1. Let O denote the origin.

Show that the def of a combination of points $A_0, \dots, A_n$:

$$t_0 A_0 + \cdots + t_n A_n := O + t_0(A_0 - O) + \cdots + t_n(A_n - O)$$

does not depend on the choice of the origin $O \Leftrightarrow t_0 + \cdots + t_n = 1$

Ans: For a point P, one has

$$P + t_0(A_0 - P) + \cdots + t_n(A_n - P)$$
$$= O + (P - O) + t_0(A_0 - O - (P - O)) + \cdots + t_n(A_n - O - (P - O))$$
$$= O + t_0(A_0 - O) + \cdots + t_n(A_n - O) + (1 - t_0 - \cdots - t_n) \cdot (P - O)$$

This expression is equal to $O + t_0(A_0 - O) + \cdots + t_n(A_K - O)$

$\Leftrightarrow t_0 + \cdots + t_n = 1$

Alternatively,

$$O_1 + t_0(A_0 - O_1) + \cdots t_n(A_n - O_1) - (O_2 + t_0(A_0 - O_2) + \cdots t_n(A_n - O_2))$$
$$= (O_1 - O_2) - (t_0 + \cdots + t_n)(O_1 - O_2)$$
$$= 0.$$

2. Show that $(E_0, E_1, \dots, E_n)$ is a affine basis $\Leftrightarrow$ $(E_1 - E_0, \dots E_n - E_0)$ is an linear basis for the asso vect sp. (direction)

Linear: lin comb

Affine: aff comb (Σ coeff $= 1$)

Ans: Let A be a point, we express affinely as

$$A = a_0 E_0 + \cdots + a_n E_n \quad , \Sigma a_i = 1 \quad , a_i \text{ lin ind}$$
$$\overset{\text{WLOG}}{=} (1 - (a_1 + \cdots + a_n)) E_0 + a_1 E_1 + \cdots + a_n E_n$$
$$= E_0 + a_1(E_1 - E_0) + \cdots + a_n(E_n - E_0)$$

is a vector in the vector space with basis $E_i - E_0$.

Let B be a point in a vector space with origin E_0, expressed as

$$B = E_0 + b_0(E_1 - E_0) + \cdots + b_n(E_n - E_0)$$
$$= (1 - (b_0 + \cdots + b_n)) E_0 + b_0 E_1 + \cdots + b_n E_n$$

in which we have $1 - (b_0 + \cdots + b_n) + b_0 + \cdots + b_n = 1 \Rightarrow$ affine

3. a) Let $B \subseteq A_n$ be an affine subspace, where

$A_n = \{(1, x_1, \dots, x_n)\} = \{(x_0, x_1, \dots, x_n) \mid x_0 = 1\}$

Find a vect space $\hat{B} \subseteq \mathbb{k}^{n+1}$ s.t. $B \subseteq \hat{B}$ as an affine hyperplane not passing through the origin.

b) For an affine subspace B described by a system of linear equations

$$B : \quad A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b, \quad x_0 = 1 \quad ,$$

parametrisation

$B \subseteq A_n$

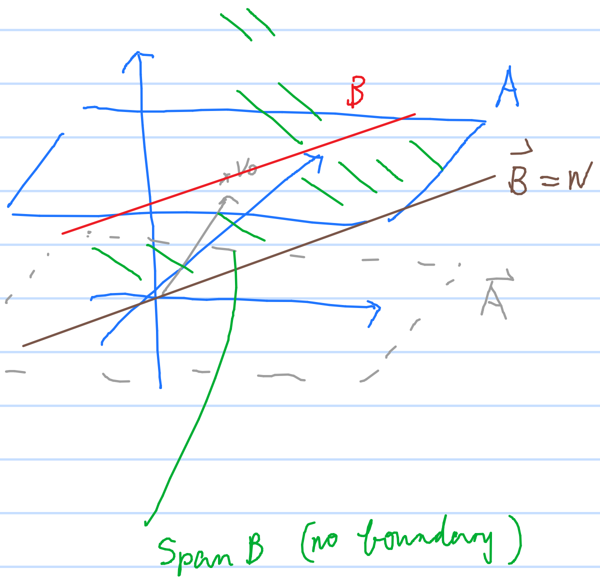
describe the subspace $\hat{B}$ in a similar way.

Ans: $B = v_0 + W$ where $W \subseteq \mathrm{Dir}\, A_n$, $v_0 \in A_n$

Let $\{w_1, \dots, w_m\}$ be a basis for W.

