Two-phase Simplex Method

We should apply the two-phase method for an artificial problem, though for most of the cases, one-phase is enough.

Mechanism:

1^{st} phase:

Artificial problem

If we have a LP

$\min (c^T x \mid Ax = b, x \geq 0)$ where $b \geq 0$,

but A does not contain identity submatrix,

we introduce artificial variables t_i, the new LP is

$\min ((1\ 1 \cdots 1)^T t \mid Ax + Et = b, x \geq 0, t \geq 0)$ where $b \geq 0$

However, this is not canonical now as $A+E$ does not contain an identity submatrix.

We can subtract all the rows of constraints

i.e., new obj fun = row of obj fun $- \sum$ rows of constraint,

$\Rightarrow$ this gives the canonical form.

2^{nd} phase: (given that all $t_i = 0$)

1. Remove the artificial columns.
2. Replace the objective row with the initial problem.
   For a basic variable x_i, if the corresponding $c_i \neq 0$, then add multiples of rows of constraint to turn it 0.
3. Repeat the simplex method.

5.8 $\quad \min\left\{ (0 \;\; 15 \;\; 30 \;\; 0 \;\; 15)\, x \;\middle|\; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 & 3 \end{pmatrix} x \geq \begin{pmatrix} 120 \\ 80 \\ 110 \end{pmatrix}, x \geq 0 \right\}$

Ans : ① Pre-table

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	t_1	t_2	t_3	$-Z$	RHS
1	1	0	0	0	-1	0	0	1	0	0	0	120
1	0	2	1	0	0	-1	0	0	1	0	0	80
0	1	0	2	3	0	0	-1	0	0	1	0	110
0	0	0	0	0	0	0	0	1	1	1	1	0

$\Rightarrow$ Table

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	t_1	t_2	t_3	$-Z$	RHS	
0	1	-2	-1	0	-1	1	0	1	-1	0	0	40	$-p$
1	0	2	1	0	0	-1	0	0	1	0	0	80	
0	1	0	2	3	0	0	-1	0	0	1	0	110	
0	-2	2	-1	-3	1	-1	1	0	2	0	1	-150	$+2p$

$c_1 < 0 \qquad \min = 80 \quad , \quad$ pivot at $(2,1)$

$\Rightarrow$ Table

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	t_1	t_2	t_3	$-Z$	RHS	
0	1	-2	-1	0	-1	1	0	1	-1	0	0	40	
1	0	2	1	0	0	-1	0	0	1	0	0	80	
0	0	2	3	3	1	-1	-1	-1	1	1	0	70	$-p$
0	0	-2	-3	-3	-1	1	1	2	0	0	1	-70	$+2p$

$c_2 < 0$, pivot at $(1,2)$

Table

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	t_1	t_2	t_3	$-Z$	RHS	
0	1	0	2	3	0	0	-1	0	0	1	0	110	$+p$
1	0	0	-2	-3	-1	0	1	1	0	-1	0	10	$-p$
0	0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	35	$/2$
0	0	0	0	0	0	0	0	2	1	1	1	0	$+p$

$c_3 < 0$, pivot at $(3,3)$

basic $= x_2, x_1, x_3, z$, set non-basic $= 0$,
so $x_2 = 110$, $x_3 = 35$, $x_1 = 10$
$x_4 = x_5 = 0$

Now proceed to the 2^{nd} phase.

New Table

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3		$-Z$	RHS
0	1	0	2	3	0	0	-1		0	110
1	0	0	-2	-3	-1	0	1		0	10
0	0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$		0	35
0	15	30	0	15	0	0	0	0	1	0

$c_2, c_3 \neq 0$, so we manipulate:

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	$-Z$	RHS
	0	1	0	2	3	0	0	-1	0	110
	1	0	0	-2	-3	-1	0	1	0	10
	0	0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	35
①	0	0	30	-30	-30	0	0	15	1	-1650
②	0	0	0	-75	-75	-15	15	30	1	-2700

Pivot at (3, 4):

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	$-Z$	RHS
0	1	$-\frac{4}{3}$	0	1	$\frac{-2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{140}{3}$
1	0	$\frac{4}{3}$	0	-1	$-\frac{1}{3}$	$\frac{-2}{3}$	$\frac{1}{3}$	0	$\frac{170}{3}$
0	0	$\frac{2}{3}$	1	1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{70}{3}$
0	0	50	0	0	10	-10	5	1	-950

Pivot at (1, 7):

x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	$-Z$	RHS
0	$\frac{3}{2}$	-2	0	$\frac{3}{2}$	-1	1	$-\frac{1}{2}$	0	95
1	1	0	0	0	-1	0	0	0	120
0	$\frac{1}{2}$	0	1	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	0	55
0	15	30	0	15	0	0	0	1	0

$\therefore x = (120 \quad 0 \quad 0 \quad 55 \quad 0)^T, \qquad z = 0$

More on the universal property of tensor products

E.g. tensor product of maps:

Given linear maps $\varphi : U \to U'$, $\psi : V \to V'$, and the bilinear map $t : U \times V \to U \otimes V$, $(u, v) \mapsto u \otimes v$, we have a unique map $f : U \otimes V \to U' \otimes V'$:

$$\begin{tikzcd} U \times V \arrow[r, "\varphi \otimes \psi"] \arrow[d, "t"'] & U' \otimes V' \\ U \otimes V \arrow[ur, dashed, "\exists!\, f"'] & \end{tikzcd}$$

where $\varphi \otimes \psi : U \times V \to U' \otimes V'$
$u, v \mapsto \varphi(u) \otimes \psi(v)$

Pf. Define $f : U \otimes V \to U' \otimes V'$
$u_i \otimes v_i \mapsto \varphi(u_i) \otimes \psi(v_i)$

where $u_i \otimes v_i$ is a basis vector.

Then f is linear:

$$\begin{aligned} & f(a u_1 \otimes v_1 + u_2 \otimes v_2) \\ &= f(a u_1 \otimes v_1) + f(u_2 \otimes v_2) \\ &= \varphi(a u_1) \otimes \psi(v_1) + \varphi(u_2) \otimes \psi(v_2) \\ &= a \varphi(u_1) \otimes \psi(v_1) + \varphi(u_2) \otimes \psi(v_2) \\ &= a f(u_1 \otimes v_1) + f(u_2 \otimes v_2). \end{aligned}$$

The diagram commutes:

$$\begin{aligned} f t(u, v) = f(u \otimes v) &= \varphi(u) \otimes \psi(v) \\ &= \varphi \otimes \psi(u, v) \end{aligned}$$

Uniqueness:

Suppose $g t = f t$, then

$g(u_i \otimes v_i) = f(u_i \otimes v_i)$ for basis vector $u_i \otimes v_i$

$\Rightarrow g = f$.

Rk. Indeed $U' \otimes V'$ can be arbitrary, since in CT, we say that the functor $\mathrm{Bilin}(U, V; -) : \mathrm{Vect}_{\mathbb{k}} \to \mathrm{Set}$, $W \mapsto U \times V \to W$

is represented by $U \otimes V$, i.e.

$$\mathrm{Bilin}(U, V; -) \cong \mathrm{Vect}_{\mathbb{k}}(U \otimes V, -)$$

so for any arbitrary W,

$$\mathrm{Bilin}(U, V; W) \cong \mathrm{Vect}_{\mathbb{k}}(U \otimes V, W)$$

$$\begin{tikzcd} U \times V \arrow[r] \arrow[d, "t"'] & W \\ U \otimes V \arrow[ur, dashed] & \end{tikzcd}$$