three geometries satisfy all of Euclid's axioms other than the Fifth Postulate. Euclid's decision to include his postulate is vindicated.

These various geometries can be modelled in several different ways. Hyperbolic geometry is especially versatile in this respect. In one model the space concerned is the upper half of the complex plane, omitting the real axis and everything below it. A line is a semicircle that meets the real axis at right angles. Topologically, this space is the same as a plane and the lines are identical to ordinary lines. The curvature of the lines reflects the negative curvature of the underlying space.

In a second model of hyperbolic geometry, introduced by Poincaré, the space is represented as the interior of a circle, not including its boundary, and the lines are circles that meet the boundary at right angles. Again, the distorted geometry reflects the curvature of the underlying space. The artist Maurits Escher produced many pictures based on this model of hyperbolic geometry, which he learned from the Canadian geometer Coxeter.

These two models hint at some deep connections between hyperbolic geometry and complex analysis. These connections relate to certain groups of transformations of the complex plane; hyperbolic geometry is the geometry of their invariants, according to Felix Klein's Erlangen Programme. Another class of transformations, called Möbius transformations, brings elliptic geometry into play as well.

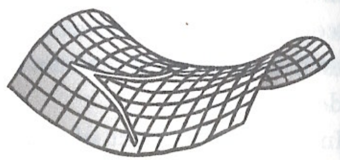
What Non-Euclidean geometry does for us

What shape is the universe? The question may seem simple but answering it is difficult – partly because the universe is so big, but mainly because we are inside it and cannot stand back and see it as a whole. In an analogy that goes back to Gauss, an ant living on a surface, and observing it only within that surface, could not easily tell if the surface was a plane, a sphere, a torus or something more complicated.

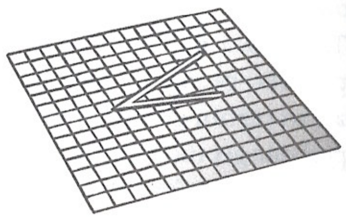
General relativity tells us that near a material body, such as a star, space–time is curved. Einstein's equations, which relate the curvature to the density of matter, have many different solutions. In the simplest ones,

Space with positive, negative and zero curvature

A *closed universe* curves back on itself. Lines that were diverging apart come back together. Density > critical density.



An *open universe* curves away from itself. Diverging lines curve at increasing angles away from each other. Density < critical density.



A *flat universe* has no curvature. Diverging lines remain at a constant angle with respect to each other. Density = critical density.

the universe as a whole has positive curvature and its topology is that of a sphere. But for all we can tell, the overall curvature of the real universe might be negative instead. We don't even know whether the universe is infinite, like Euclidean space, or is of finite extent, like a sphere. A few physicists maintain that the universe is infinite, but the experimental basis for this assertion is highly questionable. Most think it is finite.

To get Poincaré's dodecahedral space, identify opposite faces

Surprisingly, a finite universe can exist without having a boundary. The sphere is like that in two dimensions, and so is a torus. The torus can be given a *flat* geometry, inherited from a square by identifying opposite edges. Topologists have also discovered that space can be finite yet negatively curved: one way to construct such spaces is to take a finite polyhedron in hyperbolic space and identify various faces, so that a line passing out of the polyhedron across one face immediately re-enters at another face. This construction is analogous to the way the edges of the screen wrap round in many computer games.

If space is finite, then it should be possible to observe the same star in different directions, though it might seem much further away in some directions than in others, and the observable region of the universe might be too small anyway. If a finite space has hyperbolic geometry, these multiple occurrences of the same stars in different directions determine a system of gigantic circles in the heavens, and the geometry of those circles determines which hyperbolic space is being observed. But the circles could be anywhere among the billions of stars that can be seen, and so far attempts to observe them, based on statistical correlations among the apparent positions of stars, have not produced any result.

In 2003 data from the Wilkinson Microwave Anisotropy Probe led Jean-Pierre Luminet and his collaborators to propose that space is finite but *positively* curved. They found that Poincaré's dodecahedral space – obtained by identifying opposite faces of a curved dodecahedron – gives the best agreement with observations. This suggestion received wide publicity as the assertion that the universe is shaped like a football. This suggestion has not been confirmed and we currently have no idea of the true shape of space. However, we do have a much better understanding of what has to be done to find out.

The geometry of space

What of the geometry of space? We now agree with Klügel, and disdain Kant. This is a matter for experience, not something that can be deduced by thought alone. Einstein's General Relativity tells us that space (and time) can be curved; the curvature is the gravitational effect of matter. The curvature can vary from one location to another, depending on how the matter is distributed. So *the* geometry of space is not really the issue. Space can have different geometries in different places. Euclid's geometry works well on human scales, in the human world, because gravitational curvature is so small that we don't observe it in our daily lives. But out there in the greater universe, non-Euclidean geometries prevail.

To the ancients and indeed well into the 19th century, mathematics and the real world were hopelessly confused. There was a general belief that mathematics was a representation of basic and inevitable features of the real world, and that mathematical truth was absolute. Nowhere was this assumption more deeply rooted than

in classical geometry. Space was Euclidean, to virtually everyone who thought about the question. What else could it be?

This question ceased to be rhetorical when logically consistent alternatives to Euclid's geometry began to appear. It took time to recognize that they *were* logically consistent – at least, just as consistent as Euclid's geometry – and even longer to realize that our own physical space might not be perfectly Euclidean. As always, human parochialism was to blame – we were projecting our own limited experiences in one tiny corner of the universe on to the universe as a whole. Our imaginations do seem to be biased in favour of a Euclidean model, probably because, on the small scales of our experience, it is an excellent model and also the simplest one available.

Thanks to some imaginative and unorthodox thinking, often viciously contested by a less imaginative majority, it is now understood – by mathematicians and physicists, at least – that there are many alternatives to Euclid's geometry, and that the nature of physical space is a question for observation, not thought alone. Nowadays, we make a clear distinction between mathematical models of reality, and reality itself. For that matter, much of mathematics bears no obvious relation to reality at all – but is useful, all the same.