Consider $\mathrm{Span}\, B := \mathrm{span}(v_0, w_1, \dots, w_m)$

$(v_0 \neq 0$ since $v_0 = (1, x_1, \dots, x_n) \in A_n)$



4. Prove that the affine maps $\varphi : A_n \to A_m$ correspond bijectively to linear maps $\psi : \mathbb{k}^{n+1} \to \mathbb{k}^{m+1}$ satisfying $\psi(A_n) \subseteq A_m$.

Here, an affine map is $f : A \to B$ s.t.

$$\mathrm{Dir}\, f : \mathrm{Dir} A \to \mathrm{Dir} B$$
$$b-a \mapsto f(b)-f(a) \quad \text{is linear map}$$

This implies $f(a+v) = f(a) + \mathrm{Dir} f(v)$, $a \in A$, $v \in \mathrm{Dir} A$

Ans: For a linear map $\psi : \mathbb{k}^{n+1} \to \mathbb{k}^{m+1}$,

$\varphi := \psi|_{A_n} : A_n \to A_m$ is clearly a function.

Consider $\mathrm{Dir}\,\varphi := \psi|_{\mathrm{Dir} A_n} : \mathrm{Dir} A_n \to \mathrm{Dir} A_m$, which is clearly linear.

For $b-a \in \mathrm{Dir} A_n$,
$$\begin{aligned} \mathrm{Dir}\,\varphi\,(b-a) &= \psi(b-a) \\ &= \psi(b) - \psi(a) \qquad b, a \in A_n \\ &= \varphi(b) - \varphi(a) \end{aligned}$$

$\therefore$ φ is an affine map.

Conversely, let $\varphi : A_n \to A_m$ be an affine map with an induced linear map $\mathrm{Dir}\,\varphi : \mathrm{Dir} A_n \to \mathrm{Dir} A_m$.

Now let E_0 be a chosen origin in A_n and e_i be the linear basis for $\mathrm{Dir} A_n$. $E_1 - E_0$, $E_2 - E_0$, $\dots$

Define $\psi = (\psi(E_0), \psi(e_1), \dots, \psi(e_n))$

by $(\varphi(E_0), \mathrm{Dir}\,\varphi(e_1), \dots, \mathrm{Dir}\,\varphi(e_n))$, here $\varphi(E_0) \in A_m$, $\varphi(e_i) \in \mathrm{Dir} A_m$

$$= \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

This is obviously a linear map.

$$n+1\left\{\begin{pmatrix} 1 & 0\,0 \cdots 0 \\ \begin{pmatrix} \\ \end{pmatrix} & \underbrace{\begin{pmatrix} \quad \end{pmatrix}}_{n \times n} \end{pmatrix}\right.$$

It remains to show that $\psi|_{A_n} = \varphi$.

Let $(1, x_1, \dots, x_n) \in A_n$.

$$\begin{aligned} \psi(1, x_1, \dots, x_n)^T &= \psi(1 \cdot E_0 + x_1 \cdot e_1 + \cdots + x_n \cdot e_n) \\ &= \psi(E_0) + x_1 \cdot \psi(e_1) + \cdots + x_n \cdot \psi(e_n) \\ &= \varphi(E_0) + x_1 \cdot \mathrm{Dir}\,\varphi(e_1) + \cdots + x_n \cdot \mathrm{Dir}\,\varphi(e_n) \\ &= \varphi(E_0 + x_1 \cdot e_1 + \cdots + x_n \cdot e_n) \\ &= \varphi(1, x_1, \dots, x_n)^T \end{aligned}$$

Let $\{F_0, f_1, \dots, f_m\}$ be basis for $\mathbb{k}^{m+1}$,

where we set $F_0 := \varphi(E_0)$

$(\varphi(E_0), \mathrm{Dir}\,\varphi(e_1), \dots, \mathrm{Dir}\,\varphi(e_n)) =$

$\varphi(E_0) = M_{11} F_0 + M_{21} f_1 + \cdots$

$:= F_0 \Rightarrow M_{11} = 1$

$\mathrm{Dir}\,\varphi(e_1) = A_{11} f_1 + A_{21} f_2 + \cdots A_{m1} f_m$

$\mathrm{Dir}\,\varphi(e_k) = A_{1k} f_1 + A_{2k} f_2 + \cdots + A_{mk} f_m$

$\Rightarrow M_{1k} = 0$